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ROOT SQUARE MEAN LABELING OF ARROW GRAPHS WITH ENCODING AND DECODING

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ABSTRACT. A graph G with p vertices and q edges is called a Root Square mean labeling if it is possible to label the vertices $x \in v$ with distinct labels $\rho(x)$ from $1, 2, \cdots, q+1$ in such a way that each edge $e = ab$ is labeled with $\rho = \left\lceil \sqrt{\frac{\rho(a)^2+\rho(b)^2}{2}} \right\rceil$ or $\left\lfloor \sqrt{\frac{\rho(a)^2+\rho(b)^2}{2}} \right\rfloor$ then the edge labels are distinct. In this case ρ is called Root Square mean(RSM) labeling of G . In this paper we prove Arrow graphs A_n^2,A_n^3,A_n^4 admits Root Square mean (RSM) labeling.In today's world, digital data transfer is becoming more and more common in all industries. Data security plays a critical role in the delivery and storage of data. Labeling is an essential component of the cryptosystem. A new approach in the encoding and decoding process on the Root Square Mean Labeling is applied in this paper through an algorithm for encoding and decoding of secured message.

Keywords: Root Square mean (RSM) labeling, Arrow graphs

AMS Subject Classification:05078

1. Introduction

Consider (p, q) graph $G = (V, E)$, where the vertex set and edge set of the graph G are denoted, respectively, by the symbols V(G) and E(G). For the basic terminology and notations we refer to Harary [2]. Rosa proposed the concept of graph labeling in 1967.Gallian [1] offers a thorough analysis of graph labeling.The concept of Mean labeling of graph was introduced by Ponraj et al. [3].The root square mean labeling was introduced by Sandhya et al. and they have proved Path,Cycle,comb,Ladder,Triangular snake,Quadrilateral Snake,Complete graph[6, 7]. Further Meena et al.investigated in their paper studied on some cycle related graphs and theta graphs and proved that the graphs admit Root Mean Square labeling[4, 5]. Cryptography is a technique for information protection that converts an original communication into a coded text that can be decoded by a knowledgeable person. We require a secret key in order to convert a regular letter into a secret encrypted

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one. Saverio Caminiti first proposed the concept of coding languages[8]. The encoding and decoding algorithm using AUM Block sum labeling for path and other graphs were discussed by Uma Maheswari et al. [9]

2. Preliminaries

Definition 2.1 (Mean Labeling). *A function ρ is called mean labeling for a graph G=(V, E) if $\rho : V \to \{0, 1, 2, 3, \cdots, q\}$ is injective and the induce function $\rho^* : E \to \{1, 2, 3, \cdots, q\}$ defined as $\rho* = \left\lceil \frac{\rho(a)+\rho(b)}{2} \right\rceil$ or $\left\lfloor \frac{\rho(a)+\rho(b)}{2} \right\rfloor$ is bijective for every edge.A graph G is called mean labeling.*

Definition 2.2 (Root Square Mean Labeling). *A graph G with p vertices and q edges is called a RSM labeling. if it is possible to label the vertices $x \in v$ with distinct labels $\rho(x)$ from $1, 2, \cdots, q + 1$ in such a way that each edge $e = ab$ is labeled with $\rho = \left\lceil \sqrt{\frac{\rho(a)^2+\rho(b)^2}{2}} \right\rceil$ or $\left\lfloor \sqrt{\frac{\rho(a)^2+\rho(b)^2}{2}} \right\rfloor$ then the edge labels are distinct.In this case ρ is called Root Square mean(RSM) labeling of G*

Definition 2.3 (Arrow graph). *An arrow graph A_n^t with width t and length n is obtained by joining a vertex v with superior vertices of $P_m \times P_n$ by m new edges from one end.*

3. Main Results

Theorem 3.1. *Arrow Graph A_n^2 is a RSM labeling.*

Proof. Let G be an arrow graph A_n^2 .
Let $V(G) = \{a_i, b_i, c; 1 \leq i \leq n\}$ and $E(G) = \{(a_ia_{i+1}); (b_ib_{i+1}); 1 \leq i \leq n-1\}$
$\cup\{a_ib_i; 1 \leq i \leq n\} \cup \{(a_1c), (b_1c)\}$
Define a function $\rho : V \to \{1, 2, 3, \cdots, q + 1\}$
Let us label the vertices as follows
$\rho(a_1) = 3;\ \rho(a_{i+1}) = 3i + 4;\ 1 \leq i \leq n - 1$
$\rho(b_i) = 3i - 1;\ \rho(c) = 1;\ 1 \leq i \leq n$
Then the edges are labeled with
$\rho^*(a_ia_{i+1}) = 3i + 2;\ \rho^*(b_ib_{i+1}) = 3i + 1;\ 1 \leq i \leq n - 1$
$\rho^*(a_ib_i) = 3i;\ 1 \leq i \leq n\ \rho^*(cb_1) = 1;\ \rho^*(a_1c) = 2$
Then we get distinct edge labels.
Hence Arrow Graph A_n^2 is a RSM Labeling. □

