TWMS J. App. and Eng. Math. V.15, N.6, 2025, pp. 1448-1459

REGULARIZED GAP FUNCTIONS FOR p-STRONGLY MONOTONE VARIATIONAL-HEMIVARIATIONAL INEQUALITIES AND APPLICATIONS TO GLOBAL ERROR BOUNDS

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Abstract. The paper studies regularized gap functions for a general class of elliptic variational–hemivariational inequalities by employing the sum rule of Clarke's generalized directional derivatives. We first devote regularized gap functions for this class of inequalities based on the forms introduced by Yamashita and Fukushima. Global error bounds for the variational–hemivariational inequality then are formulated in terms of regularized gap functions under strongly monotone assumptions of a general order $p > 1$ on the given data. The established results are meaningful generalizations to corresponding ones in the literature.

Keywords: variational–hemivariational inequality, regularized gap function, global error bound, strong monotonicity of order p.

AMS Subject Classification: 47J20; 49J40; 49J45

1. Introduction

The gap function is well known as a valuable tool for solving variational inequalities because it can transform a variational inequality into an associated minimization problem (see Auslender [1]). However, in general the Auslender's gap function is non-differentiable. This implies a disadvantage in using iterative methods to solve associated minimization problems. To overcome this limitation, Fukushima [3] originally proposed a new gap function for variational inequalities called the regularized gap function. Yamashita and Fukushima [21] developed global error bound results for variational inequalities in terms of regularized gap functions of the Fukushima and Moreau-Yosida types. Error bound

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§ Manuscript received: March 22, 2024; accepted: July 04, 2024.

provides an upper estimation of the distance between an arbitrary feasible point and the solution set of a certain problem. It is a powerful tool for convergence analysis of iterative methods for solving of variational inequality problems. Thus, regularized gap functions and error bounds have attracted considerable attentions due to their importances in examining various kinds of variational inequalities and equilibrium problems, see e.g., [9, 10, 11, 14] and the references therein.

On the other hand, the theory of hemivariational inequalities is a natural extension of variational inequalities introduced by Panagiotopoulos for investigating various problems in mechanics by employing Clarke generalized gradients for locally Lipschitz functions involving nonconvex and nonsmooth energy potentials, see e.g., [17, 18]. Besides, hemivariational inequalities have been generalized to variational–hemivariational inequalities in both convex and nonconvex potentials. These inequalities have been applied in mechanics, engineering, especially in nonsmooth analysis and extensively studied in various directions. Recently, Hung et al. [8] developed the regularized gap function and error bound results to a class of variational–hemivariational inequalities of elliptic types. This approach has been further investigated for various kinds of variational–hemivariational inequalities, see e.g., [4, 12, 13, 19]. Very recently, Tam and Chen [20] investigated the D-gap function based on the regularized gap function of the Fukushima type for variational–hemivariational inequalities employing the sum rule of Clarke's generalized directional derivatives. However, Tam and Chen [20] only used strongly monotone assumptions of order 2 to obtain error bound results for such variational–hemivariational inequalities. Besides, regularized gap functions of the Moreau-Yosida type have not been considered in [20]. Han and Nashed [6] formulated well-posedness results for variational–hemivariational inequalities with strongly monotone assumptions of a general order $p > 1$ instead of using the order 2 in the majority of existing references.

Motivated by these works, in this paper, we consider a general class of elliptic variational–hemivariational inequalities by employing the sum of Clarke's generalized directional derivatives. Then we propose regularized gap functions in the forms of both the Fukushima type and the Moreau-Yosida type. Based on using the strongly monotone assumptions of a general order $p > 1$ in [6], we establish global error bounds for such a variational–hemivariational inequality via regularized gap functions.

The rest of this paper is structured as follows. Section 2 presents some basic definitions and properties that will be used throughout the paper. We also introduce a generalized variational-hemivariational inequality and provide its solution existence based on some imposed hypotheses on the initial data. In Section 3, we study regularized gap functions for this problem. Applications to establishing global error bounds for such a problem by virtue of regularized gap functions are devoted in Section 4. Finally, Section 5 gives some conclusions of this paper.

2. Preliminaries

Let W be a normed space with its topological dual W^*. The norm on W and the duality pairing of W and W^* are denoted by $\|\cdot\|_W$ and $\langle\cdot,\cdot\rangle_W$, respectively. For normed spaces W and V, $\mathcal{L}(W, V)$ denotes the space of all linear continuous operators from W to V. We now recall some primary definitions that will be used in the sequel (see e.g., [2, 16]).

Definition 2.1. A function $H\colon W \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be

(a): proper, if $H \not\equiv +\infty$;

(b): convex, if $H(sw + (1-s)v) \le sH(w) + (1-s)H(v)$ for all w, $v \in W$ and $s \in [0,1]$;
(c): lower semicontinuous at $w_0 \in W$, if for any sequence $\{w_n\} \subset W$ such that $w_n \to w_0$, it holds $H(w_0) \le \liminf H(w_n)$;
(d): upper semicontinuous at $w_0 \in W$, if for any sequence $\{w_n\} \subset W$ such that $w_n \to w_0$, it holds $\limsup H(w_n) \le H(w_0)$;
(e): lower semicontinuous (resp., upper semicontinuous) on W, if H is lower semicontinuous (resp., upper semicontinuous) at every $w_0 \in W$.

