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COINCIDENCE POINTS IN b-METRIC SPACES VIA $\mathcal{L}$-SIMULATION FUNCTIONS AND DIGRAPHS

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Abstract. The study of fixed point theory combining digraphs and $\mathcal{L}$-simulation functions is a new development in the domain of contractive type single valued theory. In this paper, we introduce the notion of a generalized $\mathcal{L}G$- contraction by using $\mathcal{L}$-simulation functions and digraphs. We discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings satisfying such contractions in b-metric spaces. Our result will extend and unify several comparable results in the existing literature including the well known Banach contraction theorem in metric spaces. Finally, we give some non-trivial examples to illustrate and justify the validity of our main result.

Keywords: $\mathcal{L}$-simulation function, $\mathcal{L}G$-contraction, point of coincidence, common fixed point.

AMS Subject Classification: 54H25, 47H10.

1. Introduction

Fixed point theory is an important branch of mathematics due to its applications in different fields of mathematics and applied sciences. The Banach contraction theorem [7] is one of the fundamental results that helps many researchers working in this field. This fundamental result has been generalized by several mathematicians in many directions(see [8, 17, 20, 25, 26, 27]). Bakhtin [6] introduced b-metric spaces as a generalization of metric spaces by modifying the usual triangle inequality and proved the famous Banach contraction theorem in this framework. In 2014, Jleli and Samet [22] gave a generalization of the Banach contraction theorem in generalized metric spaces by using the notion of θ-contractions. After that, Ahmad et al.[3] modified the notion of θ-contractions and extended the result of Jleli and Samet [22] to metric spaces.

Jungck [23] introduced the concept of weak compatibility. Several authors have obtained common fixed points by using this notion. In recent investigations, the study of fixed point theory combining a graph is a new development which takes a vital role in many aspects.

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In [18], Echenique studied fixed point theory by using graphs and then Espinola and Kirk [19] applied fixed point results in graph theory. In [24], Khojasteh et al. introduced $\mathcal{L}$-contractions by using the concept of simulation functions and unified some existing metric fixed point results. Motivated by the idea given in [24] and some recent works on metric spaces with a graph (see [4, 5, 9, 10, 11]), we reformulated some important coincidence point and common fixed point results in b-metric spaces endowed with a graph by using $\mathcal{L}$-simulation functions.

2. Some Basic Concepts

This section begins with some basic notations, definitions and necessary results in b-metric spaces that will be needed in the sequel.

Definition 2.1. [16] *Let X be a nonempty set and $b \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x,y) = 0$ *if and only if* $x = y$*;*
(ii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$*;*
(iii) $d(x,y) \leq b\,(d(x,z) + d(z,y))$ *for all* $x, y, z \in X$*.*

The pair (X,d) is called a b-metric space.

It is worth noting that the class of b-metric spaces is effectively larger than that of the ordinary metric spaces.

The following example supports the above remark.

Example 2.1. [26] *Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = d(y,x)$ for all $x, y \in X$, $d(x,x) = 0$, $x \in X$ and $d(-1,0) = 3$, $d(-1,1) = d(0,1) = 1$. Then (X,d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that*

$$d(-1,1) + d(1,0) = 1 + 1 = 2 < 3 = d(-1,0).$$

It is easy to verify that $b = \frac{3}{2}$.

Example 2.2. [28] *Let (X,d) be a metric space and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $b = 2^{p-1}$.*

Definition 2.2. [13] *Let (X,d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) (x_n) *converges to x if and only if* $\lim_{n\to\infty} d(x_n, x) = 0$*. We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x (n \to \infty)$*.*
(ii) (x_n) *is Cauchy if and only if* $\lim_{n,m\to\infty} d(x_n, x_m) = 0$*.*
(iii) (X,d) *is complete if and only if every Cauchy sequence in X is convergent.*

Remark 2.1. [13] *In a b-metric space (X,d), the following assertions hold:*

(i) *A convergent sequence has a unique limit.*
(ii) *Each convergent sequence is Cauchy.*
(iii) *In general, a b-metric is not continuous.*

Theorem 2.1. [2] *Let (X,d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{b^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq b^2 d(x,y).$$

In particular, if $x = y$, *then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, *we have*

$$\frac{1}{b} d(x,z) \le \liminf_{n\to\infty} d(x_n, z) \le \limsup_{n\to\infty} d(x_n, z) \le b d(x,z).$$

Definition 2.3. [1] *Let* T *and* S *be self mappings of a set* X. *If* $y = Tx = Sx$ *for some* x *in* X, *then* x *is called a coincidence point of* T *and* S *and* y *is called a point of coincidence of* T *and* S.

