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COMPLETENESS AND COMPACTNESS IN TYPE-2 FUZZY METRIC SPACES

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Abstract. In this paper completeness and compactness in type-2 fuzzy metric spaces are studied. Cantor's intersection theorem characterizing the completeness of a type-2 fuzzy metric space is proved. Baire's category theorem is also extended in this space. Notions of compactness, sequential compactness, total boundedness and Lebesgue number have been introduced. It is shown that a type-2 fuzzy metric space is compact if and only if it is totally bounded and complete. It is shown that in this space sequential compactness and compactness are equivalent to each other.

Keywords: Fuzzy metric, type-2 fuzzy metric, completeness, compactness, fuzzy numbers.

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1. Introduction

Motivating from the idea of Probabilistic metric space,in 1975, Kramosil and Michalek [9] first introduced the concept of fuzzy metric space replacing the role of probability distribution by an appropriate fuzzy set. Afterwards several authors introduced fuzzy metric in different ways. In 1979, Ercez [4] defined fuzzy metric considering distance between two fuzzy sets. Deng [5] in 1982, defined fuzzy metric as a distance between two fuzzy points. In 1984, approaching in a different way Kaleva and Seikkala [8] defined fuzzy metric on a nonempty crisp set X as a non-negative fuzzy real number valued function over $X \times X$. Afterwards in 1989, T. Bandyopadhyay et. al. [1] proposed a definition of fuzzy metric generalizing both of Deng and Kaleva-Seikkala. In 1994, with a view to obtain a Hausdorff topology from a fuzzy metric, George and Veeramani [6] made a slight modification of the fuzzy metric introduced by Kramosil-Machalek [9].
Other types of generalizations of fuzzy metric were continued to evolve. In this direction Park [12] introduced intuitionistic fuzzy metric space in 2004 and later its modified version by Sadati et al. [13] in 2008. Shen et al. [16] introduced a definition of interval-valued

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fuzzy metric in 2012.

On the other hand, in order to deal with higher order uncertainties, Zadeh [18] in 1975, proposed an idea of type-2 fuzzy sets, which generalizes his own idea of fuzzy sets [17]. Since then works are going on in the field of type-2 fuzzy setting in set theory, logic, algebra and their applications in decision making problems ([2, 7]). Recently in [14], we have first introduced a definition of type-2 fuzzy metric space and studied some of its properties. As a continuation of this, in this paper, we study completeness, compactness, sequential compactness, total boundedness and their interrelations in a type-2 fuzzy metric space. Cantor's intersection theorem and Baire's category theorem are also extended in this space. In Section 4, compactness related results are studied in type-2 fuzzy metric spaces.

The organization of the paper is as follows: In the Preliminary section we take all the definitions and results which are to be used in the rest of this paper. In Section 3, we prove Cantor's intersection theorem characterizing the completeness of a type-2 fuzzy metric space. Next the Baire's category theorem is also extended in this space. In Section 4, compactness related results are studied in type-2 fuzzy metric spaces.

2. Preliminaries

Definition 2.1. [15] *A t-norm is a function* $*[0,1]\times[0,1]\to[0,1]$ *which is commutative, associative, monotonic increasing w.r.t. both the components and* $a*1=a, \forall a\in[0,1]$.

In 1975, Kramosil and Michalek [9] introduced a definition of a fuzzy metric space which is given below.

Definition 2.2. [9] *The triplet* $(X,M,*)$ *is a fuzzy metric space if* X *is a nonempty set, * is a continuous t-norm and* M *is a fuzzy set on* $(X^2\times\mathbb{R})$ *satisfying for all* $x,y,z\in X$ *and* $t,s\in\mathbb{R}$ *the following axioms:*
(KM1) $M(x,y,t)=0, \forall t<0$.
(KM2) $M(x,y,t)=1, \forall t>0 \iff x=y$.
(KM3) $M(x,y,t)=M(y,x,t)$.
(KM4) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$.
(KM5) $M(x,y,.):(0,\infty)\to[0,1]$ *is left continuous and non -decreasing*
(KM6) $\lim_{t\to\infty} M(x,y,t)=1$.

In [6], George and Veeramani slightly changed some of the above conditions to introduce following definition of a fuzzy metric space whose induced topology is Hausdorff.

Definition 2.3. [6] *The triplet* $(X,M,*)$ *is said to be a fuzzy metric space if* X *is a nonempty set, '*' is a continuous t-norm and* M *is a fuzzy set on* $X^2\times(0,\infty)$ *satisfying for all* $x,y,z\in X$ *and* $t,s>0$ *the following axioms:*
(GV1) $M(x,y,t)>0$
(GV2) $M(x,y,t)=1\ \forall t>0 \iff x=y$
(GV3) $M(x,y,t)=M(y,x,t)$
(GV4) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$
(GV5) $M(x,y,.):(0,\infty)\to[0,1]$ *is continuous.*

On the other hand taking the value set of the fuzzy metric to be the set of non-negative fuzzy real numbers, Kaleva and Seikkala [8] introduced the definition of fuzzy metric in a

different way.
Before we present the definition of fuzzy metric introduced by Kaleva and Seikkala, we give the definition and some properties of fuzzy real numbers.