Definition 2.2. A function $H\colon W \to \mathbb{R}$ is said to be locally Lipschitz, if for every $w \in W$, there exist a neighbourhood $\mathcal{N}$ of w and a constant $l_w > 0$ such that

$$|H(w_1) - H(w_2)| \le l_w \|w_1 - w_2\|_W \quad \text{for all} \quad w_1, w_2 \in \mathcal{N}.$$

For a locally Lipschitz function $H\colon W \to \mathbb{R}$, we denote by $H^0(w;v)$ the Clarke generalized directional derivative of H at the point $w \in W$ in the direction $v \in W$ defined by

$$H^0(w;v) = \limsup_{u \to w,\ s \to 0^+} \frac{H(u+sv) - H(u)}{s}.$$

The generalized gradient of H at $w \in W$, denoted by $\partial H(w)$, is a subset of W^* given by

$$\partial H(w) = \big\{\zeta^* \in W^* \mid H^0(w;v) \ge \langle \zeta^*, v\rangle_W \text{ for all } v \in W\big\}.$$

The following lemma provides some nice properties of the Clarke generalized directional derivatives.

Lemma 2.1. (see [2, Proposition 2.1.1]) *Let W be a Banach space, and $H\colon W \to \mathbb{R}$ be a locally Lipschitz function. The following assertions hold.*

(i): *For each $w \in W$, the function $W \ni v \mapsto H^0(w;v) \in \mathbb{R}$ is finite, positively homogeneous, subadditive, and Lipschitz continuous.*
(ii): *The function $W \times W \ni (w,v) \mapsto H^0(w;v) \in \mathbb{R}$ is upper semicontinuous.*
(iii): *For every w, $v \in W$, it holds*

$$H^0(w;v) = \max\{\langle \zeta, v\rangle_W \mid \zeta \in \partial H(w)\}.$$

For each $i \in \{1,\dots,k\}$, let V be a Hilbert space, V_G and V_{H_i} be Banach spaces, $C \subset V$ and $C_G \subset V_G$. In addition, let $F\colon V \to V^*$, $\Phi\colon E \to C_G$, $\Psi_i\colon V \to V_{H_i}$ be operators, $G\colon C_G \times C_G \to \mathbb{R}$, $H_i\colon V_{H_i} \to \mathbb{R}$ be functions, $a_i > 0$ for all $i \in \{1,\dots,k\}$ and $\varphi \in V^*$. We now consider the following generalized variational-hemivariational inequality.

Problem 2.1. *Find $u \in C$ such that*

$$\langle F(u), v-u\rangle_V + G(\Phi u, \Phi v) - G(\Phi u, \Phi u) + \sum_{i=1}^{k} a_i H_i^0(\Psi_i u; \Psi_i v - \Psi_i u) \ge \langle \varphi, v-u\rangle_V \quad \forall v \in C.$$

For motivating the study Problem 2.1, we give some problems investigated in the literature as its special cases.

(a): When $a_i = 1$ for all $i \in \{1,\dots,k\}$, Problem 2.1 reduces to the abstract elliptic variational-hemivariational inequality introduced by Tam-Chen [20]:

Problem 2.2. *Find $u \in C$ such that*

$$\langle F(u), v-u\rangle_V + G(\Phi u, \Phi v) - G(\Phi u, \Phi u) + \sum_{i=1}^{k} H_i^0(\Psi_i u; \Psi_i v - \Psi_i u) \geq \langle \varphi, v-u\rangle_V \quad \forall v \in C.$$

(b): When $k=1$, $a_1=1$, $H_1=H$ and $\Psi_1=\Psi$, Problem 2.1 reduces to the following problem:

Problem 2.3. Find $u \in C$ such that

$$\langle F(u), v-u\rangle_V + G(\Phi u, \Phi v) - G(\Phi u, \Phi u) + H^0(\Psi u; \Psi v - \Psi u) \geq \langle \varphi, v-u\rangle_V \quad \forall v \in C.$$

investigated by Han et al. [7].

(c): When $k=2$, $a_1=a_2=1$, $G\equiv 0$, Problem 2.1 is equivalent to the following form introduced by Han et al. [5].

Problem 2.4. Find $u \in C$ such that

$$\langle F(u), v-u\rangle_V + H_1^0(\Psi_1 u; \Psi_1 v - \Psi_1 u) + H_2^0(\Psi_2 u; \Psi_2 v - \Psi_2 u) \geq \langle \varphi, v-u\rangle_V \quad \forall v \in C.$$

Now, we provide the following assumptions on the data of Problem 2.1 with $p>1$.