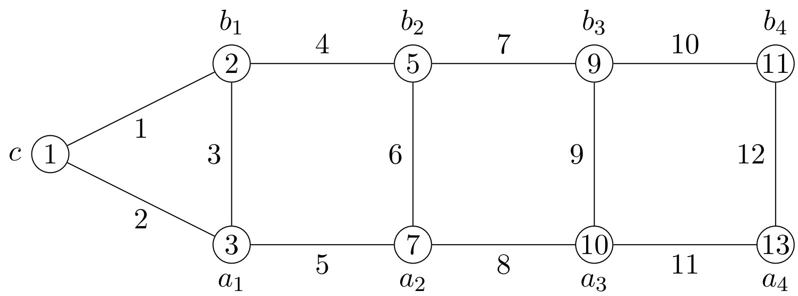

Figure 1. Arrow Graph A_3^2

Theorem 3.2. *Arrow Graph A_n^3 is a RSM labeling.*

Proof. Let G be an arrow graph A_n^3
, Let $V(G) = \{a_i, b_i, c_i, d; 1 \le i \le n\}$ and $E(G) = \{(a_i a_{i+1}); (b_i b_{i+1}); 1 \le i \le n-1\}$
$\cup\{(b_i a_i); (c_i b_i); (c_i c_{i+1}); 1 \le i \le n\} \cup\{(a_1 d), (c_1 d)\}$
Define a function $\rho : V \to \{1, 2, 3, \cdots, q+1\}$
Let us label the vertices as follows
$\rho(a_1) = 3;\ \rho(a_{i+1}) = 5i + 5;\ 1 \le i \le n-1$
$\rho(b_1) = 5;\ \rho(b_{2i}) = 10i - 2;\ \rho(b_{2i+1}) = 10i + 3;\ 1 \le i \le n-1$
$\rho(c_1) = 2;\ \rho(c_{2i}) = 10i - 3;\ \rho(c_{2i+1}) = 10i + 1;\ 1 \le i \le n-1$
Then the edges are labeled with
$\rho^*(a_1 a_2) = 7;\ \rho^*(a_{i+1} a_{i+2}) = 5i + 8;\ \rho^*(b_i b_{i+1}) = 5i + 1;\ 1 \le i \le n-1$
$\rho^*(c_i c_{i+1}) = 5i; 1 \le i \le n-1$
$\rho^*(a_i b_i) = 5i - 1;\ 1 \le i \le n$
$\rho^*(b_1 c_1) = 3;\ \rho^*(b_2 c_2) = 8;\ \rho^*(b_i c_i) = 5i + 7;\ 3 \le i \le n-2$
$\rho^*(a_1 d) = 1;\ \rho^*(c_1 d) = 2$
Then we get distinct edge labels.
Hence Arrow Graph A_n^3 is a root square mean graph. □