Definition 2.4. *A fuzzy real number u is a function from $\mathbb{R}$ to $[0,1]$ which is is upper semicontinuous, convex and normal. A non-negative fuzzy real number is a fuzzy real number u for which $u(t) = 0, \forall t < 0$. The set of all non-negative fuzzy real numbers is denoted by $\widehat{\mathbb{R}}^+$. For any $\alpha \in (0,1]$, $\alpha -$ level set $[u]_\alpha$ is a closed bounded interval $[u^-_\alpha, u^+_\alpha]$. For any $r \in \mathbb{R}, \overline{r}$ is the fuzzy real number defined by $\overline{r}(x) = 0$, for $x \neq r$ and $\overline{r}(x) = 1$ for $x = r$ As in Congxin and Cong* [3]*, it is also assumed in this paper that $[u]_0 = cl\{x \in \mathbb{R}, u(x) > 0\}$ is compact. For any fuzzy real number t, define $t^\circ = \overline{\dfrac{a+b}{2}}$, where $[t]_0 = [a,b]$, and t° is called first approximation of the fuzzy real number t.*

Following [8, 10], some algebraic operations are defined below:
$(u \oplus v)(t) = \sup\limits_{s\in\mathbb{R}}\{u(s) \wedge v(t-s)\}$
$(u \ominus v)(t) = \sup\limits_{s\in\mathbb{R}}\{u(s) \wedge v(s-t)\}$
$(u \odot v)(t) = \sup\limits_{s\in\mathbb{R},\ s\neq 0}\{u(s) \wedge v(t/s)\}$
$(u \oslash v)(t) = \sup\limits_{s\in\mathbb{R}}\{u(st) \wedge v(s)\}$

For $k > 0, ku(t) = u(t/k)$ and $0u = \overline{0}$.
It can be shown that for $u, v \in \widehat{\mathbb{R}}^+, k > 0$,
$[u \oplus v]_\alpha = [u^-(\alpha) + v^-(\alpha), u^+(\alpha) + v^+(\alpha)]$;
$[u \ominus v]_\alpha = [u^-(\alpha) - v^+(\alpha), u^+(\alpha) - v^-(\alpha)]$;
$[u \odot v]_\alpha = [u^-(\alpha)v^-(\alpha), u^+(\alpha)v^+(\alpha)]$;
$[u \oslash v]_\alpha = [u^-(\alpha)/v^+(\alpha), u^+(\alpha)/v^-(\alpha)]$;
$[ku]_\alpha = [ku^-(\alpha), ku^+(\alpha)$.
Inequalities among the members of $\widehat{\mathbb{R}}^+$ are defined as
$u \leq v$ if $u^-(\alpha) \leq v^-(\alpha)$ and $u^+(\alpha) \leq v^+(\alpha), \forall \alpha \in (0,1]$
$u < v$ if $u^-(\alpha) < v^-(\alpha)$ and $u^+(\alpha) < v^+(\alpha), \forall \alpha \in [0,1]$
$u \ll v$ if there is $\epsilon > 0$ such that $u \oplus \overline{\epsilon} \leq v$
For simplicity, whenever necessary, we shall use the symbols + and - instead of $\oplus$ and $\ominus$ respectively, and juxtaposition for the product $\odot$.

For our purpose we shall take fuzzy real numbers as considered in Congxin and Cong [3]. A subset A of $\widehat{\mathbb{R}}$ (set of all fuzzy real numbers) is said to be bounded above if there exists a fuzzy number M such that $u \leq M$, for all $u \in A$. M is called the supremum of A if M is an upper bound of A and $M \leq W$ for any upper bound W of A. It is denoted by $SupA$ or $\vee A$. A lower bound and infimum of A are defined similarly. Existence of supremum for bounded above set and existence of infimum of bounded below set of fuzzy real numbers have been established by Congxin and Cong [3]. The set of all fuzzy real numbers $\alpha > \overline{0}$ is denoted by E and the set of all fuzzy real numbers α such that $\overline{0} \leq \alpha \leq \overline{1}$ is denoted by I.