$\underline{\mathbf{A}(F)}$: For the operator $F\colon V \to V^*$,

- **(a):** F is continuous;
- **(b):** F is strongly monotone of order p, i.e., there exists $m_F>0$ such that

$$\langle F(v_1)-F(v_2), v_1-v_2\rangle_V \geq m_F\|v_1-v_2\|_V^p, \quad \forall v_1, v_2 \in V.$$

$\underline{\mathbf{A}(G)}$: For the function $G\colon C_G \times C_G \to \mathbb{R}$,

- **(a):** for each $u \in C_G$, $G(u,\cdot)\colon C_G \to \mathbb{R}$ is convex and continuous;
- **(b):** there exists $m_G>0$ such that

$$G(u_1,v_2)-G(u_1,v_1)+G(u_2,v_1)-G(u_2,v_2) \leq m_G\|u_1-u_2\|_{V_G}^{p-1}\|v_1-v_2\|_{V_G}, \quad \forall u_1,u_2,v_1,v_2 \in C_G.$$

$\underline{\mathbf{A}(H)}$: For each $i \in \{1,\dots,k\}$, for the locally Lipschitz function $H_i\colon V_{H_i} \to \mathbb{R}$,

- **(a):** $\|\xi\|_{V_{H_i}^*} \leq c_0 + c_1\|v\|_{V_{H_i}}$, $\forall v \in V_{H_i}$, $\xi \in \partial H_i(v)$ with some $c_0, c_1 \geq 0$;
- **(b):** there exists $m_{H_i} \geq 0$ such that

$$H_i^0(v_1; v_2-v_1) + H_i^0(v_2; v_1-v_2) \leq m_{H_i}\|v_1-v_2\|_{V_{H_i}}^p, \tag{1}$$

for all $v_1, v_2 \in V_{H_i}$.

$\underline{\mathbf{A}(C)}$:

- **(a):** C is a nonempty, closed and convex subset of V;
- **(b):** C_G is a nonempty, closed and convex subset of V_G with $\Phi(C) \subset C_G$.

$\underline{\mathbf{A}(\Phi)}$: The operator $\Phi\colon E \to C_G$ is linear and compact.

$\underline{\mathbf{A}(\Psi)}$: For each $i \in \{1,\dots,k\}$, the operator $\Psi_i\colon V \to V_{H_i}$ is linear and compact.

$\underline{\mathbf{A}(\varphi)}$: $\varphi \in V^*$.

$\underline{\mathbf{A}(\mathit{const.})}$: $a_i>0$ for all $i \in \{1,\dots,k\}$ and

$$m_F - m_G\|\Phi\|_{\mathcal{L}(V,V_G)}^p - \sum_{i=1}^{k} a_i m_{H_i}\|\Psi_i\|_{\mathcal{L}(V,V_{H_i})}^p > 0.$$

The following example illustrates the strong monotonicity of order p of the operator F.

Example 2.1. Let $V = \mathbb{R}^n$ and $F\colon \mathbb{R}^n \to V^* = \mathbb{R}^n$ be defined by

$$F(v) = 3\|v\|^{p-2}v + \frac{1}{2}\mathbf{e},$$

where $p \geq 2$ and $\mathbf{e} = (1, \ldots, 1)^\top \in \mathbb{R}^n$. Then for any $v_1, v_2 \in \mathbb{R}^n$, we have

$$\begin{aligned}\langle F(v_1) - F(v_2), v_1 - v_2\rangle_V &= 3\left(\|v_1\|^{p-2}v_1 - \|v_2\|^{p-2}v_2\right)\cdot(v_1 - v_2)\\ &\geq \frac{3}{2^{p-2}p}\|v_1 - v_2\|^p\\ &= \frac{3}{2^{p-2}p}\|v_1 - v_2\|_V^p,\end{aligned}$$

where we used the inequality

$$\left(\|x\|^{p-2}x - \|y\|^{p-2}y\right)\cdot(x - y) \geq \frac{1}{2^{p-2}p}\|x - y\|^p$$

for $p \geq 2$ and for all $x, y \in \mathbb{R}^n$ (see [15, Lemma 3]).

Thus, F is strongly monotone of order p with $m_F = \dfrac{3}{2^{p-2}p}$, $p \geq 2$.

To end this section, we provide the existence and uniqueness result for Problem 2.1 based on slightly modifying the arguments of [6, Theorem 4.3] employing the properties of linear and compact operators Φ, Ψ_i and the term $\sum_{i=1}^{k} a_i H_i^0(\cdot;\cdot)$, for all $i \in \{1, \ldots, k\}$.

Theorem 2.1. *Suppose that the assumptions* $\mathbf{A}(F)$*,* $\mathbf{A}(G)$*,* $\mathbf{A}(H)$*,* $\mathbf{A}(C)$*,* $\mathbf{A}(\Phi)$*,* $\mathbf{A}(\Psi)$*,* $\mathbf{A}(\varphi)$ *and* $\mathbf{A}(const.)$ *hold, then Problem 2.1 has a unique solution.*

3. Regularized gap functions

In this section, we investigate regularized gap functions in the forms of the Fukushima and Moreau-Yosida types for Problem 2.1.

Definition 3.1. A real-valued function $\Gamma\colon C \to \mathbb{R}$ is said to be a gap function for Problem 2.1, if it satisfies the following conditions:

- **(a):** $\Gamma(u) \geq 0$ for all $u \in C$;
- **(b):** $u^* \in C$ is such that $\Gamma(u^*) = 0$ if and only if u^* is a solution to Problem 2.1.