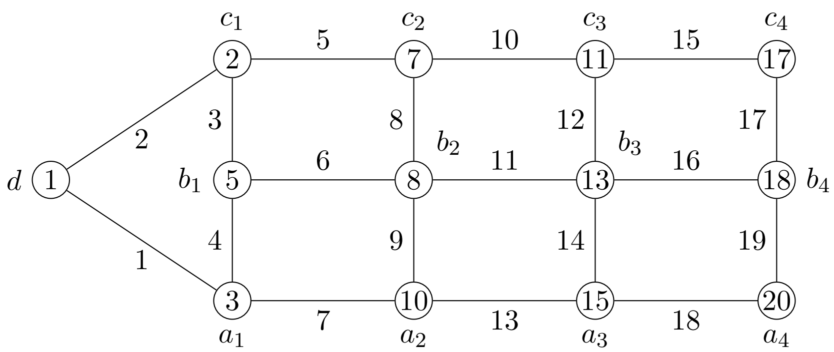

FIGURE 2. Arrow Graph A_3^3

Theorem 3.3. *Arrow Graph A_n^4 is a RSM labeling.*

Proof. Let G be an arrow graph A_n^4 .
Let $V(G) = \{a_i, b_i, c_i, d_i, e; 1 \le i \le n\}$ and $E(G) = \{(ed_1, ea_1); (a_i a_{i+1}); (b_i b_{i+1});$
$(c_i c_{i+1}); (d_i d_{i+1});\ 1 \le i \le n-1\} \cup \{(b_i a_i); (c_i b_i); (c_i d_i; 1 \le i \le n\}$
Define a function $\rho : V \to \{1, 2, 3, \cdots, q+1\}$
Let us label the vertices as follows
$\rho(a_1) = 3;\ \rho(a_{i+1}) = 7i + 6\ 1 \le i \le n-1$
$\rho(b_{3i-2}) = 21i + 16;\ \rho(b_{3i-1}) = 21i - 10;\ \rho(b_{3i}) = 21i - 3;\ 1 \le i \le n$
$\rho(c_{3i-2}) = 21i - 17;\ \rho(c_{3i-1}) = 21i - 11;\ \rho(c_{3i}) = 21i - 4;\ 1 \le i \le n$
$\rho(d_2) = 8;\ \rho(d_{2i+1}) = 14i + 1;\ \rho(d_{2i+2}) = 14i + 9;\ 1 \le i \le n$
Then the edges are labeled with $\rho^*(a_1 e) = 2;\ \rho^*(d_1 e) = 1$
$\rho^*(a_i a_{i+1}) = 7i + 2; ;\ \rho^*(b_i b_{i+1}) = 7i + 1;\ 1 \le i \le n-1$
$\rho^*(c_i c_{i+1}) = 7i;\ \rho^*(d_i d_{i+1}) = 7i - 1; 1 \le i \le n-1$
$\rho^*(a_1 b_1) = 4;\ \rho^*(a_{i+1} b_{i+1}) = 7i + 5;\ 1 \le i \le n-1$
$\rho^*(b_1 c_1) = 5;\ \rho^*(b_{i+1} c_{i+1}) = 7i + 4;\ 1 \le i \le n-1$
$\rho^*(c_i d_i) = 7i - 4;\ 1 \le i \le n$
Then we get distinct edge labels.
Hence Arrow Graph A_n^4 is a RSM labeling. □

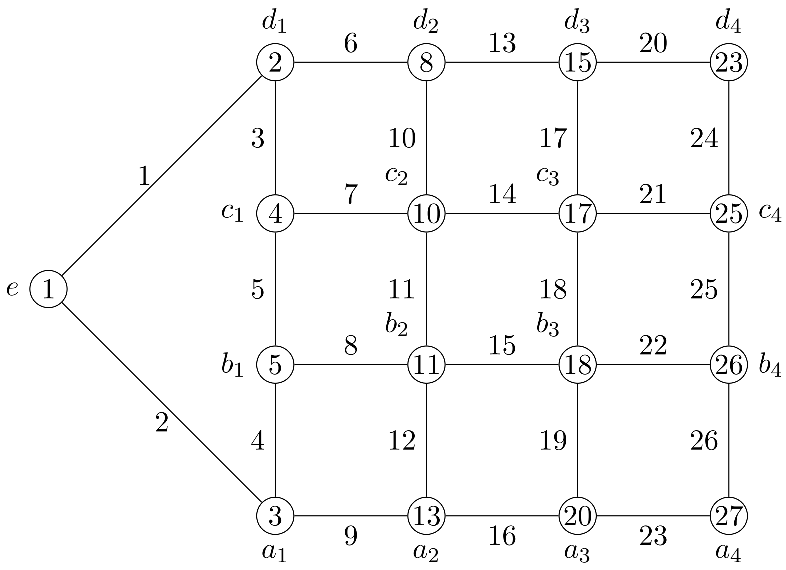

FIGURE 3. Arrow Graph A_3^4

4. APPLICATION

Definition 4.1 (Plain text)**.** *An original intelligible message is called as Plain text.*

Definition 4.2 (Cipher text)**.** *The Transformed message after coding is called as Cipher text.*

In this section, We used Root Square Mean Labeling of Arrow Graph A_2^3 to encode a message and created novel encoding and decoding techniques that increase the secrecy of the coded message.

