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PROBABILISTIC INSIGHTS INTO SPATIO-TEMPORAL GARCH MODEL WITH LOCATION-DEPENDENT PARAMETERS

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ABSTRACT. Spatial data analysis is a burgeoning field with applications ranging across ecology, finance and environmental studies, among others. Particularly, the intersection of space and time in phenomena such as environmental dynamics, energy economics and urban development necessitates sophisticated modeling techniques. In this paper, we propose a novel spatio-temporal GARCH model, extending traditional GARCH models to incorporate spatial dependencies alongside temporal dynamics. Building upon prior research, we introduce stationarity conditions and explore the asymptotic normality of the process. We also demonstrate that the process can be effectively approximated by a stationary process at a fixed location, providing valuable insights into localized behavior.

Keywords: Spatio-temporal GARCH model, stationarity conditions, asymptotic normality, spatial local stationarity.

AMS Subject Classification: 60G10, 62M30

1. INTRODUCTION

Spatial data are currently a dynamic and widely researched area, with their applications spanning various domains such as ecology, finance and environmental phenomena. In such cases, the relationship between space and time is captured and data are observed in regular or irregular geographical areas over time.

This gives rise to the concept of spatio-temporal data, which require the application of various modeling techniques to accurately predict and forecast these phenomena ([1]). The

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modelling of such spatio-temporal data is a challenging task, requiring the investigation of more sophisticated models to deal with complex non-linear, non-stationary and non Gaussian spatio-temporal real data. Pioneering earlier efforts include the works of [14], [19, 20], [10] and [21], among others. Notably, some of the popular models used in analyzing spatio-temporal data and understanding the intricate dynamics between space and time are the univariate spatio-temporal GARCH models.

In temporal data, GARCH models or the Generalized AutoRegressive Conditional Heteroskedasticity models are univariate statistical models that have become increasingly important and popular nowadays due to their simplicity and effectiveness in modeling temporal dependence. The main character of these models is that the volatility features change over time. They are used especially in analyzing financial and macroeconomic data in a very flexible way besides providing good in-sample estimations ([2]). Moreover, they are characterized by their incredible and incontestable popularity results in an array of alternative volatility models which fits financial data with stylized facts better. Indeed, the exponential GARCH and the Glosten- Jagannathan-Runkle GARCH models arise to take into account unconditional shocks in volatility and asymmetry effects. Integrated GARCH (IGARCH), fractionally IGARCH (FI- GARCH) and their different variants are long memory extensions of GARCH models that consider long memory and structural breaks simultaneously. In addition, to take into account an eventual presence of the structural breaks, [15] introduced regime switching GARCH models that have drawn considerable attention recently in several fields of science such as Financial, population dynamics, markets, traffic modeling, speech recognition and river flow analysis ([23], [13], [3], [31], [22] and [9]).

Looking at the abundant literature in prior studies on the GARCH-type models, it can be seen that all of these models consider volatility effects only from a temporal point of view. In this work, we aim to enrich the analysis by adding a dependence over space which means both time and location are not fixed. Hence, we introduce a novel spatio-temporal GARCH model by extending temporal dependence to the space framework in the same way as earlier works such as those of [27] and [4].

Several spatio-temporal extensions of the univariate GARCH models have been proposed to deal with the intricate interplay between space and time. [24] characterized a spatial ARCH model incorporating heteroscedasticity variance based on neighboring locations, applicable in spatiotemporal contexts. [28, 29, 30] introduced a process incorporating parts of GARCH and exponential GARCH (E-GARCH) processes. In addition, [16] proposed a novel circular spatio-temporal GARCH model with a translation invariant neighbour system. They gave some stationarity conditions and they investigated the spatio-temporal structure of the correlation function by an ARMA representation. A further work on GARCH models in the space-time domain, with a unified framework, has been proposed by [25] who combined pure spatial GARCH with the GARCH time series processes previously proposed. Their approach covered spatial and spatio-temporal models by considering the temporal dimension as one of the dimensions of the locations in the indexing set.

All the previous proposed spatio-temporal GARCH models are constructed with spatial stationarity assumption. However, many factors can lead to spatially non-stationarity such as the multi-directional nature of the space, therefore, in the present study we define a spatio-temporal GARCH model according to its time dynamics.That is, for particular observations at a fixed location, they can be considered as a GARCH time series. For observations at a given time, we have observations over various locations. Such an approach has been used in various applications using the location-dependent AR process to model, for instance, ground ozone and house price data (see [12] and [11] respectively).

This paper is organized as follows: In Section 2, we establish the foundation of our study by defining the model under consideration. We delve into the intricacies of the model, examining meticulously its properties and behavior. A key focus of this section is the exploration of stationarity conditions which is crucial for understanding the dynamics of the process over time. Additionally, we embark on a comprehensive investigation into the asymptotic normality of the process, shedding light on its long-term behavior and stability. In Section 3, we present a pivotal finding, where we extend the method used by [26] for varying-time GARCH models to demonstrate that the spatiotemporal process can be effectively approximated by a stationary process at a specific location s_0. This approximation not only simplifies the analysis, but also provides valuable insights into the localized behavior of the process and helps to better understand its underlying mechanisms.

In all the sequel , $\|.\|_\infty$ denotes the sup-norm of a vector, so that, for any $m \times m$ matrix $E = [e_{ij}]$, $\|E\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{m} \mid e_{ij} \mid$. If A, E and D are matrices such that E is diagonal with all diagonal elements equal in modulus to $\rho(A)$ (the spectral radius of A) and $\rho(D) \leq \rho(E)$, the $p \times p$ matrix A is said to be spectral radius diagonalizable if it is similar to the matrix $\begin{bmatrix} E & 0 \\ 0 & D \end{bmatrix}$. Then, there exists; for A, a $p \times p$ nonsingular matrix P such that $\|P^{-1}AP\|_\infty = \rho_0$ where $\rho_0 = \rho(A)$. Moreover, $C_d = C\big([0,1]^2, \mathbb{R}^d\big)$ denotes the collection of all continuous functions $f : [0,1]^2 \to \mathbb{R}^d$, equipped with the metric $d_{C_d}(f,g) = \sup_{t\in[0,1]^2} \mid f(t) - g(t) \mid$, for $f, g \in C_d$.

