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NANO TOPOLOGY INDUCED BY GRAPHS

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Abstract. The aim of this paper is to introduce new subgraph types related to given subgraphs of a graph G. Specifically, these are termed c-subgraph, i-subgraph, and b-subgraph of a subgraph H from G, denoted as c_H, i_H, and b_H, respectively. The paper explores various properties and results concerning these new subgraph types and their complements under certain binary operations. Additionally, it introduces a new type of nano topological space known as a nano graph topological space, defined in terms of these new subgraph types. The study also investigates properties of nano closure and nano interior subgraphs.

Keywords: new subgraphs, nano graph topological space, nano closure and nano interior.

AMS Subject Classification: 05C70, 54A10, 54A05.

1. Introduction

The study of graph product was introduced in 19^{th} century, some new operations (products) on graphs are defined to obtain new graphs and some properties and applications of them are discussed. Many papers of graph operations introduced new techniques or methods to create new types of graphs or subgraphs [4, 5, 6, 7, 13].
In 1982, Pawlak [12], introduced the theory of rough sets , defining a rough set as an extension of set theory where a subset of a universe is described by a pair of ordinary sets referred to as lower and upper approximation. The notion of a Nano topology was introduced by Thivagar and Richard [14] in 2013, and they presented a new type of functions called Nano continuous functions and derived their characterizations in terms of Nano closed sets, Nano closure and Nano interior. In [10], new forms of Nano topological spaces were introduced using a neighborhood system of vertices for a directed graph. The authors also explored the connection between directed graphs and Nano topological spaces, using the human heart as a real-life example. They demonstrated the practical utility of this study in addressing the blood flow system within the human heart. In [9], the authors corrected certain results previously introduced by Thivagar et al. [15]. They also introduced new forms of Nano topology and generalized Nano topology induced by graphs.

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Furthermore, they studied the approximations of various structures using relations that could potentially find applications in quantum physics and superstring theory. In [8], the authors introduced a new classification of Nano open sets, specifically Nano h_α-open set. The concept of Nano open mappings, continuous functions, and Nano h_α homeomorphism was proposed. The investigation of properties related to these functions was yielded some remarks that have been supported by examples. In [1], the author introduced and studied the concept of Grill Nano generalized closed sets within the framework of Grill Nano topological spaces. Additionally, presented the expansion of Nano generalized closed sets through grills. In [11], discussed the graphical isomorphism for undirected graphs through nano homeomorphism and also checked whether two undirected graphs have similar pattern of connections. moreover, they formalised the structural equivalence of two kinematic chains.

The main aim of this paper is to introduce novel types of subgraphs derived from random subgraphs of a graph G. It explores the properties of these subgraph types and investigates their relationships under specific binary operations. Additionally, the paper introduces a new type of Nano topology called the Nano graph topological space, defined in relation to these new types of subgraphs.

2. Preliminaries

Definition 2.1. [2] *A graph G comprises a non-empty set $V(G)$ and possibly an empty set $E(G)$ consisting of subsets of elements from $V(G)$. The elements within $V(G)$ are referred to as vertices, and those within $E(G)$ are termed edges. The cardinality of vertices (or edges) in graph G is termed its order (or size) and is denoted by $p(G)$ (or $q(G)$) respectively.*

Definition 2.2. [2] *In a graph G, when two or more edges share the same pair of different end vertices, they are referred to as multiple (or parallel) edges. Denoting an edge with end vertices u and v as $e = uv$. An edge with identical end vertices is termed a loop at the shared vertex. A graph of zero edge is called an empty (null) graph.*

Definition 2.3. [2] *The removal of a vertex v from a graph G is a subgraph $G - v$ of G has the vertex set $V(G - v) = V(G) \setminus \{v\}$ and the edge set of $G - v$ consists the set of all edges of G that are not incident with v. The removal of an edge e from G is a spanning subgraph $G - e$ has the edge set $E(G - e) = E(G) \setminus \{e\}$.*

Definition 2.4. [2] *The complement G^c of G has the same set of vertices of G and any two vertices are adjacent in G^c if and only if they are nonadjacent in G.*

A graph H is a subgraph from a graph G if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 2.5. [12] *Let U be a non-empty set and R be an equivalence relation on U. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$ and $R(x)$ denotes the equivalence class determined by x, then*