Definition 2.5. [14] *Let * be a t-norm on* $[0,1]$*. Then the induced mapping $* : I \times I \to I$, defined by $a * b = c$, where $c^-(\lambda) = a^-(\lambda) * b^-(\lambda), c^+(\lambda) = a^+(\lambda) * b^+(\lambda), \forall \lambda \in [0,1]$, is called the induced t-norm on I. Clearly if * is a continuous t-norm on* $[0,1]$*, then the induced t-norm is continuous.*

Definition 2.6. [8] *Let X be a nonempty set and $d : X \times X \to \widehat{\mathbb{R}}^+$ be a mapping satisfying the following conditions for $x, y, z \in X$*
(KS1) $d(x, y) = \overline{0}$ if and only if $x = y$
(KS2) $d(x, y) = d(y, x)$
(KS3) $d(x, y) \leq d(x, z) \oplus d(z, y)$
Then d is said to be a fuzzy metric on X and (X, d) is said to be a fuzzy metric space.

In [14], Type-2 fuzzy metric space is defined as follows:

Definition 2.7. [14] *Let X be a nonempty set. A mapping $M : X \times X \times E \to I$ satisfying the conditions for all $x, y, z \in X$ and $s, t \in E$*
$(M1) M(x, y, t) \gg \overline{0}$;
(M2) For any given $\overline{0} \ll \alpha \ll \overline{1}, M(x, y, t) \geq \alpha$ for all $t \gg \overline{0}$ if and only if $x = y$;
$(M3) M(x, y, t) = M(y, x, t)$;
(M4) For any $x, y \in X$ and any $\overline{0} \ll \alpha \ll \overline{1}$, there is $t(\gg \overline{0}) \in E$ such that $M(x, y, s) \geq \alpha, \forall s(\geq t) \in E$;
$(M5) M(x, y, t + s) \geq M(x, z, t) * M(z, y, s), t, s \in E$;
$(M6) M(x, y, .)$ *is monotonically increasing and continuous, where * is the induced t-norm on I,*
is called a type-2 fuzzy metric (or simply a fuzzy metric) on X, and (X, M) is called a type-2 fuzzy metric space (or simply a fuzzy metric space).

Definition 2.8. [14] *In a fuzzy metric space (X, M), for $x \in X, t \gg \overline{0}, \overline{0} \ll \alpha \ll \overline{1}$ the set $B_\alpha(x, t) = \{y \in X; M(x, y, t) \gg \overline{1} - \alpha\}$ is called an open ball about x with radius t and grade α.*

Definition 2.9. [14] *A set $O(\subset X)$ is said to be an open set in a fuzzy metric space (X, M) if for each $x \in O, \exists t \gg \overline{0}$ and $\overline{0} \ll \alpha \ll \overline{1}$ such that $B_\alpha(x, t) \subset O$*

Remark 2.1. *It is shown in* [14] *that the underlying topology of the fuzzy metric space is Hausdorff and first countable.*

Definition 2.10. [14] *A sequence $\{x_n\}$ in a fuzzy metric space (X, M) is said to be convergent if there is $x \in X$ such that for any $\epsilon \gg \overline{0}$ and for any $\overline{0} \ll \alpha \ll \overline{1}$ there is a positive integer N such that $M(x_n, x, t) \geq \alpha, \forall n \geq N$.*
'x' is called the limit of $\{x_n\}$.

Definition 2.11. [14] *A sequence $\{x_n\}$ in a fuzzy metric (X, M) is said to be Cauchy if for any $\epsilon \gg \overline{0}$ and for any $\overline{0} \ll \alpha \ll \overline{1}$ there is a positive integer N such that $M(x_n, y_m, \epsilon) \geq \alpha, \forall m, n \geq N$.*

Definition 2.12. [14] *A fuzzy metric space (X, M) is said to be complete if every Cauchy sequence in X is convergent.*

3. Completeness of Type-2 Fuzzy Metric Spaces

Definition 3.1. *Let (X, M) be a fuzzy metric space and $Y \subset X$. Then Y is said to be bounded if for any $\overline{0} \ll \alpha \ll \overline{1}$ there is a fuzzy real number $K \gg \overline{0}$ such that $M(x, y, K) \geq \alpha, \forall x, y \in Y$.*

Theorem 3.1. *Every Cauchy sequence is bounded.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in a fuzzy metric space (X, M). Take $\overline{1} \gg \alpha \gg \overline{0}$. By the continuity of the t-norm '$*$', there exists $\alpha' \gg \overline{0}$ such that $\alpha' * \alpha' \geq \alpha$.
Since $\{x_n\}$ is a Cauchy sequence, for $\epsilon = \overline{1}$, there exists a positive integer n_0 such that
$M(x_n, x_m, \overline{1}) \geq \alpha' \geq \alpha, \forall n, m \geq n_0$..................(i)
Again there exists a positive number K such that
$M(x_i, x_j, K) \geq \alpha' \geq \alpha \ \forall i, j = 1, 2, ..., n_0 ....(ii)$
Now for all positive integer $1 \leq i \leq n_0$ and $j \geq n_0$
$M(x_i, x_j, K + \overline{1}) \geq M(x_i, x_{n_0}, K) * M(x_{n_0}, x_j, \overline{1}) \geq \alpha' * \alpha' \geq \alpha.$ ....(iii)
Combining (i), (ii) and (iii), we get that
$M(x_i, x_j, K + \overline{1}) \geq \alpha, \forall i, j = 1.2, ...$
Hence $\{x_n\}$ is bounded. □