Definition 2.4. [23] *The mappings* $T, S : X \to X$ *are weakly compatible, if for every* $x \in X$, *the following holds:*

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.1. [1] *Let* S *and* T *be weakly compatible selfmaps of a nonempty set* X. *If* S *and* T *have a unique point of coincidence* $y = Sx = Tx$, *then* y *is the unique common fixed point of* S *and* T.

Definition 2.5. [15] *Let* $\mathcal{L}$ *be the family of all mappings* $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ *such that*

($\xi 1$) $\xi(1,1) = 1$;
($\xi 2$) $\xi(t,s) < \frac{s}{t}$ *for all* $t, s > 1$;
($\xi 3$) *for any two sequences* (t_n), $(s_n) \subseteq (1,\infty)$ *with* $t_n \le s_n$ *for all* $n \in \mathbb{N}$,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1 \Rightarrow \limsup_{n\to\infty} \xi(t_n, s_n) < 1.$$

We say that $\xi \in \mathcal{L}$ *is an* $\mathcal{L}$*-simulation function. It is to be noted that* $\xi(t,t) < 1$ *for all* $t > 1$.

Example 2.3. [15] *Let* ξ_b, ξ_w, $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ *be functions defined as follows, respectively:*

(i) $\xi_b(t,s) = \frac{s^k}{t}$ *for all* t, $s \ge 1$, *where* $k \in (0,1)$.
(ii) $\xi_w(t,s) = \frac{s}{t\phi(s)}$ *for all* t, $s \ge 1$, *where* $\phi : [1,\infty) \to [1,\infty)$ *is a nondecreasing and lower semicontinuous function such that* $\phi^{-1}(\{1\}) = 1$.
(iii)

$$\xi(t,s) = \begin{cases} 1, & \text{if } (s,t) = (1,1), \\ \frac{s}{2t}, & \text{if } s < t, \\ \frac{s^\lambda}{t}, & \text{otherwise,} \end{cases}$$

for all s, $t \ge 1$, *where* $\lambda \in (0,1)$.

Then ξ_b, ξ_w, $\xi \in \mathcal{L}$.

Definition 2.6. [3] *Let* Θ *be the set of all functions* $\theta : (0,\infty) \to (1,\infty)$ *such that*

($\theta 1$) θ *is nondecreasing;*
($\theta 2$) *for all* $(t_n) \subseteq (0,\infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \Longleftrightarrow \lim_{n\to\infty} t_n = 0^+;$$

($\theta 3$) θ *is continuous on* $(0,\infty)$.

Example 2.4. [3] *Let* $\theta : (0,\infty) \to (1,\infty)$ *be defined as* $\theta(t) = e^t$ *for all* $t > 0$. *Then* $\theta \in \Theta$.

We next review some basic notions in graph theory.

Let (X, d) be a b-metric space. We assign a reflexive digraph G with the vertex set $V(G) = X$ and the set $E(G)$ of its edges contains no parallel edges. So we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the graph obtained from G by reversing the direction of edges, i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [12, 14, 21].

Definition 2.7. *Let (X, d) be a b-metric space with the coefficient $b \geq 1$ and let $G = (V(G), E(G))$ be a graph. A mapping $f : X \to X$ is called a Banach G-contraction or simply G-contraction if there exists $\alpha \in (0, \frac{1}{b})$ such that*

$$d(fx, fy) \leq \alpha\, d(x, y)$$

for all x, $y \in X$ with $(x, y) \in E(G)$.

Any Banach contraction is a G_0-contraction, where the graph G_0 is defined by $E(G_0) = X \times X$. But it is valuable to note that a Banach G-contraction need not be a Banach contraction (see Remark 3.2).

