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ON ITALIAN DOMINATION NUMBER OF UNARY OPERATIONS OF SPECIAL GRAPHS

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Abstract. For a graph $G = (V, E)$, an Italian dominating function (IDF) $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either v is adjacent to a vertex assigned 2 under f, or v is adjacent to at least two vertices assigned 1 under f. The weight of an Italian dominating function is the $\sum_{v \in V} f(v)$, and the minimum weight of a Italian dominating function f is the Italian domination number. This study illustrates the Italian domination number of graphs that are generated when various unary operations are applied to standard graph classes.

Keywords: Domination, Roman domination, Italian dominating function, Italian domination number.

AMS Subject Classification: 05C69, 05C76

1. Introduction

All graphs we considered here are finite, simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E(G)$. The open neighborhood $N(v)$ of a vertex v consists of the vertices adjacent to v, and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of v is the cardinality of its open neighborhood. The graph's maximum degree is indicated by $\Delta(G)$. A pendant is a vertex of degree one. A universal vertex is a vertex that is adjacent to every other vertex in the graph. We generally refer to [1, 13] for more terminologies or and notations related to graph theory.

The concept of domination in graphs has evolved over time and is increasingly popular among researchers due to its broad applications. O. Ore [10] introduced the concept of domination and further described in the text Fundamentals of domination in graphs [6]. Roman domination is a new variety of domination parameter introduced by Cockayne et al. [3] drawing influence from Ian Stewart article on protecting the Roman Empire [11]. The idea of Roman $\{2\}$-domination came about as the outcome of numerous articles regarding Roman domination and its variation. The Italian domination number is a graph labeling

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problem that was first put forward by M. Chellali et al. [2] as Roman {2}-domination. Based on the defense plan of Roman empire, every location that lacks a legion must have a neighboring location with two legion or at least two neighboring locations that each carry one legion. Formally, Italian dominating function $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$, with $f(v) = 0$ either there is a vertex $u \in N(v)$ with $f(u) = 2$, or at least two vertices $u, w \in N(v)$ with $f(u) = f(w) = 1$. The weight of an Italian dominating function is the $\sum_{v\in V} f(v)$ and the minimum weight of an Italian dominating function f is the Italian domination number, denoted by $\gamma_I(G)$. An example of Italian domination function assigned for a graph G is shown in Figure 1.

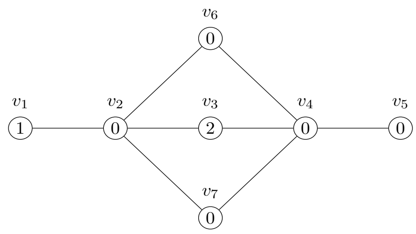

FIGURE 1. Graph G with $\gamma_I(G) = 4$.

Italian domination number was connected with other domination parameters by M. Chellali et al. [2] and also showed that Italian domination is NP-complete for bipartite graphs. The authors of [7] characterize the trees T with $\gamma(T) + 1 = \gamma_I(T)$ and $\gamma_I(T) = 2\gamma(T)$ in 2017. Subsequently, in 2021 various results based on binary operations in Italian domination are presented, particularly for Cartesian products and rooted products[5, 8]. As far as we are aware, no one has looked into the Italian domination number on unary product of graphs. So this study aims to analyze this variant of Roman domination in the various unary product of graphs. In section 2 we share some preliminary findings. In Section 3 we presents our major findings regarding the Italian domination number of graphs that are generated when various unary operations are applied to some standard graph classes.

2. Preliminary results

Theorem 2.1. [2] *If G is a connected graph of order n and maximum degree $\Delta(G) = \Delta$, then $\gamma_I(G) \geq 2n/(\Delta + 2)$.*

Theorem 2.2. [2] *For every graph $G, \gamma(G) \leq \gamma_I(G) \leq 2\gamma(G)$.*

Theorem 2.3. [2] *For the classes of paths P_n and cycles C_n, $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$ and $\gamma_I(C_n) = \lceil \frac{n}{2} \rceil$.*

Proposition 2.1. [12] *If G is a graph of order n, then $\gamma_I(G) = n$ if and only if $\Delta(G) \leq 1$.*

Proposition 2.2. [12] *If G is a graph of order $n \geq 2$, then $\gamma_I(G) = 2$ if and only if $\Delta(G) = n - 1$ or there exist two different vertices u and v such that $N(u) \cap N(v) = V(G)\backslash\{u, v\}$.*