Let $\kappa > 0$ be a fixed parameter. We now consider the function $\mathcal{D}_\kappa^f\colon C \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{D}_\kappa^f(u) = \sup_{v\in C}\Bigg(&\langle F(u) - \varphi, u - v\rangle_V - G(\Phi u, \Phi v) + G(\Phi u, \Phi u)\\ &- \sum_{i=1}^{k} a_i H_i^0(\Psi_i u; \Psi_i v - \Psi_i u) - \frac{\kappa}{p}\|u - v\|_V^p\Bigg)\end{aligned}$$

for all $u \in C$.

The following proposition shows that $\mathcal{D}_\kappa^f$ is a gap function for Problem 2.1.

Theorem 3.1. *Suppose that the conditions of Theorem 2.1 are satisfied. Then, the function* $\mathcal{D}_\kappa^f$ *is a gap function for Problem 2.1.*

Proof. We are going to verify two conditions of Definition 3.1.

(a) Let $\kappa > 0$. In view of the definition of $\mathcal{D}^f_\kappa$, it holds for any $u \in C$ that

$$\mathcal{D}^f_\kappa(u) \geq \langle F(u) - \varphi, u - u\rangle_V - G(\Phi u, \Phi u) + G(\Phi u, \Phi u) - \sum_{i=1}^k a_i H_i^0(\Psi_i u; \Psi_i u - \Psi_i u) - \frac{\kappa}{p}\|u - u\|_V^p$$

$$= -\sum_{i=1}^k a_i H_i^0(\Psi_i u; 0_{V_{\Psi_i}}) = 0.$$

Thus, $\mathcal{D}^f_\kappa(u) \geq 0$ for all $u \in C$.

(b) Assume that $u^* \in C$ satisfying $\mathcal{D}^f_\kappa(u^*) = 0$, that is,

$$\sup_{v\in C}\left(\langle F(u^*) - \varphi, u^* - v\rangle_V - G(\Phi u^*, \Phi v) + G(\Phi u^*, \Phi u^*) - \sum_{i=1}^k a_i H_i^0(\Psi_i u^*; \Psi_i v - \Psi_i u^*) - \frac{\kappa}{p}\|u^* - v\|_V^p\right) = 0.$$

Thus,

$$\langle F(u^*) - \varphi, v - u^*\rangle_V + G(\Phi u^*, \Phi v) - G(\Phi u^*, \Phi u^*) + \sum_{i=1}^k a_i H_i^0(\Psi_i u^*; \Psi_i v - \Psi_i u^*) \geq -\frac{\kappa}{p}\|u^* - v\|_V^p$$

for all $v \in C$.

For any $z \in C$ and $\mu \in (0,1)$, we substitute $v = v_\mu := (1-\mu)u^* + \mu z \in C$ into the above inequality to obtain

$$\mu\langle F(u^*) - \varphi, z - u^*\rangle_V + \mu G(\Phi u^*, \Phi z) - \mu G(\Phi u^*, \Phi u^*) + \mu\left(\sum_{i=1}^k a_i H_i^0(\Psi_i u^*; \Psi_i z - \Psi_i u^*)\right)$$

$$\geq \langle F(u^*) - \varphi, v_\mu - u^*\rangle_V + G(\Phi u^*, \Phi v_\mu) - G(\Phi u^*, \Phi u^*) + \sum_{i=1}^k a_i H_i^0(\Psi_i u^*; \Psi_i v_\mu - \Psi_i u^*)$$

$$\geq -\frac{\kappa}{p}\|u^* - v_\mu\|_V^p = -\frac{\kappa\mu^p}{p}\|u^* - z\|_V^p,$$

where the properties of operators $\Phi \in \mathcal{L}(V, V_G)$ and $\Psi_i \in \mathcal{L}(V, V_{H_i})$, the convexity of $z \mapsto G(\Phi u^*, \Phi z)$, positive homogeneity of $v \mapsto H_i^0(\Psi_i u^*; \Psi_i v - \Psi_i u^*)$ for all $i \in \{1, \ldots, k\}$ have been used in the above inequalities. Hence,

$$\langle F(u^*) - \varphi, z - u^*\rangle_V + G(\Phi u^*, \Phi z) - G(\Phi u^*, \Phi u^*) + \sum_{i=1}^k a_i H_i^0(\Psi_i u^*; \Psi_i z - \Psi_i u^*) \geq -\frac{\kappa\mu^{p-1}}{p}\|u^* - z\|_V^p$$

for all $z \in C$. Letting $\mu \to 0^+$, one obtain

$$\langle F(u^*), z - u^*\rangle_V + G(\Phi u^*, \Phi z) - G(\Phi u^*, \Phi u^*) + \sum_{i=1}^k a_i H_i^0(\Psi_i u^*; \Psi_i z - \Psi_i u^*) \geq \langle \varphi, z - u^*\rangle_V$$

for all $z \in C$. Thus, u^* is also a solution to Problem 2.1.