Remark 2.2. *If f is a G-contraction, then f is both a G^{-1}-contraction and a $\tilde{G}$-contraction.*

Definition 2.8. *Let (X, d) be a b-metric space with the coefficient $b \geq 1$ and let $G = (V(G), E(G))$ be a graph. A mapping $f : X \to X$ is called a generalized $\mathcal{L}G$-contraction if there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that*

$$\xi(\theta(bd(fx, fy)), \theta(d(x, y))) \geq 1$$

for all x, $y \in X$ with $(x, y) \in E(G)$ and $d(fx, fy) > 0$.

3. Main Result

In this section, we assume that (X, d) is a b-metric space with the coefficient $b \geq 1$ and $G = (V(G), E(G))$ is a reflexive digraph which has no parallel edges. Let f, $g : X \to X$ be such that $f(X) \subseteq g(X)$. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq g(X)$, we can find an element $x_1 \in X$ satisfying $fx_0 = gx_1$. Continuing in this way, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$.

Definition 3.1. *Let (X, d) be a b-metric space endowed with a graph G and f, $g : X \to X$ be such that $f(X) \subseteq g(X)$. We define C_{gf} the set of all elements x_0 of X such that $(gx_n, gx_m) \in E(\tilde{G})$ for m, $n = 0, 1, 2, \cdots$ and for every sequence (gx_n) such that $gx_n = fx_{n-1}$.*

Taking $g = I$, the identity map on X, C_{gf} becomes C_f which is the collection of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for m, $n = 0, 1, 2, \cdots$.

Before presenting our main result, we state a property of the graph G, call it property $(*)$.

Property $(*)$**:** If (gx_k) is a sequence in X such that $gx_k \to x$ and $(gx_k, gx_{k+1}) \in E(\tilde{G})$ for all $k \geq 1$, then there exists a subsequence (gx_{k_i}) of (gx_k) such that $(gx_{k_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Taking $g = I$, the above property reduces to property $(\acute{*})$:

Property $(\acute{*})$**:** If (x_k) is a sequence in X such that $x_k \to x$ and $(x_k, x_{k+1}) \in E(\tilde{G})$ for all $k \geq 1$, then there exists a subsequence (x_{k_i}) of (x_k) such that $(x_{k_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Theorem 3.1. *Let (X, d) be a b-metric space endowed with a graph G and let f, $g : X \to X$ be mappings with g is one to one. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that*

$$\xi(\theta(bd(fx, fy)), \theta(d(gx, gy))) \geq 1 \tag{1}$$

for all x, $y \in X$ with $(gx, gy) \in E(\tilde{G})$ and $d(fx, fy) > 0$. Suppose also that $f(X) \subseteq g(X)$, $g(X)$ is a complete subspace of X and the graph G has the property $(*)$. Then f and g have a point of coincidence in $g(X)$ if $C_{gf} \neq \emptyset$.
Moreover, f and g have a unique point of coincidence in $g(X)$ if the graph G has the following property:
$(**)$ If x, y are points of coincidence of f and g in $g(X)$, then $(x, y) \in E(\tilde{G})$.
Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. We first mention that if g is one to one and $d(fx, fy) > 0$, then it must be the case that $d(gx, gy) > 0$. Assume that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. Since $f(X) \subseteq g(X)$, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$ and $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.
We note that if $gx_n = gx_{n-1}$ for some $n \in \mathbb{N}$, then $gx_n = fx_{n-1} = gx_{n-1}$ which implies that gx_n is a point of coincidence of f and g. So, we assume that $gx_n \neq gx_{n-1}$ for every $n \in \mathbb{N}$.

We now prove that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = 0. \tag{2}$$

Since $(gx_{n-1}, gx_n) \in E(\tilde{G})$ for all $n \in \mathbb{N}$ and $d(fx_{n-1}, fx_n) > 0$, it follows from conditions (1) and $(\xi 2)$ that

$$\begin{aligned} 1 &\leq \xi(\theta(bd(gx_n, gx_{n+1})), \theta(d(gx_{n-1}, gx_n))) \\ &< \frac{\theta(d(gx_{n-1}, gx_n))}{\theta(bd(gx_n, gx_{n+1}))}, \end{aligned}$$

which gives that

$$\theta(bd(gx_n, gx_{n+1})) < \theta(d(gx_{n-1}, gx_n)).$$

As θ is nondecreasing, we have for all $n \in \mathbb{N}$,

$$d(gx_n, gx_{n+1}) \leq bd(gx_n, gx_{n+1}) < d(gx_{n-1}, gx_n). \tag{3}$$

Therefore, $(d(gx_{n-1}, gx_n))$ is a decreasing sequence of positive real numbers and so there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(gx_{n-1}, gx_n) = r$.
From condition (3), it follows that