4.1. **Algorithm for Encoding Arrow Graph. Step.1** Number the 26 alphabets in multiples of five. Five represent vowels, and the remaining letters represent consonants as follows,
The vowels a,e,i,o,u is assigned to multiples of five ie) 5,10,15,20,25 and other numbers 1,2,3,4,6,7,8,9,11,12,13,14,16,17,18,19,21,22,23,24,26 are assigned to consonants in the order.
Step.2 Each character is given a shift cipher using the formula $u_i = (u + n)(\text{mod } 26)$, which moved each number n places. Here, n is the message's length and u is the number assigned to that character.
Step.3 Find the smallest positive integer v_i such that w_j is the geometric mean of u_i and v_i, and take it as such.
Step.4 Select the Arrow Graph A_2^3 and label the edges as v_i and the remaining edges as w_j in the order.
Step.5 Reassign the edge labels asE_i=$v_i + E(j)$for $i = 1, 2, \cdots, n$ and E_i=$w_j + E(j)$for $j = n+1, n+2, \cdots, 2n$ where $E(i)$ is the edges of the A_2^3.
Step.6 The secret to decoding the message is to deliver the cipher text as A_2^3, along with the updated edge label.
Now we find that the edge labels of Arrow graph is encoded and a new cipher text is

obtained. We proceed to decode the cipher text as follows using the labels of the edges of Key Graph .

4.2. **Algorithm for Decoding the Cipher text of Arrow Graph. Step.1** Find the Root square mean labeling for the given Key graph .
Step.2 Determine the integers w_j=E_j-$E(j)$ for $j = n+1, n+2, \cdots, 2n$ and v_i=E_i-$E(i)$ for $i = 1, 2 \cdots, n$, where the labels in the edges of the key graph are $E_i^{'}s$ and the edges of the same graph are $E(i)^{'}s$.
Step.3 Identify the positive number. Using the relation,$u_i = \frac{w_j^2}{v_i}$ for $i = 1, 2, 3, 4, 5, 6$ and $j = 7, 8, 9, 10, 11, 12$
Step.4 Find the value of A_i using $A_i = [u_i - \frac{n}{2}]mod26$ Where n represents the number of edges and the alphabet for each A_i from the encoding table to get the plain text.
Now we illustrate the application through an example by assigning the edge labels of Arrow graph A_3^2 by "CHURCH". The following is the encoding procedure.
Encoding

A	B	C	D	E	F	G	H	I	J	K	L	M
5	1	2	3	10	4	6	7	15	8	9	11	12
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	20	14	16	17	18	19	25	21	22	23	24	26

TABLE 1. Encoding Table

Length of the word is 6
Using the shift cipher u_i=$(u + n)$(mod 26)
We get

C	H	U	R	C	H
2	7	25	17	2	7
8	13	5	23	8	13

Denote $u_1 = 8$, $u_2 = 13$, $u_3 = 5$, $u_4 = 23$, $u_5 = 8$, $u_6 = 13$.
For $i = 1 \ldots n$ and $j = n+1, n+2, \ldots 2n$, find the lowest positive integer V_i such that the geometric mean between u_i and v_i is a positive integer. Denote this as w_j.

$$G.M(u_1, v_1) = G.M(8, 8) = 8 = w_7;\ G.M(u_2, v_2) = G.M(13, 13) = 13 = w_8$$
$$G.M(u_3, v_3) = G.M(5, 5) = 5 = w_9;\ G.M(u_4, v_4) = G.M(23, 23) = 23 = w_{10}$$
$$G.M(u_5, v_5) = G.M(8, 8) = 8 = w_{11};\ G.M(u_6, v_6) = G.M(13, 13) = 13 = w_{12}$$

Reassign the edge labels as
$E_i = v_i + E(i)\ for\ i\ =\ 1, 2, 3, 4, 5, 6$ and $E_j = w_j + E(j)\ for\ j\ =\ 7, 8, 9, 10, 11, 12$
where $E(i)$ is the edges label of An^2.

$$E_1 = v_1 + E(1) = 8 + 1 = 9;\ E_2 = v_2 + E(2) = 13 + 2 = 15$$
$$E_3 = v_3 + E(3) = 5 + 3 = 8;\ E_4 = v_4 + E(4) = 23 + 4 = 27$$
$$E_5 = v_5 + E(5) = 8 + 5 = 13;\ E_6 = v_6 + E(6) = 13 + 6 = 19$$
$$E_7 = w_7 + E(7) = 8 + 7 = 15;\ E_8 = w_8 + E(8) = 13 + 8 = 21$$
$$E_9 = w_9 + E(9) = 5 + 9 = 14;\ E_{10} = w_{10} + E(10) = 23 + 10 = 33$$

$E_{11} = w_{11} + E(11) = 8 + 11 = 19;\; E_{12} = w_{12} + E(12) = 13 + 12 = 25$

The Arrow graph A_n^2 with the relabeled edges is given below
Send the receiver the labeled graph above so that they can decode it.