Theorem 3.2. *In a fuzzy metric space every Cauchy sequence having a convergent subsequence is convergent.*

Proof. Let (X, M) be a fuzzy metric space and $\{x_n\}$ be a Cauchy sequence in (X, M) having a convergent subsequence, say, $\{x_{n_i}\}$ converging to $x \in X$. Choose $\epsilon \gg \overline{0}$ and $\overline{0} \ll \alpha \ll \overline{1}$. Then there exists $\overline{0} \ll \alpha' \ll \overline{1}$ such that $\alpha' * \alpha' \geq \alpha$. By the Cauchyness of $\{x_n\}$, there exists a positive integer n_0 such that
$M(x_n, x_m, \epsilon/2) \geq \alpha', \forall n, m \geq n_0$.
By the convergence of $\{x_{n_i}\}$ to x, there exists a positive integer i_0 such that
$M(x_{n_i}, x, \epsilon/2) \geq \alpha', \forall i \geq i_0$.
Choose $i_1 \geq i_0$ such that $n_i \geq n_0 \forall i \geq i_1$. Then for $n \geq n_0$,
$M(x_n, x, \epsilon) \geq M(x_{n_{i_1}}, x_n, \epsilon/2) * M(x_{n_{i_1}}, x, \epsilon/2) \geq \alpha' * \alpha' \geq \alpha.$
Hence $\{x_n\}$ converges to x. □

Theorem 3.3. *Let (X, M) be a fuzzy metric space and $F \subset X$. If for any given $\epsilon \gg \overline{0}$ and $\overline{0} \ll \alpha \ll \overline{1}, M(x, y, \epsilon) \geq \alpha$ holds $\forall x, y \in F$ then $M(\xi, \eta, \epsilon) \geq \alpha$ holds $\forall \xi, \eta \in \overline{F}$($\overline{F}$ denotes the closure of F).*

Proof. Since $\xi, \eta \in \overline{F}$, there exist sequences $\{x_n\}, \{y_n\}$ in F such that $x_n \to \xi$ and $y_n \to \eta$. Since $\overline{0} \ll \alpha \ll \overline{1}$, there exists a positive real number r such that $\overline{0} \ll \alpha - \overline{r} \ll \alpha$. Then there exists $\overline{0} \ll \alpha' \ll \overline{1}$ such that $\alpha' * (\alpha - \frac{1}{2}\overline{r}) * \alpha' \gg \alpha - \overline{r}$.
Since $x_n \to \xi$ and $y_n \to \eta$, for any $\delta > 0$, there exists a positive integer $n(\delta)$ such that
$M(\xi, x_n, \overline{\delta}/2) \geq \alpha'$ and $M(\eta, y_n, \overline{\delta}/2) \geq \alpha', \forall n \geq n(\delta)$.
Then $M(\xi, \eta, \epsilon + \overline{\delta})$
$\geq M(\xi, x_{n(\delta)}, \overline{\delta}/2) * M(x_{n(\delta)}, y_{n(\delta)}, \epsilon) * M(y_{n(\delta)}, \eta, \overline{\delta}/2)$
$\geq \alpha' * \alpha * \alpha'$
$\geq \alpha' * (\alpha - \frac{1}{2}\overline{r}) * \alpha'$
$\gg \alpha - \overline{r}$.
Since r can be made a positive number as small as we please, $M(\xi, \eta, \epsilon + \overline{\delta}) \geq \alpha$.
Further letting $\delta \to 0$, by using continuity of $M(\xi, \eta, .)$,
$M(\xi, \eta, \epsilon) = lim_{\delta \to 0} M(\xi, \eta, \epsilon + \overline{\delta}) \geq \alpha.$ □

3.1. **Cantor's Theorem and Baire's Category Theorem.** In this section we shall prove Cantor's intersection theorem and Baire's category Type-2 fuzzy metric space. In this context it might be mentioned that Mazumder and Bag [11] extended Cantor's theorem in Cone fuzzy metric space and George and Veeramani [6] proved Baire's category theorem in fuzzy metric space.