$$\lim_{n\to\infty} bd(gx_n, gx_{n+1}) = r.$$

We shall show that $r = 0$. Assume that $r > 0$. Then by using condition ($\theta 2$), we get

$$\lim_{n\to\infty} \theta(d(gx_{n-1}, gx_n)) > 1$$

and

$$\lim_{n\to\infty} \theta(bd(gx_n, gx_{n+1})) > 1.$$

Let $t_n = \theta(bd(gx_n, gx_{n+1}))$ and $s_n = \theta(d(gx_{n-1}, gx_n))$ for all $n \in \mathbb{N}$. Then, $t_n < s_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1$. From ($\xi 3$), we obtain

$$1 \le \limsup_{n\to\infty} \xi(t_n, s_n) < 1,$$

which is a contradiction. Consequently, it follows that $\lim_{n\to\infty} d(gx_{n-1}, gx_n) = 0$ and so $\lim_{n\to\infty} \theta(d(gx_{n-1}, gx_n)) = 1$.

We now show that the sequence (gx_n) is bounded.
If possible, suppose that the sequence (gx_n) is not bounded. Then there exists a subsequence (gx_{n_i}) of (gx_n) such that $n_1 = 1$ and for each natural number i, n_{i+1} is the smallest integer satisfying

$$d(gx_{n_{i+1}}, gx_{n_i}) > 1$$

and

$$d(gx_m, gx_{n_i}) \le 1, \ for \ n_i \le m \le n_{i+1} - 1.$$

Then, we compute that

$$\begin{aligned} 1 &< d(gx_{n_{i+1}}, gx_{n_i}) \\ &\le bd(gx_{n_{i+1}}, gx_{n_{i+1}-1}) + bd(gx_{n_{i+1}-1}, gx_{n_i}) \\ &\le bd(gx_{n_{i+1}}, gx_{n_{i+1}-1}) + b. \end{aligned}$$

Taking limit as $i \to +\infty$ and using condition (2), we have

$$1 \le \liminf_{n\to\infty} d(gx_{n_{i+1}}, gx_{n_i}) \le \limsup_{n\to\infty} d(gx_{n_{i+1}}, gx_{n_i}) \le b. \tag{4}$$

As $d(gx_{n_{i+1}}, gx_{n_i}) > 1 \Rightarrow d(fx_{n_{i+1}-1}, fx_{n_i-1}) > 1$ and g is one to one, we have $d(gx_{n_{i+1}-1}, gx_{n_i-1}) > 0$. Using conditions (1) and ($\xi 2$), we get

$$\begin{aligned} 1 &\le \xi(\theta(bd(gx_{n_{i+1}}, gx_{n_i})), \theta(d(gx_{n_{i+1}-1}, gx_{n_i-1}))) \\ &< \frac{\theta(d(gx_{n_{i+1}-1}, gx_{n_i-1}))}{\theta(bd(gx_{n_{i+1}}, gx_{n_i}))}. \end{aligned}$$

This implies that

$$\theta(bd(gx_{n_{i+1}}, gx_{n_i})) < \theta(d(gx_{n_{i+1}-1}, gx_{n_i-1})).$$

Using condition ($\theta 1$), we get

$$bd(gx_{n_{i+1}}, gx_{n_i}) < d(gx_{n_{i+1}-1}, gx_{n_i-1}).$$

Now,

$$\begin{aligned} bd(gx_{n_{i+1}}, gx_{n_i}) &< d(gx_{n_{i+1}-1}, gx_{n_i-1}) \\ &\le bd(gx_{n_{i+1}-1}, gx_{n_i}) + bd(gx_{n_i}, gx_{n_i-1}) \\ &\le b + bd(gx_{n_i}, gx_{n_i-1}). \end{aligned}$$

Taking limit as $i \to \infty$ and using condition (4), we obtain that

$$\lim_{i\to\infty} d(gx_{n_{i+1}}, gx_{n_i}) = 1$$

and

$$\lim_{i\to\infty} d(gx_{n_{i+1}-1}, gx_{n_i-1}) = b.$$

Let $t_i = \theta(bd(gx_{n_{i+1}}, gx_{n_i}))$ and $s_i = \theta(d(gx_{n_{i+1}-1}, gx_{n_i-1}))$. Then $t_i < s_i$ for all $i \in \mathbb{N}$. As θ is continuous, we have $\lim\limits_{i\to\infty} t_i = \lim\limits_{i\to\infty} s_i > 1$. It follows from condition ($\xi 3$) that

$$1 \le \limsup_{i\to\infty} \xi(t_i, s_i) < 1,$$

which is a contradiction and hence the sequence (gx_n) is bounded.