2. Model Definition and Properties

2.1. **Model Definition.** Let $s = (x,y) \in [0,1]^2$ be spatial coordinates. The proposed STGARCH(p,q) process $\{X_t(s) \ , s \in [0,1]^2\}$ is given by

$$\begin{cases} X_t(s) = \sigma_t(s)\epsilon_t(s) \\ \sigma_t^2(s) = \omega_0(s) + \sum_{i=1}^{q} \alpha_i(s)X_{t-i}^2(s) + \sum_{j=1}^{p} \beta_j(s)\sigma_{t-j}^2(s) \ , \end{cases} \tag{1}$$

where $X_t(s)$, $\sigma_t(s)$ and $\epsilon_t(s)$ are respectively the modelled process, the volatility process and the residual process, and $\omega_0(s)$, $\alpha_i(s), i = 1,\dots,q$ and $\beta_j(s), j = 1,\dots,p$ are nonparametric functions. In (1) the innovations $\epsilon_t(s)$ are :

- Independent over time and spatially, strictly stationary.
- Strictly stationary and ergodic over time, with $E(\epsilon_t(s)) = 0$ and $var(\epsilon_t(s)) = 1$.
- Independent of $\sigma_t(s)$.

We notice that for the process $\{X_t(s) \ , s \in [0,1]^2\}$ solving the equation (1) the process $\sigma_t^2(s)$ can be considered as a sub-diagonal STBL(p,0,1,q) process having the following representation

$$\sigma_t^2(s) = \omega_0(s) + \sum_{i=1}^{q} \alpha_i(s)\eta_{t+1-i}^2(s)\sigma_{t-i}^2(s) + \sum_{j=1}^{p} \beta_j(s)\sigma_{t-j}^2(s) \ , \tag{2}$$

where $\eta_t(s) = \epsilon_{t-1}(s)$. It is hence a special case of the location-dependent bilinear process defined by Equation (1.1) in [17]. We can apply Theorems 2 and 3 of [17] to obtain the time stationarity at each point in the spatial domain, For the proposed STGARCH(p,q) as well as a Central Limit Theorem (CTL). To use these results, we need to define the following matrices,

$$\underline{\sigma}_t^2(s) = \begin{bmatrix} \sigma_t^2(s) \\ \vdots \\ \sigma_{t-p+1}^2(s) \end{bmatrix}_{p\times 1}, \quad C(s) = \begin{bmatrix} \omega_0(s) \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{p\times 1},$$

$$B(s) = \begin{bmatrix} \beta_1(s) & \beta_2(s) & \cdots & \beta_p(s) \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}_{p\times p}.$$

Without loss of generality we suppose that $q < p$ and that $A_j(s), j = 1, \ldots, q$ are $p \times p$ matrices with all components equal zero except the component in the first row and jth column being equal to $\alpha_j(s)$, for instance,

$$A_2(s) = \begin{bmatrix} 0 & \alpha_2(s) & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}_{p\times p}.$$

So, the bilinear representation in Equation (2) can be written in the following equivalent matrix form

$$\underline{\sigma}_t^2(s) = C(s) + B(s)\underline{\sigma}_{t-1}^2(s) + \sum_{j=1}^{q} A_j(s)\epsilon_{t-j}^2(s)\underline{\sigma}_{t-1}^2(s). \tag{3}$$

2.2. **Stationarity.** Now, we can apply Theorem 2 of [17] to obtain sufficient conditions for time stationarity, at each point in the spatial domain, for the STGARCH(p,q) process defined by Equation (1).

Theorem 2.2.1. *Let $\{X_t(s),\ s \in [0,1]^2\}$ be as defined in the Equation (1), and let $C(s)$, $A_j(s)$ and $B(s)$ be the matrix coefficients of Equation (3). Assume that $B(s)$ is a spectral radius diagonalizable matrix with $\|P^{-1}(s)B(s)P(s)\|_\infty = \rho_0(s)$, and set $\delta_j(s) = \|P^{-1}(s)A_j(s)P(s)\|_\infty$.*
Assume further that

$$\sup_s E\log\left(\rho_0(s) + \sum_{j=1}^{q}\delta_j(s) \mid \epsilon_{t-j}^2 \mid\right) < 0.$$

Then, for all fixed s, there exists a unique strictly stationary (over time) process $X_t(s)$ satisfying (1), such that $\sigma_t^2(s)$ is given by the first component of $\underline{\sigma}_t^2(s)$ which have the following specification

$$\underline{\sigma}_t^2(s) = C(s) + \sum_{r=1}^{\infty}\prod_{k=0}^{r-1}\left(B(s) + \sum_{j=1}^{q} A_j(s)\epsilon_{t-j-k}^2(s)\right)C(s), \tag{4}$$

with the infinite series being almost surely convergent.