Theorem 2.4. *Let $S_{r,t}$ be a double star graph on $r + t$ vertices then $\gamma_I(S_{r,t}) = 4$.*

Proof. Let $S_{r,t}$ be a double star graph on $r + t$ vertices $V = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_t\}$ where u_r and v_t are non pendant vertices. Consider the Italian dominating function

$f : V \to \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 2 & \text{if} \quad v \in \{u_r, v_t\} \\ 0 & \text{if} \quad v \in V - \{u_r, v_t\} \end{cases}$$

Then $\gamma_I(S_{r,t}) \leq w(f) = 4$. Assume $\gamma_I(S_{r,t}) = 3$ to prove $\gamma_I(S_{r,t}) \geq 4$. Then, there must be an Italian dominating function f that assigns a value of 2 to one vertex, a value of 1 to another, and 0 to the remaining vertices, or a value of 1 to three vertices and 0 to the remaining vertices. Since neither of the situation is possible, $\gamma_I(S_{r,t}) = 4$. □

Theorem 2.5. *Let $K_{m,n}$ be a complete bipartite graph on $m + n$ vertices then*

$$\gamma_I(K_{m,n}) = \begin{cases} 2 & \text{if} \quad \min\{m, n\} = 2 \\ 3 & \text{if} \quad \min\{m, n\} = 3 \\ 4 & \text{if} \quad m, n \geq 4. \end{cases}$$

Proof. Let $K_{m,n}$ be complete bipartite graph whose vertices are partitioned in to two sets X and Y with cardinality m and n respectively. For $\min\{m, n\} = 2$ or 3, we can define an IDF $f : V \to \{0, 1, 2\}$ such that $f(v) = 1$ for all vertices in the set with minimum cardinality and $f(v) = 0$ for all other vertices. For $m, n \geq 4$, we can define an IDF f that assigns 2 to one vertex each from the sets X and Y and 0 to all remaining vertices. Hence $\gamma_I(K_{m,n}) \leq w(f) = 4$. Now assume that $\gamma_I(K_{m,n}) = 3$. Then, there must be an Italian dominating function f that assigns a value of 2 to one vertex, a value of 1 to another, and 0 to the remaining vertices, or a value of 1 to three vertices and 0 to the remaining vertices. However, $\gamma_I(K_{m,n}) = 4$ because neither of the two cases are possible. □

Theorem 2.6. *Let G be a complete k-partite graph, where the cardinality of k sets are greater than or equal to 4 then $\gamma_I(G) = 4$.*

Proof. Let G be a complete k-partite graph with k partitions. We can define an IDF $f : V \to \{0, 1, 2\}$ such that f assigns 2 for one vertex from any of the two independent sets and assigns 0 for all other vertices because every pair of vertices in the k independent sets are adjacent. Alternatively, we can construct a second IDF that assigns 1 for any two vertices from either of the two independent sets and 0 for all remaining vertices. We can only define these forms of IDF on the complete k-partite graph in order to obtain the minimal weight. Hence, $\gamma_I(G) = w(f) = 4$. □

In the following section, we determine how different unary operations affect the Italian domination number of a few common graph classes.

3. Unary operations on graph

Graph operations are operations which use initial graphs and generate new ones from them. They comprise binary and unary operations. Unary operations take a single graph and develop it into a new one. Although there are numerous unary operations in the literature, we focus on a few of them such as total graph of a graph, subdivision of edges of a graph, generalized corona of a graph, duplication of vertex of graph, and Myceilskian of a graph.

3.1. Total graph Operation.

Definition 3.1. [4] ***Total graph, T(G)***: *The total graph, T(G) of a graph G is the graph whose set of vertices is the union of the set of vertices and of the set of edges of G, with two vertices of T(G) being adjacent if and only if the corresponding elements of G are adjacent or incident. For example consider the total graph of Figure 1.*