We now show that the sequence (gx_n) is Cauchy.

Put

$$\alpha_n = \sup\{d(gx_i, gx_j) > 0 : i,\, j \ge n\},\ n \in \mathbb{N}.$$

Sequence (gx_n) being bounded, $\alpha_n < \infty$ for every $n \in \mathbb{N}$. Since (α_n) is a decreasing sequence of positive real numbers, there exists $\alpha \ge 0$ such that

$$\lim_{n\to\infty} \alpha_n = \alpha. \tag{5}$$

Assume that $\alpha > 0$. Then it follows from the definition of α_n that for every natural number k, there exist $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \ge k$, $d(gx_{m_k}, gx_{n_k}) > 0$ and

$$\alpha_k - \frac{1}{k} < d(gx_{m_k}, gx_{n_k}) \le \alpha_k.$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} d(gx_{m_k}, gx_{n_k}) = \alpha > 0. \tag{6}$$

Since g is one to one, for every $k \in \mathbb{N}$,

$$d(gx_{m_k}, gx_{n_k}) > 0 \Rightarrow d(fx_{m_k-1}, fx_{n_k-1}) > 0 \Rightarrow d(gx_{m_k-1}, gx_{n_k-1}) > 0.$$

Also, it follows from (1), ($\xi 2$) and the definition of α_n that

$$\begin{aligned} 1 &\le \xi(\theta(bd(gx_{m_k}, gx_{n_k})), \theta(d(gx_{m_k-1}, gx_{n_k-1}))) \\ &< \frac{\theta(d(gx_{m_k-1}, gx_{n_k-1}))}{\theta(bd(gx_{m_k}, gx_{n_k}))}, \end{aligned}$$

which gives that

$$\theta(bd(gx_{m_k}, gx_{n_k})) < \theta(d(gx_{m_k-1}, gx_{n_k-1})).$$

Hence,

$$bd(gx_{m_k}, gx_{n_k}) < d(gx_{m_k-1}, gx_{n_k-1}) \le \alpha_{k-1}.$$

Letting $k \to \infty$ and using conditions (5) and (6), we obtain that

$$b\alpha \le \liminf_{k\to\infty} d(gx_{m_k-1}, gx_{n_k-1}) \le \limsup_{k\to\infty} d(gx_{m_k-1}, gx_{n_k-1}) \le \alpha. \tag{7}$$

If $b > 1$, then it follows from condition (7) that $\alpha = 0$.

If $b = 1$, then $\lim\limits_{k\to\infty} d(gx_{m_k-1}, gx_{n_k-1}) = \alpha > 0$.

Let $t_k = \theta(d(gx_{m_k}, gx_{n_k}))$ and $s_k = \theta(d(gx_{m_k-1}, gx_{n_k-1}))$. Then $t_k < s_k$ for every $k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} t_k = \lim\limits_{k\to\infty} s_k > 1$. Using ($\xi 3$), we get

$$1 \le \limsup_{k\to\infty} \xi(t_k, s_k) < 1,$$

which is a contradiction and hence we have $\alpha = 0$, i.e., $\lim_{n\to\infty} \alpha_n = 0$. Therefore, (gx_n) is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $gx_n \to u = gv$ for some $v \in X$.