Now, let's provide an example to illustrate this theorem

Exemple 2.2.1. *Consider the STGARCH* $(1,1)$ *process*

$$\begin{cases} X_t(s) = \sigma_t(s)\epsilon_t(s)\,, \quad s=(x,y)\in[0,1]^2 \\ \sigma_t^2(s) = \omega(s) + a(s)\,X_{t-1}^2(s) + c(s)\,\sigma_{t-1}^2(s)\,, \end{cases}$$

where $a:[0,1]^2 \to [-1,1]$, $c:[0,1]^2 \to [-1,1]$ *and*

$$\begin{aligned} a(x,y) &= a_1 \sin x \cos y, \\ c(x,y) &= a_2(5-x)\cos y, \end{aligned}$$

and the innovations $\{\epsilon_t(s)\}$ *are independent over time and spatially, strictly stationary and ergodic over time, with* $E\left[\epsilon_t(s)\right]=0$ *and* $\operatorname{var}\left[\epsilon_t(s)\right]=1$. *In this case* $\underline{\sigma}_t^2(s)=\sigma_t^2(s)$, $B(s)=c(s)$, $C(s)=\omega(s)$ *and* $A_1(s)=a(s)$. *It is clear that* $A(s)$ *is a spectral radius diagonalizable* 1×1 *matrix with* $\rho_0(s) = \mid c(s)\mid$ *and* $\delta_1(s) = \mid a(s)\mid$. *The sufficient condition for the existence of a strictly stationary (over time) process* $\sigma_t^2(s)$ *is*

$$\sup_s E\log(\mid c(s)\mid + \mid a(s)\mid\mid \epsilon_1^2(s)\mid) < 0,$$

which becomes

$$\mid a_2\mid + 5\mid a_1\mid < 1.$$

Moreover, the solution is given by

$$\sigma_t^2(s) = \omega(s) + \omega(s)\sum_{r=1}^{\infty}\prod_{k=0}^{r-1}[a(s) + c(s)\,\epsilon_{t-1-k}^2(s)].$$

2.3. **Central limit theorem (CLT).** In the following, we investigate the asymptotic normality of the sample mean derived from our process. In the sequel, we shall assume that:

- (H1) $\epsilon_t(s)$ is an i.i.d sequence with $E\left(\epsilon_t(s)\right)=0$, $E\left(\epsilon_t(s)^2\right)=1$, and $E\left(\epsilon_t(s)^4\right)$ is finite. Moreover we set $E\left(\epsilon_t(s)^r\right)=\mu_r(s)$, for $r=3,4$.

- (H2) $\omega_0(s)$, $\{\alpha_i(s); i=1,\ldots,p\}$ and $\{\beta_i(s); i=1,\ldots,q\}$ are continuous functions.

- (H3) $\sup_s \rho(B(s)\otimes A(s)) = \rho_1 < 1$ and $\sup_s \rho(G(s)) = \rho_2 < 1$, where

$$G(s) = B(s)\otimes B(s) + \sum_{j=1}^{q} A_j(s)\otimes A_j(s).$$

First we put

$$\wp_{T,n} = \sum_{t=1}^{n}\sum_{u=1}^{T}\frac{\sigma_t^2(s_u)-\mu}{(nT)^{\frac{1}{2}}},$$

(H1) and (H3) allow us to apply the Chanda Central Limit Theorem ([8]) associated with bilinear time series on $\Sigma_n(s) = \sum_{t=1}^{n}\frac{\sigma_t^2(s)-\mu}{(n)^{\frac{1}{2}}}$, while we use (H2) to have $\Sigma_n(.)\in C_1$

Theorem 2.3.1. *Assume that the conditions (H1), (H2) and (H3) hold, then*

$$\mathcal{L}\left(\wp_{T,n}\right) \to \mathcal{N}\left(0,\eta^2\right),$$

as $n, T\to\infty$, *where*

$$\eta^2 = \lim_{T\to\infty}\frac{1}{T}\sum_{u=1}^{T}\left[\lim_{m\to\infty}\left\{E\sigma_{1,m}^4\left(s_u\right) + 2\sum_{v=1}^{m}\left|\sigma_{1,m}^2\left(s_u\right)\sigma_{1+v,m}^2\left(s_u\right)\right|\right\}\right].$$

3. The Approximated Spatially Stationary STGARCH(p,q) Process

In this section, we explore additional probabilistic aspects of our process. So, we demonstrate that the spatially nonstationary process $X_t(s)$ can be effectively approximated by a spatially stationary process $X_{t,s_0}(s)$. First, we note that the STGARCH(p,q) process defined in (1) admits the following presentation

$$\mathbf{X_t}(s) = \mathbf{A_t}(s)\mathbf{X_{t-1}}(s) + \mathbf{b_t}(s), \tag{5}$$

where

$$\mathbf{X_t}(s)^T = \left(\sigma_t^2(s), \ldots, \sigma_{t-q+1}^2(s), X_{t-1}^2(s), \ldots, X_{t-p+1}^2(s)\right),$$
$$\boldsymbol{b_t}(s)^T = (\omega_0(s), 0, \ldots, 0) \in \mathbb{R}^{p+q-2},$$
$$\mathbf{A_t}(s) = \begin{pmatrix} \Im_t(s) & \beta_q(s) & \underline{\alpha}(s) & \alpha_p(s) \\ \boldsymbol{I}_{q-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \underline{\epsilon}_{t-1}^2(s) & 0 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I}_{p-2} & \mathbf{0} \end{pmatrix}, \in \mathbb{R}^{(p+q-1)\times(p+q-1)}.$$

$\Im_t(s) = \left(\beta_1(s) + \alpha_1(s)\epsilon_{t-1}^2(s), \beta_2(s), \ldots, \beta_{q-1}(s)\right)$, $\underline{\alpha}(s) = (\alpha_2(s), \ldots, \alpha_{p-1}(s))$ and $\underline{\epsilon}_{t-1}^2(s) = \left(\epsilon_{t-1}^2(s), 0, \ldots, 0\right) \in \mathbb{R}^{q-1}$.
By iterating (5), we arrive at a result

$$\mathbf{X_t}(s) = \sum_{k=0}^{\infty} \prod_{l=0}^{k-1} \mathbf{A_{t-l}}(s)\mathbf{b_{t-k}}(s). \tag{6}$$

In the sequel, we need the following assumptions

Assumptions 2. The STGARCH(p,q) process satisfies the representation (5), and for all $s \in [0,1]^2$ the random matrices $\mathbf{A_t}(s)$ and the vector $\mathbf{b_t}(s)$ satisfy the next assumptions:

- For each $v \in \{1, \ldots, V\}$ where $V \in \mathbb{N}$, there is a sequence of a random matrix $\mathcal{A}(v)$, that are identically distributed, positive and such that
$$\sup_s \mid \mathbf{A_t}(s) \mid \leq \mathcal{A}_t(v), \text{ for all } s \in \left[\frac{v-1}{V}, \frac{v}{V}\right)^2.$$
In addition $E(\|\mathcal{A}_t(v)\|_\infty^\lambda) < \infty$ and for some $\lambda < 0$, the Lyapunov exponent of those matrices is less than λ .
- There is a stationary sequence b_t such as
$$\sup_s \mid \mathbf{b_t}(s) \mid < b_t.$$
- For some $\lambda' > 0$; $E(\|\mathcal{A}_t\|_\infty^{\lambda'}) < \infty$ and $E(\|b_t\|_\infty^{\lambda'}) < \infty$.
- For all s , $v \in [0,1]^2$
$$\mid \mathbf{A_t}(s) - \mathbf{A_t}(s') \mid \leq C \mid s - s' \mid \mathcal{A}_t, \text{ and } \quad E[\mathcal{A}_t] < \infty.$$
$$\mid \dot{\mathbf{A}}_\mathbf{t}(s) - \dot{\mathbf{A}}_\mathbf{t}(s') \leq C \mid s - s' \mid^2 \mathcal{A}_t, \text{ with } \sup_s \mid \dot{\mathbf{A}}_\mathbf{t}(s) \mid \leq C\mathcal{A}_t(v).$$
$$\mid \mathbf{b_t}(s) - \mathbf{b_t}(s') \mid \leq C \mid s - s' \mid b_t.$$
$$\mid \dot{\mathbf{b}}_\mathbf{t}(s) - \dot{\mathbf{b}}_\mathbf{t}(s') \mid \leq C \mid s - s' \mid^2 b_t, \quad \text{with } \sup_s \mid \dot{\mathbf{b}}_\mathbf{t}(s) \mid \leq Cb_t.$$
This is a kind of Holder continuity of order $(1+2)$ for random matrices. This assumption allows us to use Taylor expansion for approximating the spatio-temporal

GARCH model, aligning with the methodology that [26] employed for varying-time GARCH models.
To simplify matters, we assume that for $k \geq 1$, $\prod_{j=0}^{-k} \mathcal{A}_j = I$ where I is the identity matrix.

In the following theorem, we aim to demonstrate that the spatially nonstationary process $X_t^2(s)$ could be approximated by a spatially stationary process $X_{t,s_0}^2(s)$ in a fixed location s_0, where

$$\begin{cases} X_{t,s_0}(s) = \sigma_{t,s_0}(s)\epsilon_t(s) \\ \sigma_{t,s_0}^2(s) = \omega_0(s_0) + \sum_{i=1}^{q} \alpha_i(s_0) X_{t-i,s_0}^2(s) + \sum_{j=1}^{p} \beta_j(s_0)\sigma_{t-j,s_0}^2(s), \end{cases}$$

Theorem 3.1. *Let $X_t^2(s)$ be a STGARCH(p,q) model that satisfies the representation* (1), (6). *We define W_t as follows*

$$W_t = \sum_{k=0}^{\infty} \sum_{v=1}^{V} \prod_{l=0}^{k-1} \mathcal{A}_{t-l}(v) b_{t-k}.$$

If Assumptions 2 *hold and if for some $\lambda' > 0$,*

$$\sup_s E \mid X_t^2(s) \mid^{\lambda'} < \infty, \qquad \sup_s E \mid W_t \mid^{\lambda'} < \infty,$$

Then, we have

$$|X_t^2(s) - X_{t,s_0}^2(s)| \leq \|s - s_0\|_\infty N_t,$$

$$N_t = C \sum_{k=0}^{\infty} \sum_{v=1}^{V} \prod_{l=0}^{k-1} \mathcal{A}_{t-l}(v)(\mathcal{A}_{t-k} W_{t-k-1} + b_{t-k}).$$

N_t is stationary process in both space and time.

The fact that N_t is a stationary process in both space and time is supported by a non-stationary extension of [6] (Theorem 1) and [5] (Theorem 2.5), combined with the almost sure convergence of N_t.
To refine the previous approximation, we begin by introducing the concept of a derivative spatial process. This derivative spatial process corresponds to the derivative of the spatially stationary process $X_{t,s_0}(s)$ with respect to s_0 and it is given by :

$$\dot{\mathbf{X}}_\mathbf{t}(s) = \dot{\mathbf{A}}_\mathbf{t}(s)\mathbf{X}_{\mathbf{t-1}}(s) + \mathbf{A}_\mathbf{t}(s)\dot{\mathbf{X}}_{\mathbf{t-1}}(s) + \dot{\mathbf{b}}_\mathbf{t}(s). \tag{7}$$

Let

$$\mathbf{X}_{\mathbf{t,2,s_0}}(s)^T = \big(\frac{\partial \mathbf{X}_{\mathbf{t,s_0}}(s)}{\partial x_0}, \mathbf{X}_{\mathbf{t,s_0}}(s)\big), \;\; \mathbf{b}_{\mathbf{t,2}}(s_0)^T = \big(\partial_{x_0}\mathbf{b}_\mathbf{t}(s_0), \mathbf{b}_\mathbf{t}(s_0)\big).$$

Let also

$$\mathbf{b}_{\mathbf{t,2}} = \begin{pmatrix} C\mathbf{b_t}^T \\ \mathbf{b_t}^T \end{pmatrix} \qquad \mathcal{A}_{t,2}(v) = \begin{pmatrix} \mathcal{A}_t(v) & C\mathcal{A}_t(v) \\ 0 & \mathcal{A}_t(v) \end{pmatrix} \text{ and } \mathcal{A}_{t,2} = \begin{pmatrix} \mathcal{A}_t & C\mathcal{A}_t \\ 0 & \mathcal{A}_t \end{pmatrix}.$$