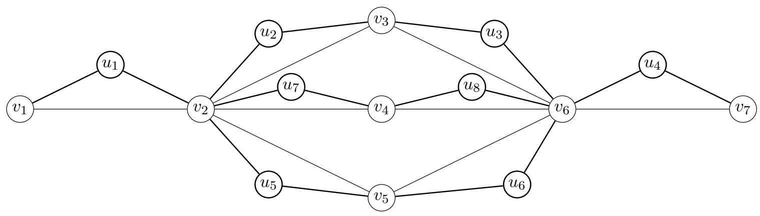

FIGURE 2. Total graph $T(G)$ for the graph shown in Figure 1

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Graph class	$\lvert V(G)\rvert$	$\lvert V(T(G))\rvert$	$\gamma_I(G)$	$\gamma_I(T(G))$
C_n	n	$2n$	$\lceil \frac{n}{2} \rceil$	n
P_n	n	$2n-1$	$\lceil \frac{n+1}{2} \rceil$	$2\lfloor \frac{n}{2} \rfloor$
W_n	n	$3n-2$	2	n
K_n	n	$\frac{n(n+1)}{2}$	2	n
$K_{1,n}$	$n+1$	$2n+1$	2	2
Sr,t	$r+t$	$2(r+t)-1$	4	4
$K_{m,n}$	$m+n$	$m(n+1)+n$	$4, m, n \geq 4$	$2\min\{m,n\}$

TABLE 1. Impact of total graph operation on Italian domination number of standard graph classes.

Observation 3.1. *Let G be a graph of order n and $T(G)$ be the total graph. Then $\gamma_I(T(G)) \leq n$.*

Proof. Let G be a connected graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges. Let $T(G)$ be the graph obtained by adding the vertices $u_1, u_2, \ldots, u_m$ corresponding to each edge in G. Consider an Italian dominating function $f : V \to \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 1 & \text{if} \quad v \in V(G) \\ 0 & \text{if} \quad v \in V(T(G)) - V(G) \end{cases}$$

which implies $\gamma_I(T(G)) \leq w(f) = n$. □

Theorem 3.1. *Let $G = C_n$ or W_n or K_n with $n \geq 3$. Then $\gamma_I(T(G)) = n$.*

Proof. Let G be a cycle, wheel, or complete graph with m edges and n vertices. Define $V = \{v_1, v_2, \ldots, v_n\}$ as vertices of G. The graph $T(G)$ has extra vertices $U = \{u_1, u_2, \ldots, u_m\}$ that correspond to each edge in G. Assume $\gamma_I(T(G)) = n - 1$. Then there exists an IDF, $f : V(T(G)) \to \{0, 1, 2\}$ with $w(f) = n - 1$. Since $deg(u_j) = 2 < deg(v_i)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, the IDF assigning 0 for all vertices in U produces the most minimal weight. As a result, either one of u_j's neighbors must assign 2, or both neighbors must assign 1. In both cases we have $w(f) \geq n$, which implies $\gamma_I(T(G)) \geq n$. According to Observation 3.1, $\gamma_I(T(G)) = n$, where G represents a cycle, wheel, or complete graph. □

Theorem 3.2. *Let $G = P_n$ be a path of order n. Then $\gamma_I(T(G)) = 2\lfloor \frac{n}{2} \rfloor$.*

Proof. Let G be a path of order n with vertices $V = \{v_1, v_2, \ldots, v_n\}$, where v_1 and v_n are leaf vertices, and $T(G)$ be the total graph obtained by adding vertices, $U = \{u_1, u_2, \ldots, u_{n-1}\}$. Consider the IDF $f : V(T(G)) \to \{0, 1, 2\}$, which assigns 0 and 2 to the vertices in V alternatively and 0 to all the vertices in U. Thus $\gamma_I(T(G)) \leq w(f) = 2\lfloor \frac{n}{2} \rceil$. As $deg(u_j) = 2 < deg(v_i)$, for $2 \leq i \leq n - 1$ and $1 \leq j \leq n - 1$, the IDF assigning 0 for all two degree vertices provides the least weight. When n is odd, the f defined above is the only IDF that offers the minimal weight, whereas when n is even, $f(v_i) = 1$ and $f(u_i) = 0$ for $1 \leq i \leq n$ and $1 \leq j \leq n-1$ is also an IDF that gives the same total weight. Thus, $\gamma_I(T(G)) = 2\lfloor \frac{n}{2} \rfloor$. $\square$

Theorem 3.3. *Let $G = K_{1,n}$ be a star of order $n+1$. Then $\gamma_I(T(G)) = 2$.*

Proof. Let G be a star of order $n+1$ and $T(G)$ is the total graph of star. The universal vertex in G is connected to the one end of the n newly added vertices that correspond to each edge. Thus, $T(G)$ has a universal vertex and $\gamma_I(T(G)) = 2$ based on proposition 2.2. $\square$