As $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 0$. By property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Let $S = \{p_i : p_i = d(fx_{n_i}, fv) > 0,\ i \in \mathbb{N}\}$. For $p_i \in S$, we have

$$\begin{aligned} 1 &\leq \xi(\theta(bp_i), \theta(d(gx_{n_i}, gv))) \\ &< \frac{\theta(d(gx_{n_i}, gv))}{\theta(bd(fx_{n_i}, fv))}, \end{aligned}$$

which implies that

$$\theta(bd(fx_{n_i}, fv)) < \theta(d(gx_{n_i}, gv)).$$

As θ is nondecreasing, we have

$$bd(fx_{n_i}, fv) < d(gx_{n_i}, gv).$$

If $p_i \notin S$, then

$$0 = p_i = d(fx_{n_i}, fv) \leq d(gx_{n_i}, gv).$$

Thus,

$$bd(fx_{n_i}, fv) \leq d(gx_{n_i}, gv) \ for\ all\ i \in \mathbb{N}.$$

Now,

$$\begin{aligned} d(fv, gv) &\leq bd(fv, fx_{n_i}) + bd(fx_{n_i}, gv) \\ &\leq d(gx_{n_i}, gv) + bd(gx_{n_i+1}, gv). \end{aligned}$$

Taking limit as $i \to \infty$, we have $d(fv, gv) = 0$ and hence, $fv = gv = u$. Therefore, u is a point of coincidence of f and g.

For uniqueness, we assume that there exists u^* in $g(X)$ such that $fx = gx = u^*$ for some $x \in X$ and $u \neq u^*$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$1 \leq \xi(\theta(bd(fv, fx)), \theta(d(gv, gx))) = \xi(\theta(bd(u, u^*)), \theta(d(u, u^*))) < \frac{\theta(d(u, u^*))}{\theta(bd(u, u^*))}.$$

This gives that $\theta(bd(u, u^*)) < \theta(d(u, u^*))$ and so $bd(u, u^*) < d(u, u^*)$, a contradiction. This proves that $u = u^*$. Therefore, f and g have a unique point of coincidence in $g(X)$.

If f and g are weakly compatible, then by Proposition 2.1, f and g have a unique common fixed point in $g(X)$. □

We give some examples to illustrate our main result.

Example 3.1. *Let $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = \mid x - y \mid^4$ for all $x,\ y \in X$. Then (X, d) is a complete b-metric space with the coefficient $b = 8$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{n}) : n = 1,\ 2,\ 3,\ \cdots\}$.*

Let $f,\ g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{5},\ if\, x \neq \frac{2}{5}, \\ \\ 1,\ if\, x = \frac{2}{5} \end{cases}$$

and $gx = 3x$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$.

Let $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ be defined by $\xi(t,s) = \frac{s^{\frac{1}{3}}}{t}$ for all $t,\ s \geq 1$ and $\theta : (0,\infty) \to (1,\infty)$ be defined by $\theta(t) = e^t$ for all $t > 0$.

If $x = 0,\ y = \frac{1}{3n}$, then $gx = 0,\ gy = \frac{1}{n}$ and so $(gx, gy) \in E(\tilde{G})$.

For $x = 0,\ y = \frac{1}{3n}$, we have
$d(fx,fy) = d(0,\frac{1}{15n}) = \frac{1}{15^4n^4} > 0,\ d(gx,gy) = d(0,\frac{1}{n}) = \frac{1}{n^4}$
and so $(\theta(d(gx,gy)))^{\frac{1}{3}} = e^{\frac{1}{3n^4}},\ \theta(bd(fx,fy)) = e^{\frac{8}{15^4n^4}}$.

Since $\frac{1}{3n^4} > \frac{8}{15^4n^4} \Rightarrow e^{\frac{1}{3n^4}} > e^{\frac{8}{15^4n^4}}$, we have

$$\begin{aligned} \xi(\theta(bd(fx,fy)),\theta(d(gx,gy))) &= \frac{(\theta(d(gx,gy)))^{\frac{1}{3}}}{\theta(bd(fx,fy))} \\ &= \frac{e^{\frac{1}{3n^4}}}{e^{\frac{8}{15^4n^4}}} \\ &> 1. \end{aligned}$$

Therefore,
$$\xi(\theta(bd(fx,fy)),\theta(d(gx,gy))) > 1$$
for all $x,\ y \in X$ with $(gx,gy) \in E(\tilde{G})$ and $d(fx,fy) > 0$.

It is easy to verify that $0 \in C_{gf}$. Also, any sequence (gx_n) with the property $(gx_n, gx_{n+1}) \in E(\tilde{G})$ must be either a constant sequence or a sequence of the following form

$$\begin{aligned} gx_n &= 0, \text{ if } n \text{ is odd} \\ &= \frac{1}{n}, \text{ if } n \text{ is even} \end{aligned}$$

where the words 'odd' and 'even' are interchangeable. Consequently it follows that property $(*)$ holds true. Furthermore, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.1 and 0 is the unique common fixed point of f and g in $g(X)$.