We suppose that $\mathbf{A}_{\mathbf{t},\mathbf{2}}(\mathbf{s})$ and $\mathcal{A}_{t,2}(v)$ have negative Lyapunov exponent.
Since, $\mid \mathbf{A}_{\mathbf{t},\mathbf{2}}(\mathbf{s}) \mid < \mathcal{A}_{t,2}(v)$ and $\sup_s \mid \mathbf{b}_{\mathbf{t},\mathbf{2}}(\mathbf{s}) \mid < \mathbf{b}_{\mathbf{t},\mathbf{2}}$, we can rewrite (7) as follows:

$$\begin{pmatrix} \frac{\partial \mathbf{X_t}(s)}{\partial s} \\ \mathbf{X_t}(s) \end{pmatrix} = \mathbf{A}_{\mathbf{t},\mathbf{2}}(\mathbf{s}) \begin{pmatrix} \frac{\partial \mathbf{X_{t-1}(s)}}{\partial s} \\ \mathbf{X_{t-1}}(s) \end{pmatrix} + \begin{pmatrix} \partial_s \mathbf{b_t}(s) \\ \mathbf{b_t}(s) \end{pmatrix}, \tag{8}$$

where

$$\mathbf{A_{t,2}(s)} = \begin{pmatrix} \mathbf{A_t(s)} & \partial_s \mathbf{A_t(s)} \\ 0 & \mathbf{A_t(s)} \end{pmatrix}.$$

The primary purpose of defining the process outlined in equation (8) is to enable the estimation of the derivative process, in addition $\mathbf{X_{t,2}}(s)$ satisfies all the previous conditions. By adding week assumptions on the moments of $\mathbf{X_{t,2}(s)}$ we can apply directly Theorem 3.1 . Define

$$W_{t,2} = \sum_{k=0}^{\infty} \sum_{v=1}^{V} \prod_{l=0}^{k-1} \mathcal{A}_{t-l,2}(v) b_{t-k,2}.$$

Corollary 3.1. *If all the pervious assumptions hold and if for some $\lambda' > 0$,*

$$\sup_s E \mid X_{t,2}(s) \mid^{\lambda'} < \infty \quad \text{and} \quad \sup_s E \mid Y_{t,2} \mid^{\lambda'} < \infty,$$

then,

$$|X_{t,2} - X_{t,2,s_0}(s)| \leq \|s - s_0\|_\infty N_{t,2},$$

and $\| N_t \| \leq \| N_{t,2} \|$, with

$$N_{t,2} = C \sum_{k=0}^{\infty} \sum_{v=1}^{V} \prod_{l=0}^{k-1} \mathcal{A}_{t-l,2}(v) (\mathcal{A}_{t-k,2} W_{t-k-1,2} + b_{t-k,2}).$$

$N_{t,2}$ is a stationary process in both space and time.

By utilizing equation (7), we have

$$\begin{aligned}\dot{\mathbf{X}}_\mathbf{t}(s) &= \dot{\mathbf{A}}_\mathbf{t}(s)\Big(\mathbf{A_{t-1}}(s)\mathbf{X_{t-2}}(s) + \mathbf{b_{t-1}}(s)\Big) \\ &\quad + \mathbf{A_t}(s)\left(\dot{\mathbf{A}}_\mathbf{t-1}(s)\mathbf{X_{t-2}}(s) + \mathbf{A_{t-1}}(s)\dot{\mathbf{X}}_\mathbf{t-2}(s) + \dot{\mathbf{b}}_\mathbf{t-1}(s)\right) + \dot{\mathbf{b}}_\mathbf{t}(s),\end{aligned}$$

By iteration, we conclude that

$$\dot{\mathbf{X}}_\mathbf{t}(s) = \sum_{k=0}^{\infty}\sum_{v=0}^{k-1}\left[\prod_{j=0}^{v-1}\mathbf{A_{t-j}}(s)\right]\dot{\mathbf{A}}_\mathbf{t-v}(\mathbf{s})\left[\prod_{j=v+1}^{k-1}\mathbf{A_{t-j}}(s)\right]\mathbf{b_{t-k}}(s)$$

$$+\sum_{k=0}^{\infty}\left[\prod_{j=0}^{k-1}\mathbf{A_{t-j}}(s)\right]\dot{\mathbf{b}}_\mathbf{t-k}(s).$$

Now by corollary 3.1 and since that all the paths of $X_t^2(s)$ are almost surely Lipchitzian with order $(1+2)$, we can write the following Taylor expansion of $X_t^2(s)$ around s_0

$$X_t^2(s) = X_t^2(s_0) + (s - s_0)\,\dot{X}_{t,s_0}(s) + O_p\left(||s - s_0|\,|^2\right). \tag{9}$$

One notable characteristic, easily verifiable, is that the partial derivatives of $\mathbf{X_{t,s_0}(s)}$ concerning s_0 generally yield $\dot{\mathbf{X}}_\mathbf{t,s_0}(\mathbf{s})$. We employ this insight below, illustrating that the spatially nonstationary process $\mathbf{X_t(s)}$ can be approximated by a linear combination of two spatially stationary processes.

Theorem 3.2. *We suppose that $X_t^2(s)$ satisfies model* (1) *and Assumption* 2, *we have*

$$\left|X_t^2(s) - X_{t,s_0}^2(s) - (s - s_0)\,\dot{X}_{t,s_0}^2(s)\right| \leq \|s - s_0\|_\infty^2\, N_t.$$

We observe that Theorem 3.2 . is a representation of a spatially nonstationary process in terms of spatially stationary processes. Moreover, if s is in the proximity of s_0, then $X_t^2(s)$ can be approximated by $X^2_{t,s_0}(s) + (s - s_0)\dot{X}^2_{t,s_0}(s)$.