Theorem 3.4. *(Cantor's theorem): A fuzzy metric space (X, M) is complete if and only if for every decreasing sequence of closed sets $F_1 \supset F_2 \supset ..... \supset F_n \supset ...$ such that there exist $\epsilon_n \gg \overline{0}$ and $\overline{1} \gg \alpha_n \gg \overline{0}, n \in \mathbb{N}$ with $\epsilon_n \to \overline{0}$ and $\alpha_n \to \overline{1}$ and $M(x, y, \epsilon_n) \geq \alpha_n, \forall x, y \in F_n, \forall n \in \mathbb{N}$, the intersection $\bigcap_{n=1}^{\infty} F_n$ is a singleton set.*

Proof. Suppose (X, M) is complete. Let $F_1 \supset F_2 \supset .... \supset F_n \supset ....$ be a decreasing sequence of closed sets such that there exist $\epsilon_n \gg \overline{0}$ and $\overline{1} \gg \alpha_n \gg \overline{0}, n \in \mathbb{N}$ with $\epsilon_n \to \overline{0}$ and $\alpha_n \to \overline{1}$ and $M(x, y, \epsilon_n) \geq \alpha_n, \forall x, y \in F_n, \forall n \in \mathbb{N}$. Choose $x_n \in F_n, n \in \mathbb{N}$. Then for any $n, m \geq n_0, x_n, x_m \in F_{n_0}$ and hence $M(x_n, x_m, \epsilon_{n_o}) \geq \alpha_{n_0}$.
Take an $\epsilon \gg \overline{0}$ and $\overline{0} \ll \alpha \ll \overline{1}$. Since $\alpha_n \to \overline{1}$ and $\epsilon_n \to \overline{0}$, there exists $n_0 \in \mathbb{N}$ such that $\overline{0} \ll \epsilon_n \ll \epsilon$ and $\overline{0} \ll \alpha \ll \alpha_n \ll \overline{1}, \forall n \geq n_0$. Then $M(x_n, x_m, \epsilon) \gg \alpha, \forall n, m \geq n_0$. So $\{x_n\}$ is a Cauchy sequence in (X, M). By the completeness of (X, M), there exists $x \in X$ such that $\{x_n\}$ converges to x. Now $x_n \in F_m, \forall n \geq m$ and F_m is closed. So $x \in F_m$. This is true $\forall m \in \mathbb{N}$. So $x \in \bigcap_{m=1}^{\infty} F_m$. If possible, let $y(\neq x) \in \bigcap_{m=1}^{\infty} F_m$. Then $M(x, y, \epsilon_m) \geq \alpha_m, \forall m \in \mathbb{N}$. As $\epsilon_n \to \overline{0}$ and $\alpha_n \to \overline{1}$, for any $\epsilon \gg \overline{0}$ and $\overline{0} \ll \alpha \ll \overline{1}, M(x, y, \epsilon) \geq \alpha$. Then $x = y$. So $\bigcap_{m=1}^{\infty} F_m$ is a singleton set.
Conversely, suppose that the condition holds. Let $\{x_n\}$ be a Cauchy sequence in (X, M). Then for any $\epsilon \gg \overline{0}$ and $\overline{1} \gg \alpha \gg \overline{0}$, there exists $n(\epsilon, \alpha) \in \mathbb{N}$ such that $M(x_n, x_m, \epsilon) \geq \alpha, \forall n, m \geq n(\epsilon, \alpha)$.
Let $F_n = \overline{\{x_n, x_{n+1}, ...\}}$. Then $F_1 \supset F_2 \supset .... \supset F_n \supset ....$ is a decreasing sequence of closed sets and for $\overline{0} \ll \epsilon$ and $\overline{0} \ll \alpha \ll \overline{1}$, there exists $n(\epsilon, \alpha) \in \mathbb{N}$ such that $M(x, y, \epsilon) \geq \alpha, \forall x, y \in F_n, \forall n \geq n(\epsilon, \alpha)$. Hence by the given condition $\bigcap_{n=1}^{\infty} F_n$ is a singleton set, say, $\{x\}$. Then $M(x, x_n, \epsilon) \geq \alpha, \forall n \geq n(\epsilon, \alpha)$. So $\{x_n\}$ converges x. □

Definition 3.2. *A set B is said to be nowhere dense in (X, M) if $int(cl(B)) = \phi$ A set Y in (X, M) is said to be of 1st category if it can be expressed as a countable union of nowhere dense sets. If a set Y is not of 1st category then it is called a set of 2nd category.*

Theorem 3.5. *(Baire's category theorem): A complete fuzzy metric space is of 2nd category.*