The following remark ensures that the weak compatibility condition in Theorem 3.1 can not be relaxed.

Remark 3.1. *In Example 3.1, if we take $gx = 3x - 14$ for all $x \in X$ instead of $gx = 3x$, then $5 \in C_{gf}$ and $f(5) = g(5) = 1$ but $g(f(5)) \neq f(g(5))$, i.e., f and g are not weakly compatible. However, all other conditions of Theorem 3.1 are fulfilled. We observe that 1 is the unique point of coincidence of f and g without being any common fixed point.*

Remark 3.2. *In Example 3.1, f is a Banach G-contraction with constant $k = \frac{1}{125}$ but it is not a Banach contraction. In fact, for $x = 0,\ y = \frac{2}{5}$, we have*

$$\begin{aligned} d(fx,fy) &= d(0,1) \\ &= 1 \\ &= \frac{625}{16}.\frac{16}{625} \\ &> k\,d(x,y), \end{aligned}$$

for any $k \in (0,\frac{1}{b})$. This implies that f is not a Banach contraction.

The next example supports that the property $(*)$ is necessary in Theorem 3.1.

Example 3.2. *Let $X = [0,\infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) =\mid x-y \mid^3$ for all $x, y \in X$. Then (X,d) is a complete b-metric space with the coefficient $b = 4$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x,y) : (x,y) \in (0,1]\times(0,1],\, x \leq y\}$.*

Let f, $g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{4}, & \text{if } x \neq 0, \\ \\ 1, & \text{if } x = 0 \end{cases}$$

and $gx = \frac{x}{2}$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$.

Let $\xi : [1,\infty)\times[1,\infty) \to \mathbb{R}$ be defined by $\xi(t,s) = \frac{s^{\frac{3}{4}}}{t}$ for all t, $s \geq 1$ and $\theta : (0,\infty) \to (1,\infty)$ be defined by $\theta(t) = e^t$ for all $t > 0$.

For x, $y \in X$ with $(gx, gy) \in E(\tilde{G})$ and $d(fx, fy) > 0$, we have $bd(fx, fy) = \frac{1}{2}d(gx, gy)$.

Since $\frac{3}{4}d(gx, gy) > \frac{1}{2}d(gx, gy) \Rightarrow e^{\frac{3}{4}d(gx,gy)} > e^{\frac{1}{2}d(gx,gy)}$, we obtain

$$\begin{aligned} \xi(\theta(bd(fx,fy)),\theta(d(gx,gy))) &= \frac{(\theta(d(gx,gy)))^{\frac{3}{4}}}{\theta(4d(fx,fy))} \\ &= \frac{e^{\frac{3}{4}d(gx,gy)}}{e^{\frac{1}{2}d(gx,gy)}} \\ &> 1. \end{aligned}$$

We now show that the property $(*)$ does not hold true. For $x_n = \frac{2}{n}$, $gx_n = \frac{1}{n}$ and hence $gx_n \to 0$. Also, $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \in \mathbb{N}$. But there exists no subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, 0) \in E(\tilde{G})$. As a result, we get no point of coincidence of f and g in $g(X)$.

4. Some Consequences of the Main Result

Theorem 4.1. *Let (X,d) be a complete b-metric space endowed with a graph G and let $f : X \to X$ be a mapping. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that*

$$\xi(\theta(bd(fx,fy)),\theta(d(x,y))) \geq 1$$

for all x, $y \in X$ with $(x,y) \in E(\tilde{G})$ and $d(fx,fy) > 0$. Suppose also that the triple (X,d,G) has the property $(\acute{*})$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the following property:
$(\acute{**})$ *If x, y are fixed points of f in X, then $(x,y) \in E(\tilde{G})$.*

Proof. The proof follows from Theorem 3.1 by taking $g = I$, the identity map on X. □

Theorem 4.2. *Let (X,d) be a b-metric space and let f, $g : X \to X$ be mappings with g is one to one. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that*

$$\xi(\theta(bd(fx,fy)),\theta(d(gx,gy))) \geq 1$$

for all x, $y \in X$ and $d(fx,fy) > 0$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in $g(X)$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof can be obtained from Theorem 3.1 by considering $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. $\square$

Theorem 4.3. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that*

$$\xi(\theta(bd(fx, fy)), \theta(d(x, y))) \geq 1$$

for all x, $y \in X$ and $d(fx, fy) > 0$. Then f has a unique fixed point in X.