Conversely, suppose that $u^* \in C$ is a solution of Problem 2.1, namely,

$$\langle F(u^*) - \varphi, z - u^* \rangle_V + G(\Phi u^*, \Phi z) - G(\Phi u^*, \Phi u^*) + \sum_{i=1}^{k} a_i H_i^0(\Psi_i u^*; \Psi_i z - \Psi_i u^*) \geq 0$$

for all $v \in C$. This implies

$$\sup_{v \in C} \Bigg(\langle F(u^*) - \varphi, u^* - v \rangle_V - G(\Phi u^*, \Phi v) + G(\Phi u^*, \Phi u^*) - \sum_{i=1}^{k} a_i H_i^0(\Psi_i u^*; \Psi_i v - \Psi_i u^*) - \frac{\kappa}{p} \|u^* - v\|_V^p \Bigg) \leq 0,$$

and so $\mathcal{D}_\kappa^f(u^*) \leq 0$. However, $\mathcal{D}_\kappa^f(u) \geq 0$ for all $u \in C$ (by (a)). This implies that $\mathcal{D}_\kappa^f(u^*) = 0$. This completes the proof. □

We now prove that the regularized gap function $\mathcal{D}_\kappa^f$ is lower semicontinuous on C.

Lemma 3.1. *Assume that the conditions of Theorem 2.1 are satisfied. If, in addition, $G\colon C_G \times C_G \to \mathbb{R}$ is continuous, then, for each $\kappa > 0$, the function $\mathcal{D}_\kappa^f$ is lower semicontinuous on C.*

Proof. Consider the function $\widehat{\mathcal{D}}_\kappa \colon C \times C \to \mathbb{R}$ defined by

$$\widehat{\mathcal{D}}_\kappa(u, v) = \langle F(u) - \varphi, u - v \rangle_V - G(\Phi u, \Phi v) + G(\Phi u, \Phi u) - \sum_{i=1}^{k} a_i H_i^0(\Psi_i u; \Psi_i v - \Psi_i u) - \frac{\kappa}{p} \|u - v\|_V^p.$$

Assumption $\mathbf{A}(F)(a)$ implies the continuity of the function $u \mapsto \langle F(u), u \rangle_V$. Combining with the lower semicontinuity of $(u, v) \mapsto -\sum_{i=1}^{k} a_i H_i^0(\Psi_i u; \Psi_i v - \Psi_i u)$ and the continuity of $(u, v) \mapsto G(\Phi u, \Phi v)$ and $u \mapsto \|u\|_V^p$, we obtain that $u \mapsto \widehat{\mathcal{D}}_\kappa^f(u, v)$ is lower semicontinuous for all $v \in C$.

By the definition of $\mathcal{D}_\kappa^f$, we have

$$\mathcal{D}_\kappa^f(u) = \sup_{v \in C} \widehat{\mathcal{D}}_\kappa(u, v) \quad \text{for all} \quad u \in C.$$

Let $\{u_n\} \subset C$ be such that $u_n \to u_0$ as $n \to \infty$. Then,

$$\begin{aligned} \liminf_{n \to \infty} \mathcal{D}_\kappa^f(u_n) &= \liminf_{n \to \infty} \sup_{v \in K} \widehat{\mathcal{D}}_\kappa(u_n, v) \\ &\geq \liminf_{n \to \infty} \widehat{\mathcal{D}}_\kappa(u_n, z) \geq \widehat{\mathcal{D}}_\kappa(u_0, z) \end{aligned}$$

for all $z \in C$. Passing to supremum with $z \in C$ for the above inequality, it gives

$$\liminf_{n \to \infty} \mathcal{D}_\kappa^f(u_n) \geq \sup_{z \in C} \widehat{\mathcal{D}}_\kappa(u_0, z) = \mathcal{D}_\kappa^f(u_0),$$

so, the function $\mathcal{D}_\kappa^f$ is lower semicontinuous. This completes the proof. □

Let κ, $\rho > 0$ be two parameters. We now consider the following function $\mathcal{M}_{\mathcal{D}_\kappa^f, \rho}$ defined by

$$\mathcal{M}_{\mathcal{D}_\kappa^f, \rho}(u) = \inf_{z \in C} \left\{ \mathcal{D}_\kappa^f(z) + \rho \|u - z\|_V^p \right\}$$

for all $u \in C$.

Next, we will show that the function $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}$ is a gap function for Problem 2.1. Then, $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}$ is called the Moreau-Yosida regularized gap function for Problem 2.1.

Theorem 3.2. *Suppose that the hypotheses of Lemma 3.1 are satisfied. Then, for all κ, $\rho > 0$, the function $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}$ is a gap function for Problem 2.1.*

Proof. (a) For any κ, $\rho > 0$ fixed, since $\mathcal{D}^f_\kappa$ is a gap function for Problem 2.1, $\mathcal{D}^f_\kappa(u) \geq 0$ for all $u \in C$. Thus, $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}(u) \geq 0$ for all $u \in C$.