Let us introduce the process $\{\tilde{X}^2_{t,s_0}(s)\}_t$. Considering s_0 as a fixed location, we assume that the process $\{\tilde{X}^2_{t,u_0}(u)\}_t$ conforms to the provided representation:

$$\tilde{\mathbf{X}}_{\mathbf{t},\mathbf{s_0}}(s) = \tilde{\mathbf{A}}_{\mathbf{t},\mathbf{s_0}}(s)\tilde{\mathbf{X}}_{\mathbf{t-1},\mathbf{s_0}}(s) + \tilde{\mathbf{b}}_{\mathbf{t},\mathbf{s_0}}(\mathbf{s}),$$

where $\tilde{\mathbf{X}}_{t,s_0}(s)^T = (\tilde{\mathbf{X}}_{t,s_0}(s), \ldots, \tilde{\mathbf{X}}_{t-p+1,s_0}(s))$. The modified matrix $\tilde{\mathbf{A}}_{s_0}(s)$ is like $\mathbf{A_t}(\mathbf{s})$ but with each $\alpha_j(s)$ and $\beta_j(s)$ substituted by $\alpha_j(s_0) + (s - s_0)\dot{\alpha_j}(s_0)$ and $\beta_j(s_0) + (s - s_0)\dot{\beta_j}(s_0)$ respectively.
Additionally, $\tilde{\mathbf{b}}_{\mathbf{t},\mathbf{s_0}}(s)^T = (\omega_0(s_0) + (s - s_0)\dot{\omega_0}(s_0), 0, \ldots, 0)$. Consequently, it can be shown that the solution to $\tilde{\mathbf{X}}_{\mathbf{t},\mathbf{s_0}}(\mathbf{s})$ is almost surely given by

$$\tilde{\mathbf{X}}_{t,s_0}(s) = \sum_{k=0}^{\infty}\prod_{l=0}^{k-1}\tilde{\mathbf{A}}_{\mathbf{t-l},\mathbf{s_0}}(s)\tilde{\mathbf{b}}_{\mathbf{t-k},\mathbf{s_0}}(\mathbf{s}).$$

In the upcoming theorem, we demonstrate that $X_t^2(s)$ can also be represented closely by $\tilde{X}_{t,s_0}(s)^2$.

Theorem 3.3. *We suppose that $X_t^2(s)$ satisfies model* (1) *and Assumption* 2. *Assume that* $\sup_s \mid \tilde{\mathbf{A}}_{\mathbf{t}}(s) \mid \leq \tilde{\mathcal{A}}_t(v)$ *where the Lyapunov exponent of $\tilde{\mathcal{A}}_t(v)$ is less than certain $\tilde{\lambda} < 0$ and that* $\sup_s \mid \tilde{\mathbf{b}}_{\mathbf{t}}(s) \mid < \tilde{\mathbf{b}}_{\mathbf{t}}$. *Moreover , we assume that for some $\tilde{\lambda}' > 0$*

$$E(\|\tilde{\mathcal{A}}_t\|_\infty^{\tilde{\lambda}'}) < \infty \ \text{ and } E(\|\tilde{\mathbf{b}}_{\mathbf{t}}\|_\infty^{\tilde{\lambda}'}) < \infty.$$

Then,

$$|X_t^2(s) - \tilde{X}^2_{t,s_0}(s)| \leq \|s - s_0\|_\infty \tilde{N}_t,$$

where

$$\tilde{N}_t = C\sum_{k=0}^{\infty}\sum_{v=1}^{V}\prod_{l=0}^{k-1}\tilde{\mathcal{A}}_{t-l}(v)(\tilde{\mathcal{A}}_{t-k}\tilde{W}_{t-k-1} + \tilde{b}_{t-k}).$$

$\tilde{N}_t$ is a stationary process in both space and time.

Corollary 3.2. *We suppose that $X_t^2(s)$ satisfies model (1) and all the conditions of Theorem 3.3 . We have*

$$|\sigma_t^2(s) - \tilde{\sigma}^2_{t,s_0}(s)| \leq \|s - s_0\|_\infty \tilde{N}_{t,1},$$

where

$$\tilde{N}_{t,1} = C\sum_{k=0}^{\infty}\sum_{v=1}^{V}\prod_{l=0}^{k-1}\tilde{\mathcal{A}}_{t-l}(v)\left(\tilde{\mathcal{A}}_{t-k}\tilde{W}_{t-k-1} + \tilde{b}_{t-k}\right)\frac{1}{\epsilon_t(s)}.$$

$\tilde{N}_{t,1}$ is a stationary process in both space and time.

4. Conclusion

In this article, we introduced a novel spatio-temporal GARCH model that extends traditional univariate GARCH models to account for both spatial and temporal dependencies. By capturing the intricate dynamics between space and time, this model offers a more comprehensive framework for analyzing complex spatio-temporal data. Building on the foundational works of [26, 27] and others, we examined stationarity conditions and asymptotic normality. We also demonstrated how the spatio-temporal process can be approximated by a stationary process at a fixed location, simplifying the analysis. This work lays a solid foundation for future research and applications in several areas such as finance and environmental science.

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Appendix A. This appendix includes supplementary material that supports the main text. Specifically, it presents two results, which are crucial for the proof of Theorem 2.3.1. These results are available in [18] and [7] (Problem 6.16, p. 216).

Theorem A-1. *Suppose that $Y_n \Rightarrow Y_\infty$, (where $\Rightarrow$ denotes the weak convergence) as elements of C_d. Then, for any continuous functional Ψ on C_d, $\Psi(Y_n)$ converges in distribution to $\Psi(Y_\infty)$.*

Lemma A-1. *Suppose That $Y_n \sim \mathcal{N}(\mu_n, v_n)$ where $\mu_n \to \mu$, $v_n \to v$ and $0 < v < \infty$, so $Y_n \Rightarrow Y$, where $Y \sim \mathcal{N}(\mu, v)$.*

Appendix B. In this appendix, we give the proofs of our results.