Theorem 3.4. *Let $G = S_{r,t}$ be a double star of order $r+t$. Then $\gamma_I(T(G)) = \gamma_I(G)$.*

Proof. Let G be double star graph with $r+t$ vertices. The total graph $T(G)$ is obtained by adding $r+t-1$ vertices to G. The IDF f, which is defined in Theorem 2.4, can be extended in $V(T(G))$ by assigning 0 to each newly added vertices. This extended IDF thus implies $\gamma_I(T(G)) = 4 = \gamma_I(G)$. $\square$

Theorem 3.5. *Let $G = K_{m,n}$ where $m, n \geq 3$ be complete bipartite graph of order $m+n$. Then $\gamma_I(T(G)) = 2\min\{m, n\}$.*

Proof. Let G be a complete bipartite graph with vertices partitioned into two sets, X and Y with respective cardinality m and n. The total graph $T(G)$ is generated by adding mn vertices. Assume that $m \leq n$ without loss of generality. Now define an IDF $f : V(T(G)) \to \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 2 & \text{if} \quad v \in X \\ 0 & \text{if} \quad v \in V(T(G)) - X \end{cases}$$

which implies $\gamma_I(T(G)) \leq w(f) = 2\min\{m, n\}$. To prove $\gamma_I(T(G)) \geq 2\min\{m, n\}$. Assume that there exist an IDF with $w(f) = 2m-1$, where $m \leq n$. Since the degree of the newly introduced vertices is 2, assigning 1 to them does not result in the minimum weight. If f assigns 1 to a vertex in X, then the n newly added vertices must assign 1, which increases the weight. Hence $\gamma_I(T(G)) = 2\min\{m, n\}$. $\square$

3.2. **Subdivision operation.**

Definition 3.2. [13] ***Subdivision S(G):*** *$S(G)$ is obtained by splitting each edge of G by introducing a new vertex. Figure 3 shows an example of subdivision of the graph in Figure 1 with newly added vertices $\{u_1, u_2, \ldots, u_8\}$ corresponding to each edge.*

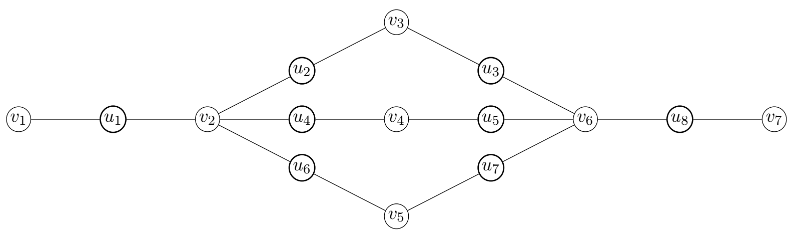

FIGURE 3. Subdivision of the graph shown in Figure 1.

Graph class	$\|V(G)\|$	$\|V(S(G))\|$	$\gamma_I(G)$	$\gamma_I(S(G))$
C_n	n	$2n$	$\lceil \frac{n}{2} \rceil$	n
P_n	n	$2n-1$	$\lceil \frac{n+1}{2} \rceil$	n
W_n	n	$3n+1$	2	n
K_n	n	$\frac{n(n+1)}{2}$	2	n
$K_{1,n}$	$n+1$	$2n+1$	2	$n+1$
Sr,t	$r+t$	$2(r+t)-1$	4	$r+t$
$K_{m,n}$	$m+n$	$m(n+1)+n$	$4, m, n \geq 4$	$m+n$

Table 2. Impact of subdivision operation on Italian domination number of standard graph classes.

Theorem 3.6. *Let G be a connected graph of order n. Then $\gamma_I(S(G)) = n$.*

Proof. Let G be a connected graph with m edges and n vertices and $S(G)$ be the graph generated by splitting each edge of G with a new vertex. Define V' as the set of n vertices in G and V'' as the set of m new vertices in $S(G)$. Consider the IDF $f : V = V' \cup V'' \to \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 1 & \text{if} \quad v \in V' \\ 0 & \text{if} \quad v \in V'' \end{cases}$$

Here, every vertex assigned 0 is adjacent to exactly two vertices assigned one, and all neighbors of the vertices $v \in V'$ have the assignment zero. Also $deg(v) = 2$, $\forall v \in V''$, therefore assigning 1 or 2 for these vertices does not result in the minimum weight. Hence $\gamma_I(S(G)) = w(f) = n$. □