Proof. Let (X, M) be a complete fuzzy metric space. Let $Y \subset X$ be such that $Y = \cup_{n=1}^{\infty} A_n$, where A_n is nowhere dense in $X, n = 1, 2, ...$ Let U be any non-empty open set in (X, M). Since A_1 is nowhere dense in $(X, M), U \cap (X \setminus \overline{A_1}) \neq \phi$. So there exists an open ball $B_{\alpha_1}(x_1, t_1) \subset U \cap (X \setminus \overline{A_1})$ and $\overline{B_{\alpha_1}(x_1, t_1)} \subset U$. Again A_2 being nowhere dense in $X, B_{\alpha_1}(x_1, t_1) \cap (X \setminus \overline{A_2}) \neq \phi$. Then there exists an open ball $B_{\alpha_2}(x_2, t_2) \subset B_{\alpha_1}(x_1, t_1) \cap (X \setminus \overline{A_2}), \alpha_2 \ll \alpha_1, t_2 \ll t_1$ and $\overline{B_{\alpha_2}(x_2, t_2)} \subset B_{\alpha_1}(x_1, t_1)$. Choosing suitable $t_1 \gg t_2 \gg .... \gg .... \gg \overline{0}$ and $\overline{1} \gg \alpha_1 \gg \alpha_2 \gg .... \gg \alpha_n \gg .... \gg \overline{0}$ such that $t_n \to \overline{0}$ and $\alpha_n \to \overline{0}$ and $\overline{B_{\alpha_{n+1}(x_{n+1}, t_{n+1})}} \subset B_{\alpha_n}(x_n, t_n) \cap (X \setminus A_{n+1})$. Then by Cantor's intersection theorem for completeness of a fuzzy metric space, $\cap_{n=1}^{\infty} \overline{B_{\alpha_n}(x_n, t_n)} \neq \phi$. and $(\cap_{n=1}^{\infty} \overline{B_{\alpha_n}(x_n, t_n)}) \cap (\cup_{n=1}^{\infty} A_n) = \phi$. Hence $Y \neq X$ So X is not of 1st category. i.e., X is of 2nd category.

□

4. Compactness in fuzzy metric space

Definition 4.1. *A fuzzy metric space (X, M) is said to be compact if every open cover of X has a finite subcover.*

Definition 4.2. *A fuzzy metric space (X, M) is said to be sequentially compact if every sequence $\{x_n\}$ in X has a convergent subsequence.*

Theorem 4.1. *A compact fuzzy metric space is sequentially compact.*

Proof. Let (X, M) be a compact fuzzy metric space. Let $\{x_n\}$ be a sequence in (X, M). If possible, let $\{x_n\}$ have no convergent subsequence. Then for each $x \in X$ there exists $(\epsilon(x), \alpha(x))(\epsilon(x) \gg \overline{0}, \overline{1} \gg \alpha(x) \gg \overline{0})$ such that the open ball $B_{\alpha(x)}(x, \epsilon(x))$ contains only finitely many terms of $\{x_n\}$.
Now $\mathcal{C} = \{B_{\alpha(x)}(x, \epsilon(x)); x \in X\}$ is an open cover of X. By the compactness of X, there is a finite subfamily, say, $\{B_{\alpha(x_1)}(x_1, \epsilon_1(x_1)), B_{\alpha(x_2)}(x_2, \epsilon_2(x_2), ...,$
$B_{\alpha(x_n)}(x_n, \epsilon_n(x_n))\}$ of $\mathcal{C}$ covering X. Then $X = \cup B_{\alpha(x_i)}(x_i, \epsilon_i(x_i))$ will contain finitely many terms of $\{x_n\}$, a contradiction. Hence every compact fuzzy metric space is sequentially compact. □

Definition 4.3. *Let $\mathcal{C}$ be a cover of a fuzzy metric space (X, M). Then a pair $(\epsilon, \alpha), \epsilon \gg \overline{0}, \overline{0} \ll \alpha \ll \overline{1}$ is said to be a Lebesgue number for $\mathcal{C}$ if for any subset E of X with the property $M(x, y, \epsilon) \geq \alpha$ for all $x, y \in E$ is contained in at least one member of $\mathcal{C}$.*

Theorem 4.2. *In a sequentially compact fuzzy metric space every open cover has a Lebesgue number.*