Proof. The proof follows from Theorem 3.1 by taking $g = I,\ G = G_0$. $\square$

Theorem 4.4. *Let (X, d) be a b-metric space and let f, $g : X \to X$ be mappings with g is one to one. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$\theta(bd(fx, fy)) \leq (\theta(d(gx, gy)))^k$$

for all x, $y \in X$ and $d(fx, fy) > 0$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in $g(X)$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof can be obtained from Theorem 3.1 by considering $G = G_0$ and $\xi = \xi_b$. $\square$

Theorem 4.5. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists $\theta \in \Theta$ such that*

$$\theta(bd(fx, fy)) \leq \frac{\theta(d(x, y))}{\phi(\theta(d(x, y)))}$$

for all x, $y \in X$ and $d(fx, fy) > 0$, where $\phi : [1, \infty) \to [1, \infty)$ is a nondecreasing and lower semicontinuous function such that $\phi^{-1}(\{1\}) = 1$. Then f has a unique fixed point in X.

Proof. The proof follows from Theorem 3.1 by taking $G = G_0,\ g = I$ and $\xi = \xi_w$. $\square$

The following theorem gives fixed point of Banach G-contraction mappings in b-metric spaces.

Theorem 4.6. *Let (X, d) be a complete b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \leq \alpha\, d(x, y)$$

for all x, $y \in X$ with $(x, y) \in E(\tilde{G})$, where $\alpha \in (0, \frac{1}{b})$ is a constant. Suppose the triple (X, d, G) has the property $(*)'$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(**)'$.

Proof. The proof follows from Theorem 3.1 by taking $g = I,\ \xi = \xi_b$ and $\theta(t) = e^t$ for all $t > 0$. $\square$

The following is the b-metric version of Banach contraction theorem.

Theorem 4.7. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping satisfying*

$$d(fx, fy) \leq \alpha d(x, y)$$

for all x, $y \in X$, where $\alpha \in (0, \frac{1}{b})$ is a constant. Then f has a unique fixed point in X.

Proof. The conclusion of the theorem follows from Theorem 3.1 by taking $G = G_0,\ g = I,\ \xi = \xi_b$ and $\theta(t) = e^t$ for all $t > 0$ $\square$

Remark 4.1. *Theorem 4.7 shows that our main result is a generalization of the Banach contraction theorem.*

Theorem 4.8. *Let (X, d) be a complete b-metric space endowed with a partial ordering $\preceq$ and let $f : X \to X$ be a mapping. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that*

$$\xi(\theta(bd(fx, fy)), \theta(d(x, y))) \geq 1$$

for all $x, y \in X$ with $x \preceq y$ or, $y \preceq x$ and $d(fx, fy) > 0$. Suppose also that the triple $(X, d, \preceq)$ has the following property:

If (x_n) is a sequence in X such that $x_n \to x$ and x_n, x_{n+1} are comparable for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i}, x are comparable for all $i \geq 1$. If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \cdots$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the following property holds:

If x, y are fixed points of f in X, then x, y are comparable.

Proof. The proof can be obtained from Theorem 3.1 by taking $g = I$ and $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

Theorem 4.9. *Let (X, d) be a metric space and let f, $g : X \to X$ be mappings with g is one to one. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that*

$$\xi(\theta(d(fx, fy)), \theta(d(gx, gy))) \geq 1$$

for all $x, y \in X$ and $d(fx, fy) > 0$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in $g(X)$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof can be obtained from Theorem 3.1 by considering $G = G_0$ and $b = 1$. □

Remark 4.2. *It is worth mentioning that we can obtain several important fixed point results in metric and b-metric spaces by suitable choices of ξ and θ.*

5. Conclusions

In this work, we obtained some new coincidence point and common fixed point results combining digraphs, $\mathcal{L}$-simulation and θ-functions in b-metric spaces. As a consequence of this study, we have been able to establish some important and well known fixed point results in metric and b-metric spaces.

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