3.3. **Generalized corona of a graph.**

Definition 3.3. [9]***Generalized Corona of a graph,*** $G \circ H_i$**:** *Given simple graphs $G, H_1, , H_n$, where $n = |V(G)|$, the generalized corona, denoted $G \circ H_i; 1 \leq i \leq n$ is the graph obtained by taking one copy of graphs $G, H_1, , H_n$ and joining the i^{th} vertex of G to every vertex of H_i. For example, consider the generalized corona of the graph in Figure1 and $\overline{K_3}$.*

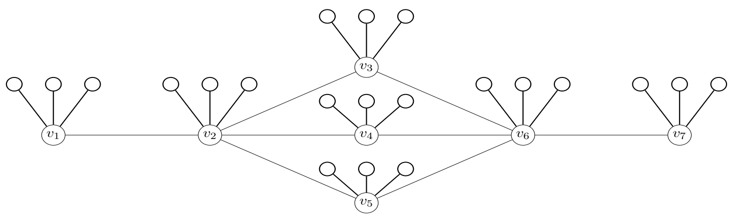

Figure 4. Generalized corona for the graph shown in Figure 1 and $\overline{K_3}$.

Theorem 3.7. *Let G be a connected graph of order n. Then $\gamma_I(G \circ \overline{K_l}) = 2n$, for $l \geq 2$.*

Proof. Let G be any graph of order n and $G \circ \overline{K_l}$ be the graph obtained by adding l pendant vertices $v_{1,l}, v_{2,l}, \ldots v_{n,l}$ to each vertex $v_1, v_2, \ldots, v_n$ in G respectively. Consider

Graph class	$\lvert V(G)\rvert$	$\lvert V(G\circ\overline{K_l})\rvert$	$\gamma_I(G)$	$\gamma_I(G\circ\overline{K_l}), l\geq 2$
C_n	n	ln	$\lceil\frac{n}{2}\rceil$	$2n$
P_n	n	ln	$\lceil\frac{n+1}{2}\rceil$	$2n$
W_n	$n+1$	$l(n+1)$	2	$2(n+1)$
K_n	n	ln	2	$2n$
$K_{1,n}$	$n+1$	$l(n+1)$	2	$2(n+1)$
Sr,t	$r+t$	$l(r+t)$	4	$2(r+t)$
$K_{m,n}$	$m+n$	$l(m+n)$	$4, m,n\geq 4$	$2(m+n)$

TABLE 3. Impact of generalized corona operation on Italian domination number of standard graph classes.

the IDF $f: V\to\{0,1,2\}$ such that

$$f(v)=\begin{cases}2 & \text{if} \quad v\in V(G)\\ 0 & \text{if} \quad v\in V(G\circ\overline{K_l})-V(G)\end{cases}$$

which implies $\gamma_I(G\circ\overline{K_l})\leq w(f)=2n$.
Now assume that $w(f)=2n-1$ for an IDF f, then $f(v)=1$ for at least one vertex in $V(G\circ\overline{K_l})$. If $f(v_i)=1$ for some $v_i\in V(G)$, then $f(v_{i,l})=1$ for all $l\geq 2$. If $f(v_{i,l})=1$ for any i and $l\geq 2$, then either $f(v_i)=2$ or $f(v_{i,k})=1$ for all $k\neq l$. That is in both cases all the pendant vertices adjacent to v_i should assign 1, resulting $w(f)\geq 2n$. Since $deg(v_i)\geq 3, \forall v_i\in V(G\circ\overline{K_l})$, there does not exist an IDF with $w(f)=2n-1$. Hence $\gamma_I(G\circ\overline{K_l})=2n$. □

3.4. **Duplication of a vertex.**

Definition 3.4. [13]***Duplication of a vertex,*** $D(vG)$: *Duplication of a vertex v of graph G produces a new graph $D(vG)$ by adding a new vertex v' such that $N(v')=N(v)$. Duplication of each vertex of the graph in Figure 1 is illustrated in 5.*

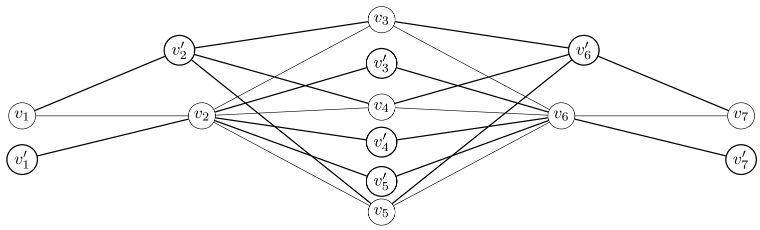

FIGURE 5. Duplication of every vertex of the graph shown in Figure 1.