Proof. Let (X, M) be a sequentially compact fuzzy metric space and $\mathcal{C}$ be an open cover of X. A set U in X is said to be a big set for $\mathcal{C}$ if it is not contained in any member of $\mathcal{C}$. Let $\mathcal{B}$ be the collection of all big sets for $\mathcal{C}$. If possible, let for each $\epsilon \gg \overline{0}$ and for each $\overline{0} \ll \alpha \ll \overline{1}$ there exists a big set B for $\mathcal{C}$ such that $M(x, y, \epsilon) \gg \alpha$ for all $x, y \in B$. Then for each $n \in \mathbb{N}$, there is a big set B_n with
$M(u, v, \overline{1/2n}) \geq \overline{n/(n+1))}$ for all $u, v \in B_n$.
Choose $x_n \in B_n, n \in \mathbb{N}$. Then $\{x_n\}$ is a sequence in (X, M). By the sequential compactness of (X, M) there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging to x, say. Now x belongs to some member O of $\mathcal{C}$. Since O is open, for any given $\overline{1} \gg \alpha \gg \overline{0}$ there is $\epsilon \gg \overline{0}$ such that
$B_\alpha(x, \epsilon) = \{y \in X; M(x, y, \epsilon) \geq \overline{1} - \alpha\} \subset O$. Choose $\overline{0} \ll \beta \ll \overline{1}$ such that $\beta \star \beta \geq \overline{1} - \alpha$. Choose a positive integer n_0 such that $\overline{1/2n_0} \ll \epsilon/2$ and $\overline{n_0/(1+n_0)} \gg \beta$. Using the convergence of $\{x_{n_i}\}$ to x, there exists i_0 such that $n_{i_0} \geq n_0$ and
$M(x, x_{n_{i_0}}, \overline{(1/2n_0))} \geq \overline{n_0/(1+n_0)} \geq \beta$.
Then for $u \in B_{n_{i_0}}$,
$M(x, u, \overline{1/n_0}) \geq M(x, x_{n_{i_0}}, \overline{1/2n_0}) * M(x_{n_{i_0}}, u, \overline{1/2n_0})$.
$\geq \overline{n_0/(1+n_0)} * \overline{M/(x_{n_{i_0}}, u, 1/2n_{i_0})} \geq \overline{n_0/(1+n_0)} * \overline{n_{i_0}/(1+n_{i_0})}$
$\geq \overline{n_0/(1+n_0)} * \overline{n_0/(1+n_0)} \geq \beta * \beta \geq \overline{1} - \alpha$.
So $u \in B_\alpha(x, \overline{1/n_0}) \subset B_\alpha(x, \epsilon), \forall u \in B_{n_{i_0}} \implies B_{n_{i_0}} \subset B_\alpha(x, \epsilon) \subset O \in \mathcal{C}$, a contradiction to the fact that $B_{n_{i_0}}$ is a big set for $\mathcal{C}$. So there exist an $\epsilon_0 \gg \overline{0}$ and $\overline{0} \ll \alpha_0 \ll \overline{1}$ such that for any big set B there exists $u, v \in B$ such that $M(u, v, \epsilon_0) \not\geq \alpha_0$. In other words, for any set $P \subset X$ with the property $M(u, v, \epsilon_0) \geq \alpha_0, \forall u, v \in P$, is not a big set and hence contained in at least one member of $\mathcal{C}$. Thus (ϵ_0, α_0) is a Lebesgue number for the open cover $\mathcal{C}$. □

Definition 4.4. *A fuzzy metric space (X, M) is said to be totally bounded if for any $\epsilon \gg \overline{0}$ and for any $\overline{0} \gg \alpha \gg \overline{1}$ there is a finite $(\epsilon, \alpha)-$ chain i.e., there is a finite set, say, $\{x_1, x_2, ..., x_n\} \subset X$ such that $\cup_{i=1}^n B_\alpha(x_i, \epsilon) = X$.*

Theorem 4.3. *Every sequentially compact fuzzy metric space is totally bounded.*

Proof. Let (X, M) be a sequentially compact fuzzy metric space. If possible, let there exist $(\epsilon, \alpha), \overline{0} \ll \epsilon$ and $\overline{0} \ll \alpha \ll \overline{1}$ such that X has no finite $(\epsilon, \alpha)-$ chain. Choose $x_1 \in X$ and consider $B_\alpha(x_1, \epsilon)$. Then $X \neq B_\alpha(x_1, \epsilon)$. Choose $x_2 \in X \setminus B_\alpha(x_1, \epsilon)$ and consider $B_\alpha(x_2, \epsilon)$.

Then $X \neq B_\alpha(x_1, \epsilon) \cup B_\alpha(x_2, \epsilon)$. The process can be continued to get a sequence $\{x_n\}$ such that $M(x_i, x_j, \epsilon) \ngeq \overline{1} - \alpha$. Then $\{x_n\}$ is a sequence in (X, M) having no convergent subsequence, a contradiction to the fact that (X, M) is sequentially compact space. Hence (X, M) is totally bounded. □

Remark 4.1. *Every compact metric space is bounded. but the converse is not necessarily true. For example let $X = l^2$, the metric space of all square summable sequences of real numbers. For any fuzzy real number $t \gg \overline{0}$, define $M(x, y, t) = \dfrac{t^\circ}{t^\circ + \overline{d(x, y)}}$, where d is the l^2 metric on X. Then $(X, M, *)$, where $*$ is the product t-norm, is a Type-2 fuzzy metric space. Now for any real number r with $0 < r < 1, M(x, y, t) \geq \overline{r}$ iff $d(x, y) \leq \dfrac{a+b}{2}(\dfrac{1-r}{r})$, where $t^\circ = \dfrac{a+b}{2}, [a, b] = ClSupp\ t$. Then the closed unit ball of (X, d) is closed and bounded in (X, M). Also for the sequence $\{x_n\}$ with $x_n = (0, 0, ..., 0, 1, 0, ..., 0), M(x_n, x_m, t) = \dfrac{t^\circ}{t^\circ + \overline{\sqrt{2}}}$, which shows that $\{x_n\}$ has no convergent subsequence, so that the unit ball is not compact in $(X.M)$. Also the sequence $\{x_n\}$ is bounded but not Cauchy, which justifies that the converse of the Theorem 3.1 does not hold.*