1. *The lower approximation of X with respect to R is denoted by $L_R(X)$ and $L_R(X) = \cup_{x\in U}\{R(x) : R(x) \subseteq X\}$.*
2. *The upper approximation of X with respect to R is denoted by $U_R(X)$ and $U_R(X) = \cup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$*
3. *The boundary of X with respect to R is denoted by $B_R(X)$ and $B_R(X) = U_R(X) \setminus L_R(X)$.*

Definition 2.6. [14] *Let U be a non-empty set and R be an equivalence relation on U. If $X \subseteq U$, then the family $\tau_R(X) = \{\phi, U, L_R(X), U_R(X), B_R(X)\}$ forms a topology on U called the nano topology. The elements of $\tau_R(X)$ are called nano-open sets.*

3. New Types of Subgraphs and their Properties

In this section, we present some new types of subgraphs generated by a given subgraph of a graph G. Some properties and results of these new types of subgraphs and their complements under some binary operations are investigate.

Definition 3.1. *Let $G = (V(G), E(G))$ be a finite non-empty graph and $H = (V(H), E(H))$ be a subgraph from G. We say that S is a minimal subgraph of H if $V(H) = V(S)$ and $E(H) \subset E(S)$. The family of all minimal subgraphs of H is denoted by S_j.*

Definition 3.2. *Let $G = (V(G), E(G))$ be a finite non-empty graph and $H = (V(H), E(H))$ be a subgraph from G and let $G_k = G \cup G^c$, G and G^c are subgraphs in G_k. Let S_j be the family of minimal subgraphs containing H and S_j^c be a family of complements of members of S_j, then we define the following concepts:*

1. *The c-subgraph of a subgraph H in G is denoted by c_H and defined as $c_H = \cap\{(S_j)^c\}$.*
2. *The i-subgraph of a subgraph H in G is denoted by i_H and defined as $i_H = \cup\{(S_j)^c\}$.*
3. *The b-subgraph of a subgraph H in G is denoted by b_H and defined as $b_H = i_H - c_H$, the operation "$-$" is the deletion of edges in c_H from edges in i_H. Therefore, c_H, i_H and b_H are subgraphs in G_k and G_k with these subgraphs is called subgraph space and it is denoted by (G_H, G_k). The subgraph H from a graph G is called exact subgraph in G if and only if $c_H = i_H$ and is called rough subgraph in G if and only if $c_H \neq i_H$.*

Definition 3.3. *If H is any subgraph of G, then we define the following:*

1. $(c_H)^c = G_k - c_H$.
2. $(i_H)^c = G_k - i_H$.
3. $(b_H)^c = G_k - b_H$.
4. $(S_H)^c = G_k - S_H$.
5. $(H_k)^c = G_k - H_k$.
6. $G^c = G_k - G$.

Remark 3.1. *Throughout this study, the symbols $\phi = \phi_k = \phi_G = \phi_{G^c}$ and $\phi_{H_k} = \phi_H = \phi_{H^c}$ are denoted to be the null graph of G_k and the null graph of $H_k = H \cup H^c$, respectively.*

Definition 3.4. *Let $e = (u, v)$ be an edge in a graph G and let H be a subgraph in G, if $e \in H$, then H is called an open subgraph of e and H is called a closed subgraph of e if $e \notin H$.*

Example 3.1. (1) *Consider the graphs G_1, G_1^c, G_{k_1} and five selected subgraphs from G_1 are shown in Figure 1 and Figure 2, respectively.*

The family of minimal subgraphs of H_i for $i = 1, 2, 3, 4, 5$ are obtained by using Definitions 3.1 and 3.2 as follows:

(a) *Minimal subgraphs containing $H_1 = \{e_1, e_3, v_1, v_2, v_3, v_4\}$ are: $S_1 = \{e_1, e_2, e_3\}$, $S_2 = \{e_1, e_3, e_4\}$ and $S_3 = \{e_1, e_2, e_3, e_4\}$.*
Their complements are: $(S_1)^c = \{e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$,
$(S_2)^c = \{e_2, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ and $(S_3)^c = \{e_5, e_6, e_7, e_8, e_9, e_{10}\}$. Then, we have
$c_{H_1} = \{e_5, e_6, e_7, e_8, e_9, e_{10}\}$, $i_{H_1} = \{e_2, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ and $b_{H_1} = \{e_2, e_4\}$.