Theorem 3.8. *Let G be a graph with universal vertex of order n, then $\gamma_I(D(vG))=$ $\gamma_I(G)+1$.*

Proof. Let G be a graph of order n with vertices $V'=\{v_1,v_2,\ldots,v_n\}$ with v_1 as universal vertex. Let $V''=\{v_1',v_2',\ldots,v_n'\}$ be the duplicated vertices added to G to get $D(vG)$ where v_i' is the copy of v_i. Consider the IDF $f: V=V'\cup V''\to\{0,1,2\}$ such that

$$f(v)=\begin{cases}2 & \text{if} \quad v=v_1\\ 1 & \text{if} \quad v=v_1'\\ 0 & \text{if} \quad v\in V-\{v_1,v_1'\}\end{cases}$$

which implies $\gamma_I(D(vG))\leq w(f)=3$ and by proposition 2.2, $\gamma_I(D(vG))=3$. □

Graph class	$\lvert V(G)\rvert$	$\lvert V(D(vG))\rvert$	$\gamma_I(G)$	$\gamma_I(D(vG))$
C_n	n	$2n$	$\lceil \frac{n}{2} \rceil$	n
P_n	n	$2n$	$\lceil \frac{n+1}{2} \rceil$	$n+1$ if $n = 3k+2$ n if $n \neq 3k+2$
W_n	$n+1$	$2(n+1)$	2	3
K_n	n	$2n$	2	3
$K_{1,n}$	$n+1$	$2(n+1)$	2	3
Sr,t	$r+t$	$2(r+t)$	4	4
$K_{m,n}$	$m+n$	$2(m+n)$	$4, m, n \geq 4$	$4, m, n \geq 4$

Table 4. Impact of duplication of vertex on Italian domination number of standard graph classes.

Theorem 3.9. *Let $G = C_n$ be a cycle of order n. Then $\gamma_I(D(vG)) = n$.*

Proof. Let G be a cycle of order n with vertices $V' = \{v_1, v_2, \ldots, v_n\}$ and $V'' = \{v'_1, v'_2, \ldots, v'_n\}$ be the duplicated vertices in the graph $D(vG)$ where v'_i is the copy of v_i. Consider the IDF $f : V = V' \cup V'' \to \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 1 & \text{if} \quad v \in V' \\ 0 & \text{if} \quad v \in V'' \end{cases}$$

which implies $\gamma_I(D(vG)) \leq w(f) = n$.
To prove $\gamma_I(D(vG)) \geq n$. Assume that there exist an IDF f such that $w(f) = n-1$. Assume $f(v) = 1$ for $n-1$ vertices in V' and 0 for remaining one. Then the condition of IDF fails for the two vertices in V'' which is adjacent to the vertex in V' labeled 0. Assigning 2 for some vertex having maximum degree also fails to get $w(f) = n-1$. Hence $\gamma_I(D(vG)) = n$. □

Theorem 3.10. *Let P_n, $n \geq 4$ be a path of order n. Then*

$$\gamma_I(D(vP_n)) = \begin{cases} n+1 & \textit{if} \quad n = 3k+2 \\ n & \textit{if} \quad n \neq 3k+2. \end{cases}$$

Proof. Let P_n be a path with n vertices $V' = (v_1, v_2, \ldots, v_n)$ where v_1 and v_n are leaf. Let $D(v(P_n)$ is the graph obtained after duplication of each vertex with the vertices $V'' = \{v'_1, v'_2, \ldots, v'_n\}$, where $N(v'_i) = N(v_i)$.
Case 1: $n = 3k+2$
Consider the IDF, $f : V = V' \cup V'' \to \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 2 & \text{if} \quad v \in \{v_2, v_5, \cdots, v_n\} \\ 1 & \text{if} \quad v \in \{v'_2, v'_5, \cdots, v'_n\} \\ 0 & \text{otherwise} \end{cases}$$