Proof of Theorem 2.2.1. We suppose that there exists a strictly stationary process $\underline{\sigma}_t^2(s)$, satisfying (3), given by

$$\underline{\sigma}_t^2(s) = C(s) + B(s)\underline{\sigma}_{t-1}^2(s) + \sum_{j=1}^{q} A_j(s)\epsilon_{t-j}^2(s)\underline{\sigma}_{t-1}^2(s).$$

By iterating, the previous equation we get for all $n \geq 1$

$$\underline{\sigma}_t^2(s) = C(s) + \left(\sum_{r=1}^{n}\prod_{k=0}^{r-1} B(s) + \sum_{j=1}^{q} A_j(s)\epsilon_{t-j-k}^2(s)\right)C(s)$$
$$+ \prod_{k=0}^{n}\left(B(s) + \sum_{j=1}^{q} A_j(s)\epsilon_{t-j-k}^2(s)\right)\underline{\sigma}_{t-n-1}^2(s).$$

Put $H_{t,n}(s) = \prod_{k=0}^{n}\left(B(s) + \sum_{j=1}^{q} A_j(s)\epsilon_{t-j-k}^2(s)\right)\underline{\sigma}_{t-n-1}^2(s)$. And let ξ be an arbitrary $p \times 1$ real numbers vector, we have

$$\mid \xi' H_{t,n}(s) \mid \leq N\|P(s)\|_\infty \prod_{k=0}^{n}\left\|P^{-1}(s)\left(B(s) + \sum_{j=1}^{q} A_j(s)\epsilon_{t-j-k}^2(s)\right)P(s)\right\|_\infty \|P^{-1}(s)\|,$$

where $N = \|\xi'\|_\infty$ is a finite positive constant independent of n. As a result

$$\mid \xi' H_{t,n}(s) \mid \leq N \prod_{k=0}^{n}\left(\rho_0(s) + \sum_{j=1}^{q}\delta_j(s) \mid \epsilon_{t-j-k}^2(s) \mid \right),$$

and

$$\lim \frac{1}{n} \log \mid \xi' H_{t,n}(s) \mid \leq \lim \frac{1}{n} \log N + \lim \frac{1}{n} \sum_{k=0}^{n} \log \Big(\rho_0(s) + \sum_{j=1}^{q} \delta_j(s) \mid \epsilon^2_{t-j-k}(s) \mid \Big).$$

By the strong law of large numbers

$$\lim \frac{1}{n} \log \mid \xi' H_{t,n}(s) \mid \leq E \log \Big(\rho_0(s) + \sum_{j=1}^{q} \delta_j(s) \mid \epsilon^2_{t-j-k}(s) \mid \Big)$$

$$\leq \sup_s E \log \Big(\rho_0(s) + \sum_{j=1}^{q} \delta_j(s) \mid \epsilon^2_{t-j-k}(s) \mid \Big), \quad a.s.$$

which implies that $H_{t,n}(s) \to 0$ a.s. as $n \to \infty$.
Since $\underline{\sigma}^2_t(s)$ is strictly stationary then, for all $n \geq 1$, $\underline{\sigma}^2_{t-n-1}(s)$ have the same distribution. In addition $H_{t,n}(s) \to 0$ a.s. implies that

$$\prod_{k=0}^{n} \Big(B(s) + \sum_{j=1}^{q} A_j(s) \epsilon^2_{t-j-k}(s) \Big) \underline{\sigma}^2_{t-m-1}(s) \quad \to \quad 0\,,$$

in probability as $n \to \infty$. Hence

$$\underline{\sigma}^2_t(s) = C(s) + \Big(\sum_{r=1}^{\infty} \prod_{k=0}^{r-1} B(s) + \sum_{j=1}^{q} A_j(s) \epsilon^2_{t-j-k}(s) \Big) C(s).$$

Conversely, we suppose that $\underline{\sigma}^2_t(s)$ satisfies (4). We must first prove that for all fixed $t \in \mathbb{Z}$ and $s \in [0,1]^2$ the following series of random vectors converges almost surely

$$\sum_{r=1}^{\infty} \prod_{k=0}^{r-1} \Big(B(s) + \sum_{j=1}^{q} A_j(s) \epsilon^2_{t-j-k}(s) \Big) C(s).$$

We put

$$Z_{rt}(s) = \prod_{k=0}^{r-1} \Big(B(s) + \sum_{j=1}^{q} A_j(s) \epsilon^2_{t-j-k}(s) \Big) C(s).$$

So,

$$\mid Z_{rt}(s) \mid \leq N \prod_{k=0}^{r-1} \Big\| P^{-1}(s) \Big(B(s) + \sum_{j=1}^{q} A_j(s) \epsilon^2_{t-j-k}(s) \Big) P(s) \Big\|_{\infty},$$

and

$$\frac{1}{r} \log \mid Z_{rt}(s) \mid \leq \frac{1}{r} N + \frac{1}{r} \sum_{k=1}^{r} \log \Big(\rho_0(s) + \sum_{j=1}^{q} \delta_j(s) \epsilon^2_{t-j-k}(s) \Big).$$

Since,

$$\frac{1}{r} \sum_{k=1}^{r} \log \Big(\rho_0(s) + \sum_{j=1}^{q} \delta_j(s) \epsilon^2_{t-j-k}(s) \Big),$$

tends to

$$E \log \Big(\rho_0(s) + \sum_{j=1}^{q} \delta_j(s) \epsilon^2_{t-j-k}(s) \Big) \; a.s.,$$

when $r \to \infty$,
we conclude that for every t and s, $\limsup \mid Z_{rt}(s) \mid^{\frac{1}{r}} < 1$ a.s, which implies that $\sum_{r\geq 1} Z_{rt}(s)$ converges absolutely almost surely. As a result $\underline{\sigma}_t^2(s)$ is strictly stationary, ergodic and satisfies (3). □