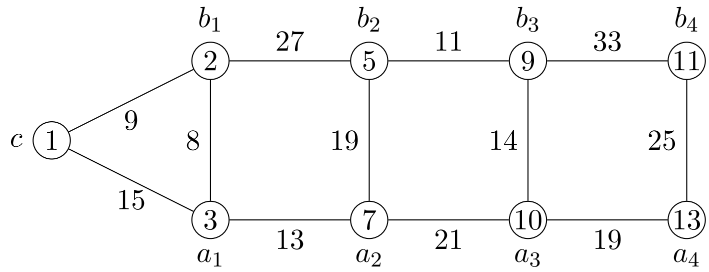

FIGURE 4. Arrow Graph A_3^2

Key for decoding
The message can be decoded using the labeled graph seen in figure 4.
Decoding

- After receiving the labeled graph find the plain text using the above decoding algorithm.
- Finding the integers v_i and w_j using the key graph figure 4 and the relation, $v_i = E_i - E(i)$ for $i = 1, 2, 3, 4, 5, 6$ and $w_j = E_j - E(j)$ for $j = 7, 8, 9, 10, 11, 12$
  We get

$$v_1 = E_1 - E(1) = 9 - 1 = 8;\; v_2 = E_2 - E(2) = 15 - 2 = 13$$
$$v_3 = E_3 - E(3) = 8 - 3 = 5;\; v_4 = E_4 - E(4) = 27 - 4 = 23$$
$$v_5 = E_5 - E(5) = 13 - 5 = 8;\; v_6 = E_6 - E(6) = 19 - 6 = 13$$
$$w_7 = E_7 - E(7) = 15 - 7 = 8;\; w_8 = E_8 - E(8) = 21 - 8 = 13$$
$$w_9 = E_9 - E(9) = 14 - 9 = 5;\; w_{10} = E_{10} - E(10) = 33 - 10 = 23$$
$$w_{11} = E_{11} - E(11) = 19 - 11 = 8;\; w_{12} = E_{12} - E(12) = 25 - 12 = 13$$

- Identify the positive number. Using the relation,
  $u_i = \frac{w_j^2}{v_i}$ for $i = 1, 2, 3, 4, 5, 6$ *and* $j = 7, 8, 9, 10, 11, 12$

$$u_1 = \frac{w_7^2}{v_1} = \frac{8^2}{2} = 8;\; u_2 = \frac{w_8^2}{v_2} = \frac{13^2}{13} = 13$$
$$u_3 = \frac{w_9^2}{v_3} = \frac{5^2}{5} = 5;\; u_4 = \frac{w_{10}^2}{v_4} = \frac{23^2}{23} = 23$$
$$u_5 = \frac{w_{11}^2}{v_5} = \frac{8^2}{8} = 8;\; u_6 = \frac{w_{12}^2}{v_6} = \frac{13^2}{13} = 13$$

- To find the plain text $A_i = [u_i - \frac{n}{2}] mod 26$

$$A_1 = [u_1 - \frac{12}{2}] mod 26 = [8 - \frac{12}{2}] mod 26 = 2$$
$$A_2 = [u_2 - \frac{12}{2}] mod 26 = [13 - \frac{12}{2}] mod 26 = 7$$

$$A_3 = [u_3 - \frac{12}{2}]mod26 = [5 - \frac{12}{2}]mod26 = 25$$
$$A_4 = [u_4 - \frac{12}{2}]mod26 = [23 - \frac{12}{2}]mod26 = 17$$
$$A_5 = [u_5 - \frac{12}{2}]mod26 = [8 - \frac{12}{2}]mod26 = 2$$
$$A_6 = [u_6 - \frac{12}{2}]mod26 = [13 - \frac{12}{2}]mod26 = 7$$

Therefore, we deduce from the encoding table that the plain text is **CHURCH**.

5. Conclusion

In this paper we have identified A_n^2,A_n^3,A_n^4 Arrow graphs and proved that the graphs are RSM labeling. We have applied encryption coding by using a revised Graph Message Jumble Code technique with new labeling and numbering of alphabets based on vowels.In the future, we intend to introduce new labeling technique and prove coding,utilising various graphs in conjunction with various methods of alphabet numbering.

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