which implies $w(f) = 2\lceil \frac{n}{3} \rceil + \lceil \frac{n}{3} \rceil = 3\lceil \frac{n}{3} \rceil = n+1$. Thus, $\gamma_I(D(vP_n)) \leq n+1$.
To prove $\gamma_I(D(vP_n)) \geq n+1$. Assume that there exist an IDF, f such that $w(f) = n$. If $f(v) = 0$, $\forall v \in V''$ then f should assign 2 for the vertices v_2 and v_{n-1} and assign 1 for all other vertices in V'. If $f(v) = 1$, $\forall v \in V''$ then f should assign 1 for the vertices v_1 and v_n and assign 0 for all other vertices in V'. In both cases $w(f) = n+2$. Therefore V'' must have vertices that have assignments of 1 and 0. Since $\deg(v) = 4$, $\forall\, v \in V' - \{v_1, v_n\}$, the IDF defined in the first section of the proof provides the minimal weight. Implies in the case when $n = 3k+2$, $\gamma_I(D(vP_n)) = n+1$.

Case 2: $n \neq 3k+2$. Then arises two subcases. For $n = 3k$, consider the IDF, $f : V = V' \cup V'' \to \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 2 & \text{if} \quad v \in \{v_2, v_5, \cdots v_{n-1}\} \\ 1 & \text{if} \quad v \in \{v'_2, v'_5, \cdots v'_{n-1}\} \\ 0 & \text{otherwise} \end{cases}$$

which implies $w(f) = 2\lceil \frac{n}{3} \rceil + \lceil \frac{n}{3} \rceil = 3\lceil \frac{n}{3} \rceil = n$. For $n = 3k+1$, consider the IDF, $f : V = V' \cup V'' \to \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 2 & \text{if} \quad v \in \{v_2, v_5, \cdots v_{n-1}\} \\ 1 & \text{if} \quad v \in \{v'_2, v'_5, \cdots\} - \{v'_{n-2}, v'_{n-1}\} \\ 0 & \text{otherwise} \end{cases}$$

which implies $w(f) = 2\lceil \frac{n}{3} \rceil + \lceil \frac{n}{3} \rceil - 2 = 3\lceil \frac{n}{3} \rceil - 2 = n$. Hence for case 2, $\gamma_I(D(vP_n)) \leq n$. Assuming $w(f) = n-1$, we have the same scenario as in case 1, so this IDF provides the minimal weight. Hence, when $n \neq 3k+2$, $\gamma_I(D(vP_n)) = n$. □

Theorem 3.11. *For the graphs $G = S_{r,t}$ or $K_{m,n}$, $\gamma_I(D(vG)) = \gamma_I(G)$.*

Proof. Let G be a complete bipartite graph or double star graph, and let $D(vG)$ be the graph that results from vertices being duplicated. In $D(vG)$, every newly added vertices is adjacent to at least one vertex assigned 2 under the function f defined in Theorem 2.4 and Theorem 2.5 (in case $m, n \geq 4$). Assigning 0 to every duplicate vertex allows the function in these theorems to extend to IDF in $V(D(vG))$. Hence $\gamma_I(D(vG)) = \gamma_I(G)$. □

3.5. Myceilskian operation.

Definition 3.5. [13] ***Mycielskian operation, $\mu(G)$:*** *Addition of $n+1$ vertices to a graph G of order n. A vertex v'_i corresponding to each vertex v_i of G where adjacent vertices of v_i is connected to v'_i and an extra vertex w where each vertex v'_i is connected by an edge to w.*

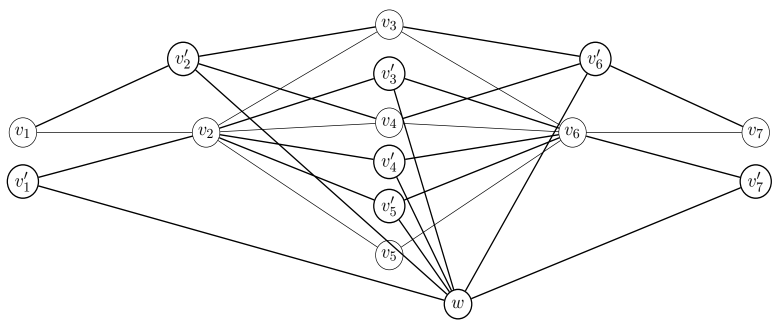

FIGURE 6. Myceilskian operation on the graph shown in Figure 1.