(b) Let $u^* \in C$ be a solution to Problem 2.1. Theorem 3.1 indicates that $\mathcal{D}^f_\kappa(u^*) = 0$. Then, we have

$$\begin{aligned}\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}(u^*) &= \inf_{z\in C}\left\{\mathcal{D}^f_\kappa(z) + \rho\|u^* - z\|^p_V\right\}\\ &\leq \mathcal{D}^f_\kappa(u^*) + \rho\|u^* - u^*\|^p_V = 0.\end{aligned}$$

Combining the above estimation with the fact that $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}(u^*) \geq 0$, one obtain $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}(u^*) = 0$.

Conversely, let $u^* \in C$ be such that $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}(u^*) = 0$, i.e.,

$$\inf_{z\in C}\left\{\mathcal{D}^f_\kappa(z) + \rho\|u^* - z\|^p_V\right\} = 0.$$

Then, there exists a minimizing sequence $\{z_n\} \subset C$ such that

$$0 \leq \mathcal{D}^f_\kappa(z_n) + \rho\|u^* - z_n\|^p_V < \frac{1}{n}. \tag{2}$$

Hence, (2) leads to $\mathcal{D}^f_\kappa(z_n) \to 0$ and $\|u^* - z_n\|_V \to 0$, as $n \to \infty$. So $z_n \to u^*$, as $n \to +\infty$. Thanks to Lemma 3.1 and the nonnegativity of $\mathcal{D}^f_\kappa$, we arrive at

$$0 \leq \mathcal{D}^f_\kappa(u^*) \leq \liminf_{n\to+\infty} \mathcal{D}^f_\kappa(z_n) = 0,$$

that is, $\mathcal{D}^f_\kappa(u^*) = 0$. Using the properties of the gap function $\mathcal{D}^f_\kappa$, we conclude that u^* is a solution to Problem 2.1. The proof is complete. □

Remark 3.1. (i) When $p = 2$ and $a_i = 1$ for all $i \in \{1, \dots, k\}$, our regularized gap function $\mathcal{D}^f_\kappa$ reduces to the corresponding regularized gap function considered in Tam-Chen [20]. Hence, Theorem 3.1 is an extension of [20, Theorem 3.5]. However, the method of proof in Theorem 3.1 for the regularized gap function $\mathcal{D}^f_\kappa$ is different from the corresponding result in [20, Theorem 3.5] without using the formulation of the optimality condition.

(ii) Furthermore, for Problem 2.1, the Moreau-Yosida regularized gap function $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}$ has not been studied in previous literature. Thus, our result in Theorem 3.2 is new even in the case of $p = 2$ and $a_i = 1$ for all $i \in \{1, \dots, k\}$.

4. Applications to global error bounds

Using the regularized gap functions $\mathcal{D}^f_\kappa$ and $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}$ established in Section 3, this section devotes to global error bounds for Problem 2.1 under assumptions of general order $p > 1$ on the data of Problem 2.1.

Theorem 4.1. *Let $u^* \in C$ be the unique solution to Problem 2.1. Suppose that the hypotheses $\mathbf{A}(F)$, $\mathbf{A}(G)$, $\mathbf{A}(H)$, $\mathbf{A}(C)$, $\mathbf{A}(\Phi)$, $\mathbf{A}(\Psi)$ and $\mathbf{A}(\varphi)$ hold. Then, for each $u \in C$, for any $\kappa > 0$ satisfying*

$$m_F - m_G\|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i}\|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p} > 0,$$

where $a_i > 0$ for all $i \in \{1, \ldots, k\}$, we have

$$\|u - u^*\|_V \leq \left(\frac{\mathcal{D}^f_\kappa(u)}{m_F - m_G\|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i}\|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p}} \right)^{\frac{1}{p}}. \tag{3}$$

Proof. Let $u^* \in C$ be the unique solution to Problem 2.1. For any $u \in C$ fixed, by the definition of $\mathcal{D}^f_\kappa$, we have

$$\begin{aligned} \mathcal{D}^f_\kappa(u) \geq \langle F(u) - \varphi, u - u^*\rangle_V - G(\Phi u, \Phi u^*) + G(\Phi u, \Phi u) \\ - \sum_{i=1}^{k} a_i H_i^0(\Psi_i u; \Psi_i u^* - \Psi_i u) - \frac{\kappa}{p}\|u - u^*\|^p_V. \end{aligned} \tag{4}$$

By assumption $\mathbf{A}(F)$(b), we get

$$\begin{aligned} \langle F(u) - \varphi, u - u^*\rangle_V &- \langle F(u^*) - \varphi, u - u^*\rangle_V \\ &= \langle F(u) - F(u^*) - \varphi, u - u^*\rangle_V \\ &\geq m_F\|u - u^*\|^p_V. \end{aligned} \tag{5}$$

Thanks to the conditions $\mathbf{A}(H)$(b), $\mathbf{A}(\Psi)$ and $a_i > 0$ for all $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned} a_i &\left[H_i^0(\Psi_i u; \Psi_i u^* - \Psi_i u) + H_i^0(\Psi_i u^*; \Psi_i u - \Psi_i u^*)\right] \\ &\leq a_i m_{H_i}\|\Psi_i u^* - \Psi_i u\|^p_{V_{H_i}} \\ &\leq a_i m_{H_i}\|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})}\|u - u^*\|^p_V. \end{aligned}$$