Theorem 4.4. *Every sequentially compact fuzzy metric space is compact.*

Proof. Let (X, M) be a sequentially compact fuzzy metric space. Let $\mathcal{C}$ be an open cover of X. Then $\mathcal{C}$ has a Lebesgue number, say, (ϵ, α), where $\epsilon \gg \overline{0}$ and $\overline{0} \ll \alpha \ll \overline{1}$. Since (X, M) is sequentially compact, it is totally bounded and hence there is a finite $(\epsilon, \overline{1} - \alpha)-$ chain, say, $\{x_1, x_2, ..., x_n\}$ for X. Then $\cup_{i=1}^{n} B_{\overline{1}-\alpha}(x_i, \epsilon) = X$. Further as (ϵ, α) is a Lebesgue number for $\mathcal{C}$, each $B_{\overline{1}-\alpha}(x_i, \epsilon)$ is contained in some member $B_i \in \mathcal{C}$. Then $\{B_1, B_2, ..., B_n\}$ is a finite subfamily of $\mathcal{C}$ covering X. So (X, M) is compact. □

Theorem 4.5. *Every compact fuzzy metric space is complete.*

Theorem 4.6. *A fuzzy metric space is compact if and only if it is totally bounded and complete.*

Proof. Let (X, M) be a fuzzy metric space and let it be totally bounded and complete. Let $\{x_n\}$ be a sequence in X. For each $i = 1, 2, ..., X$ has a finite $(\overline{1/2i}, \overline{1/i})$-chain. So for $i = 1$, there is an open ball $B_{\overline{1}}(\xi_1, \overline{1/2})$ containing infinitely many terms of $\{x_n\}$. Let $S_1 = \{n \in \mathcal{N}; x_n \in B_{\overline{1}}(\xi_1, \overline{1/2})\}$. Then S_1 is infinite. Choose an element of S_1 and rename it as n_1. Then $x_{n_1} \in B_{\overline{1}}(\xi_1, \overline{1/2})$. Arguing in a similar manner for some $\xi_2, B_{\overline{1/2}}(\xi_2, \overline{1/4})$ contains infinitely many terms x_n for $n \in S_1$ Let $S_2 = \{n \in S_1, x_n \in B_{\overline{1/2}}(\xi_2, \overline{1/4})$. Choose an element of $S_2 \setminus \{n_1\}$ and rename it as n_2. Then $x_{n_2} \in B_{1/2}(\xi_2, \overline{1/4})$. Continuing the process, we get a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, where $n_i \in S_i \setminus \{n_1, n_2, ..., n_{i-1}\}$. and
$x_{n_i} \in B_{\overline{1-(i-1)/i}}(\xi_i, \overline{1/2i}), i = 1, 2, ...$
Then for $j \geq i, M(x_{n_i}, x_{n_j}, \overline{1/i}) \geq M(x_{n_i}, \xi_i, \overline{1/2i}) * M(x_{n_j}, \xi_i, \overline{1/2i})$
$\geq \overline{(i-1)/i} * \overline{(i-1)/i}$.
As $\overline{(i-1)/i} * \overline{(i-1)/i} \to \overline{1} * \overline{1} = \overline{1}$ and $\overline{1/i} \to \overline{0}$, for any $\overline{0} \ll \alpha \ll \overline{1}$ and for any $\epsilon \gg \overline{0}$, there exists $i_0 \in \mathcal{N}$ such that $M(x_{n_i}, x_{n_j}, \epsilon) \geq \alpha, \forall i, j \geq i_0$.
So $\{x_{n_i}\}$ is a Cauchy sequence. By the completeness of $(X, M), \{x_{n_i}\}$ is convergent. So (X, M) is sequentially compact and hence, by a previous theorem, compact. Converse part is immediate. □

5. Conclusions

After the introduction of Type-2 fuzzy metric space in [15], in this paper we study completeness and compactness of this fuzzy metric space. In doing so we have extended Cantor's intersection theorem, Baire's category theorem in this space. Theorem concerning characterizing compactness in terms of completeness and total boundedness is also established in this Type-2 setting. For the further study establishment of Ascoli's theorem in this setting will be the next problem. Study of fixed point theorems in this space can also be done. There is a scope for handling decision making problems applying this Type-2 fuzzy metric.

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