Proof of Theorem 2.3.1. We consider the following sum

$$\Sigma_n(s) = \sum_{t=1}^{n} \frac{\sigma_t^2(s) - \mu}{(n)^{\frac{1}{2}}}.$$

By the Central Limits Theorem (Chanda 1989), we have for all fixed $s \in [0,1]^2$, $\mathcal{L}(\Sigma_n(s_u)) \to \mathcal{N}(0, \eta^2(s_u))$ as $n \to \infty$, where

$$\eta^2(s_u) = \lim_{m\to\infty} \left\{ E\sigma_{1,m}^4(s_u) + 2\sum_{v=1}^{m} \left|\sigma_{1,m}^2(s_u)\,\sigma_{1+v,m}^2(s_u)\right| \right\}.$$

Let Θ a continuous function whereas :

$$\Theta : \{[0,1]^2\}^T \to \mathbb{R}.$$

For each fixed $s_1 \dots s_T \in [0,1]^2$ and for all $g \in C_1$, we define $\Psi(g) = \Theta(g(s_1)\dots g(s_T)) = \frac{1}{T^{\frac{1}{2}}}\sum_{i=1}^{T} g(s_i)$, $\Psi(g)$ is a continuous functional .
By (H2) and (4) we deduce that $\Sigma_n(.) \in C_1$. Moreover, we have for $u = 1, \dots, T$, $\Sigma_n(s_u) \Rightarrow \Sigma_\infty(s_u)$, where $\Sigma_\infty(s_u) \sim \mathcal{N}(0, \eta^2(s_u))$, then, by using Theorem A-1 in Appendix A, $\Psi(\Sigma_n(s_u)) \Rightarrow \Psi(\Sigma_\infty(s_u))$. Thus

$$\mathcal{L}(\wp_{T,n}) \to \mathcal{L}\left(\frac{1}{T^{\frac{1}{2}}}\sum_{i=1}^{T} \Sigma_\infty(s_i)\right).$$

In addition, since $\epsilon_t(s)$ are independent over time and spatially, as a result $\Sigma_\infty(s_1), \dots,$ $\Sigma_\infty(s_T)$ are also independent.
Furthermore, $\Sigma_\infty(s_i) \to \mathcal{N}(0, \eta^2(s_i))$, for all $i = 1, \dots, T$, so,

$$\frac{1}{T^{\frac{1}{2}}}\sum_{i=1}^{T} \Sigma_\infty(s_i) \sim \mathcal{N}\left(0, \frac{1}{T}\left[\sum_{i=1}^{T} \eta^2(s_i)\right]\right).$$

Then, it is clear to see that $\frac{1}{T}\left[\sum_{i=1}^{T} \eta^2(s_i)\right]$ tends to the finite constant

$$\eta^2 = \lim_{T\to\infty} \frac{1}{T}\sum_{u=1}^{T}\left[\lim_{m\to\infty}\left\{ E\sigma_{1,m}^4(s_u) + 2\sum_{v=1}^{m} \left|\sigma_{1,m}^2(s_u)\,\sigma_{1+v,m}^2(s_u)\right| \right\}\right].$$

So, Lemma A-1 in Appendix A, entails that

$$\mathcal{L}(\wp_{T,n}) \to \mathcal{N}(0, \eta^2).$$

□

Proof of Theorem 3.1. We have

$$\begin{aligned}
|X_t^2(s) - X_{t,s_0}^2(s)| &\leq \|\, \mathbf{A_t}(s)\mathbf{X_{t-1}}(s) - \mathbf{A_t}(s_0)\mathbf{X_{t-1}}(s_0) \,\|_2 + C\|s - s_0\|_\infty b_t \\
&\leq \|\, \mathbf{A_t}(s) - \mathbf{A_t}(s_0) \,\|_2 \|\, \mathbf{X_{t-1}}(s) \,\|_2 \\
&+ \|\mathbf{X_{t-1}}(s) - \mathbf{X_{t-1}}(s_0)\|_2 \,\|\, \mathbf{A_t}(s_0) \,\|_2 + C\|s - s_0\|_\infty b_t \\
&\leq C \,\|\, s - s_0 \,\|_2\, \mathcal{A}_t \,\|\, \mathbf{X_{t-1}}(s) \,\|_2 + \|\mathbf{X_{t-1}}(s) - \mathbf{X_{t-1}}(s_0)\|_2\, \mathcal{A}_t(v) \\
&+ C\|s - s_0\|_\infty b_t.
\end{aligned}$$

By iteration, we find that

$$\mid X_t^2(s) - X_{t,s_0}^2(s) \mid \leq C \mid\mid s - s_0 \mid\mid_\infty \sum_{k=0}^{\infty} \sum_{v=1}^{V} \prod_{l=0}^{k-1} \mathcal{A}_{t-l}(v) \big(\mathcal{A}_{t-k} W_{t-k-1} + b_{t-k} \big),$$

where

$$W_t = \sum_{k=0}^{\infty} \sum_{v=1}^{V} \prod_{l=0}^{k-1} \mathcal{A}_{t-l}(v) b_{t-k}.$$

□

Proof of Corollary 3.1. This corollary follows immediately from Theorem 3.1 . □

Proof of Theorem 3.2. By using equation (9) we get

$$X_t^2(s) - X_t^2(s_0) = (s - s_0)\, \dot{X}_{t,s_0}(s) + R_t(s, s_0),$$

where

$$\mid R_t(s, s_0) \mid \leq ||s - s_0| \,|^2 N_t,$$

N_t being defined as in Theorem3.1. □

Proof of Theorem 3.3. This theorem can be proved in the same way as Theorem 3.1. □

Proof of Corollary 3.2. The proof of Corollary 3.2 follows immediately from Theorem 3.3 and the definition of $X_t^2(s)$. □

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