Theorem 3.12. *Let G be a graph with a universal vertex of order n, then $\gamma_I(\mu(G))$= 3.*

Proof. Let G be a graph of order n with vertices $v_1, v_2, \ldots, v_n$. Let $v'_1, v'_2, \ldots, v'_n$ be the corresponding vertices of each v_i and w be the extra vertex where each v'_i is connected by an edge in the graph $\mu(G)$. Assume v_1 as the universal vertex in G and v'_1 as its copy. The vertex v_i is adjacent to v_1 and v'_1 and v'_i is adjacent to v_1 and w for all $i = 1, 2, \ldots n$.

Graph class	$\lvert V(G)\rvert$	$\lvert V(\mu(G))\rvert$	$\gamma_I(G)$	$\gamma_I(\mu(G))$
C_n	n	$2n$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil + 2$
P_n	n	$2n$	$\lceil \frac{n+1}{2} \rceil$	$\lceil \frac{n+1}{2} \rceil + 2$
W_n	n	$2n+1$	2	3
K_n	n	$2n+1$	2	3
$K_{1,n}$	$n+1$	$2n+3$	2	3
Sr,t	$r+t$	$2(r+t)+1$	4	5
$K_{m,n}$	$m+n$	$2(m+n)+1$	$4, m,n \geq 4$	5

TABLE 5. Impact of Myceilskian operation on Italian domination number of standard graph classes.

So we get an Italian dominating function $f : V \to \{0, 1, 2\}$ with minimum weight defined as

$$f(v) = \begin{cases} 1 & \text{if} \quad v \in \{v_i, v_i', w\} \\ 0 & \quad \text{otherwise} \end{cases}$$

which implies $\gamma_I(\mu(G)) \leq w(f) = 3$ and $\gamma_I(\mu(G)) \neq 2$, since the condition in Preposition 2.2 does not hold for the graph $\mu(G)$, implying $\gamma_I(\mu(G)) = 3$. □

Theorem 3.13. *For the classes of paths P_n and cycles C_n, $\gamma_I(\mu(G)) = \gamma_I(G) + 2$, for $n \geq 4$.*

Proof. Let G be a path or cycle of order $n \geq 4$ with vertices $v_1, v_2, \ldots, v_n$ and f be the Italian dominating function of G which gives the minimum weight. By Theorem 2.3, $\gamma_I(P_n) = \lceil (n+1)/2 \rceil$ and $\gamma_I(C_n) = \lceil n/2 \rceil$. That is f is assigned 1 and 0 alternatively for the vertices of G to get the result. Let $\mu(G)$ be the graph obtained by Myceilskian operation by the addition of $n+1$ vertices $v_1', v_2', \ldots, v_n'$ and w. Consider an IDF f' as an extension of f in $\mu(G)$ such that $f'(v_i') = 0$ for all $i = 1, 2, \ldots n$ and $f'(w) = 2$. Hence $\gamma_I(\mu(G)) = \gamma_I(G) + 2$. □

Theorem 3.14. *Let $G = S_{r,t}$ be a double star of order $r + t$. Then $\gamma_I(\mu(G)) = 5$.*

Proof. Let G be double star graph with $r + t$ vertices and $\mu(G)$ be the graph obtained after Myceilskian operation by adding extra $r + t + 1$ vertices. Let f be the IDF defined in Theorem 2.4 and this can be extended in $V(\mu(G))$. The $r + t$ newly added vertices in $\mu(G)$ are adjascent to atleast one of the two vertices assigned 2 under the function f. So for the newly added $r + t$ vertices the extended IDF can assign 0 and 1 for the remaining one vertex. Hence $\gamma_I(\mu(G)) = 5$. □

Theorem 3.15. *Let $G = K_{m,n}$ be a complete bipartite graph of order $m + n$. Then $\gamma_I(\mu(G)) = 5$.*

Proof. Let G be complete bipartite graph with $m + n$ vertices and $\mu(G)$ be the graph obtained after Myceilskian operation by adding $m + n + 1$ vertices. Let f be the IDF defined for the case $m, n \geq 4$ in Theorem 2.5 and this can be extended in $V(\mu(G))$. The $m + n$ newly added vertices in $\mu(G)$ are adjascent to one of the two vertices assigned 2 under the function f. So the extended IDF can assign 0 for $m + n$ vertices and 1 for the remaining one vertex. Hence $\gamma_I(\mu(G)) = 5$. □

4. CONCLUSIONS

This paper identifies the Italian domination number of graphs that are generated across various graph classes by applying different unary operations, and then examines the effects of those unary operations on Italian domination number. In the near future, additional unary products can be used to examine the effects of Italian domination number.

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