This implies that

$$\begin{aligned} &- \sum_{i=1}^{k} a_i H_i^0(\Psi_i u; \Psi_i u^* - \Psi_i u) - \sum_{i=1}^{k} a_i H_i^0(\Psi_i u^*; \Psi_i u - \Psi_i u^*) \\ &\geq - \sum_{i=1}^{k} a_i m_{H_i}\|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})}\|u - u^*\|^p_V. \end{aligned} \tag{6}$$

It follows from the conditions $\mathbf{A}(G)$(b) and $\mathbf{A}(\Phi)$ that

$$\begin{aligned} &- [G(\Phi u, \Phi u^*) - G(\Phi u, \Phi u) + G(\Phi u^*, \Phi u^*) - G(\Phi u^*, \Phi u)] \\ &\geq -m_G\|\Phi u^* - \Phi u\|^p_{V_G} \\ &\geq -m_G\|\Phi\|^p_{\mathcal{L}(V,V_G)}\|u - u^*\|^p_V. \end{aligned} \tag{7}$$

Moreover, since u^* is a solution to Problem 2.1, we have

$$\langle F(u^*) - \varphi, u - u^* \rangle_V + G(\Phi u^*, \Phi u) - G(\Phi u^*, \Phi u^*) + \sum_{i=1}^{k} a_i H_i^0(\Psi_i u^*; \Psi_i u - \Psi_i u^*) \geq 0. \tag{8}$$

Combining relations (5)–(8), it follows that

$$\langle F(u) - \varphi, u - u^* \rangle_V - G(\Phi u, \Phi u^*) + G(\Phi u, \Phi u) - \sum_{i=1}^{k} a_i H_i^0(\Psi_i u; \Psi_i u^* - \Psi_i u)$$
$$\geq \left(m_F - m_G \|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i} \|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} \right) \|u - u^*\|^p_V. \tag{9}$$

From (4) and (9), one has

$$\mathcal{D}^f_\kappa(u) \geq \left(m_F - m_G \|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i} \|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p} \right) \left\|u - u^*\right\|^p_V. \tag{10}$$

which implies

$$\left\|u - u^*\right\|^p_V \leq \left(\frac{\mathcal{D}^f_\kappa(u)}{m_F - m_G \|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i} \|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p}} \right)^{\frac{1}{p}}.$$

Thus, inequality (3) is valid. The proof is complete. □

Theorem 4.2. *Let $u^* \in C$ be the unique solution to Problem 2.1, and $\kappa > 0$ be such that*

$$m_F - m_G \|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i} \|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p} > 0,$$

where $a_i > 0$ for all $i \in \{1, \ldots, k\}$. Assume that the conditions of Theorem 3.2 hold. Then, for each $u \in C$ and all $\rho > 0$, one has

$$\|u - u^*\|_V \leq \left(\frac{2^{p-1} \mathcal{M}_{\mathcal{D}^f_\kappa, \rho}(u)}{\min \left\{ m_F - m_G \|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i} \|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p}, \rho \right\}} \right)^{\frac{1}{p}}. \tag{11}$$

Proof. Let $u^* \in C$ be the unique solution to Problem 2.1. For any $u \in C$ fixed, it follows from the definition of $\mathcal{M}_{\mathcal{D}^f_\kappa,\rho}$ and inequality (10) that

$$\begin{aligned}
& \mathcal{M}_{\mathcal{D}^f_\kappa,\rho}(u) \\
= \; & \inf_{z \in C} \left\{ \mathcal{D}^f_\kappa(z) + \rho \|u - z\|^p_V \right\} \\
\geq \; & \inf_{z \in C} \left\{ \left(m_F - m_G \|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i} \|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p} \right) \left\|z - u^*\right\|^p_V + \rho \|u - z\|^p_V \right\} \\
\geq \; & \min \left\{ m_F - m_G \|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i} \|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p}, \rho \right\} \times \\
& \inf_{z \in C} \left\{ \|z - u^*\|^p_V + \|u - z\|^p_V \right\}
\end{aligned}$$

Applying the following inequality

$$a^p + b^p \geq \frac{1}{2^{p-1}}(a+b)^p, \quad \forall a, b \geq 0, \ p \geq 1$$

leads to

$$\|z - u^*\|_V^p + \|u - z\|_V^p \geq \frac{1}{2^{p-1}}(\|z - u^*\|_V + \|u - z\|_V)^p \geq \frac{1}{2^{p-1}}\|u - u^*\|_V^p.$$

Hence, we have

$$\mathcal{M}_{\mathcal{D}^f_{\kappa},\rho}(u) \geq \frac{1}{2^{p-1}} \min\left\{ m_F - m_G\|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i}\|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p}, \rho \right\} \|u - u^*\|_V^p,$$

and so

$$\|u - u^*\|_V \leq \left(\frac{2^{p-1}\mathcal{M}_{\mathcal{D}^f_{\kappa},\rho}(u)}{\min\left\{ m_F - m_G\|\Phi\|^p_{\mathcal{L}(V,V_G)} - \sum_{i=1}^{k} a_i m_{H_i}\|\Psi_i\|^p_{\mathcal{L}(V,V_{H_i})} - \frac{\kappa}{p}, \rho \right\}} \right)^{\frac{1}{p}}$$

for all $u \in C$. The proof is complete. □

Remark 4.1. (i) In view of Remark 3.1(i), Theorem 4.1 extends the corresponding error bound established in [20, Theorem 4.3] under the assumption of general order $p > 1$ on the data of Problem 2.1.

(ii) Thanks to Remark 3.1(ii), the error bound results for Problem 2.1 in Theorem 4.2 with respect to the Moreau-Yosida regularized gap function $\mathcal{M}_{\mathcal{D}^f_{\kappa},\rho}$ is new.

5. Conclusions

The paper extends regularized gap functions and error bounds to a class of p-strongly monotone variational–hemivariational inequalities by employing the sum rule of Clarke's generalized directional derivatives (Problem 2.1). The novelty of these results include the construction of regularized gap functions for Problem 2.1 in the forms of the Fukushima and Moreau-Yosida types (Theorems 3.1 and 3.2). Furthermore, our error bound results for Problem 2.1 are established under strongly monotone assumptions of a general order $p > 1$ and regularized gap functions (Theorems 4.1 and 4.2).

Acknowledgement. The authors are grateful to the anonymous referees for their valuable remarks which improved the presentation of the paper.

References

[1] Auslender, A., (1976), Optimisation: Méthodes Numériques, Masson, Paris.
[2] Clarke, F. H., (1983), Optimization and Nonsmooth Analysis, Wiley, New York.
[3] Fukushima, M., (1992), Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, Math. Program., 53, 99–110.
[4] Gibali, A. and Salahuddin., (2021), Error bounds and gap functions for various variational type problems. RACSAM., 115, 123.
[5] Han, W., Huang, Z., Wang, C. and Xu, W., (2019), Numerical analysis of elliptic hemivariational inequalities for semipermeable media. J. Comput. Math., 37, 545–562.
[6] Han, W. and Nashed, M. Z., (2023), On variational-hemivariational inequalities in Banach spaces, Nonlinear Science and Numerical Simulation, 124, 107309.
[7] Han, W., Sofonea, M. and Danan, D., (2018), Numerical analysis of stationary variational–hemivariational inequalities. Numer. Math., 139, 563–592.

[8] Hung, N. V., Migórski, S., Tam, V. M. and Zeng, S. D., (2020), Gap functions and error bounds for variational-hemivariational inequalities. Acta. Appl. Math., 169, 691–709.

[9] Hung, N. V., Novo, V. and Tam, V. M., (2022), Error bound analysis for vector equilibrium problems with partial order provided by a polyhedral cone. J Glob Optim., 82, 139–159.

[10] Hung, N. V. and Tam, V. M., (2021), Error bound analysis of the D-gap functions for a class of elliptic variational inequalities with applications to frictional contact mechanics. Z Angew Math Phys., 72:Art, 173.

[11] Hung, N. V., Tam, V. M., Tuan, N. H. and O'Regan, D., (2020), Regularized gap functions and error bounds for generalized mixed weak vector quasi variational inequality problems in fuzzy environments. Fuzzy Sets Syst., 400, 162–176.

[12] Hung, N. V., Tam, V. M. and Pitea, A., (2020), Global error bounds for mixed Quasi-Hemivariational inequality problems on Hadamard manifolds, Optimization, 69, 2033–2052.

[13] Hung, N. V, Li, L., Migórski, S. and Tam, V. M., (2023), Well-posedness and global error bound for generalized mixed quasi-variational-hemivariational inequalities via regularized gap functions, Optimization, 24p, in press.

[14] Mastroeni, G., (2023), Gap functions for equilibrium problems. J Glob Optim., 27, 411–426.

[15] Matei, A., Micu, S. and Niţă C., (2018), Optimal control for antiplane frictional contact problems involving nonlinear elastic materials of Hencky type. Math Mech Solids., 23, 308–328.

[16] Migórski, S., Ochal, A. and Sofonea, M., (2013), Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems. Advances in Mechanics and Mathematics, 26, Springer, New York.

[17] Panagiotopoulos, P. D., (1993), Hemivariational Inequalities, Applications in Mechanics and Engineering. Springer-Verlag, Berlin.

[18] Panagiotopoulos, P. D., (1985), Nonconvex problems of semipermeable media and related topics. Z. Angew. Math. Mech. (ZAMM), 65, 29–36.

[19] Tam, V. M., (2022), Upper-bound error estimates for double phase obstacle problems with Clarke's subdifferential. Numer. Funct. Anal. Optim., 43(4), 463–485.

[20] Tam, V. M. and Chen, J.-S., (2024), On the D-gap functions for variational-hemivariational inequalities with an application to contact mechanics, Pacific J. Optim, 18 pages, DOI: 10.61208/pjo-2023-033.

[21] Yamashita, N. and Fukushima, M., (1997), Equivalent unconstrained minimization and global error bounds for variational inequality problems. SIAM J. Control Optim., 35, 273–284.

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