(b) *Minimal subgraph containing $H_2 = \{e_1, e_5, v_1, v_3, v_4, v_5\}$ is: $S_1 = \{e_1, e_4, e_5\}$. And their complement is:*

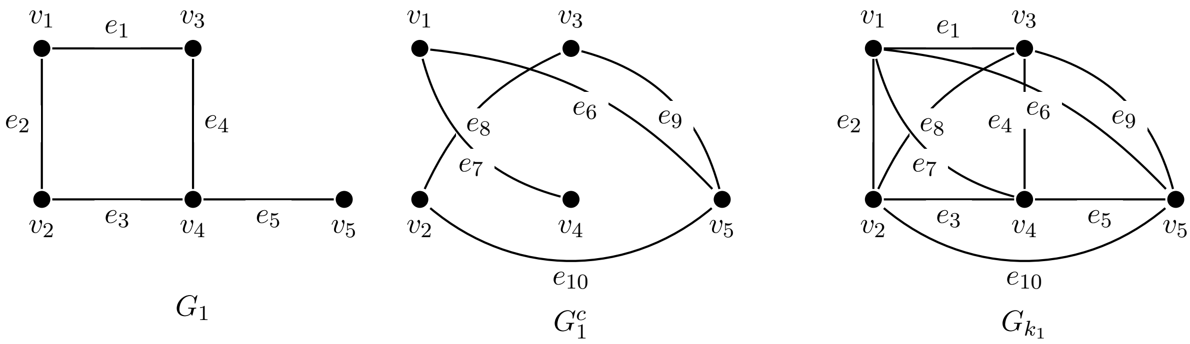

FIGURE 1. Graphs G_1, G_1^c and G_{k_1}.

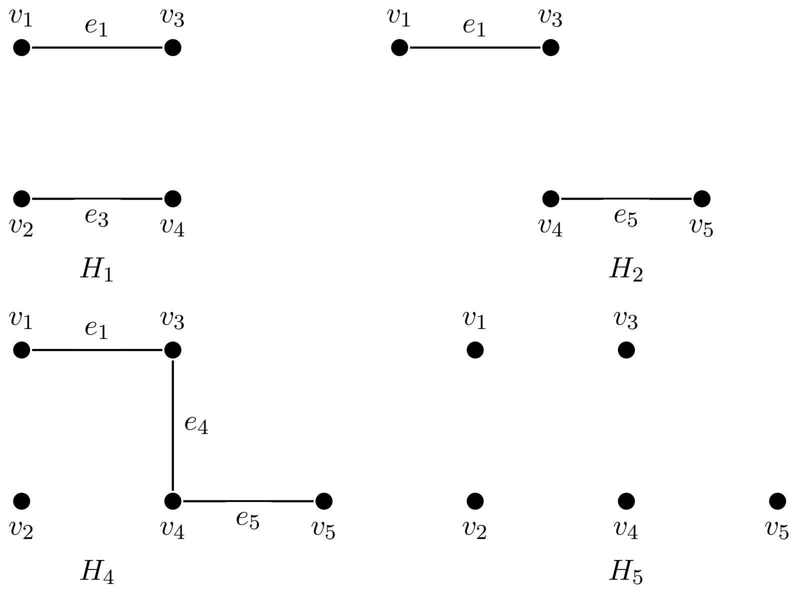

FIGURE 2. Five selected subgraphs from G_1 in Figure 1.

$(S_1)^c = \{e_2, e_3, e_6, e_7, e_8, e_9, e_{10}\}$.
Then, we have $c_{H_2} = i_{H_2} = \{e_2, e_3, e_6, e_7, e_8, e_9, e_{10}\}$ *and* $b_{H_2} = \phi$.

(c) *Minimal subgraph containing* $H_3 = \{e_1, v_1, v_3, v_4\}$ *is:* $S_1 = \{e_1, e_4\}$. *And their complement is:*
$(S_1)^c = \{e_2, e_3, e_5, e_6, e_7, e_8, e_9, e_{10}\}$. *Then, we have*
$c_{H_3} = i_{H_3} = \{e_2, e_3, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ *and* $b_{H_3} = \phi$.

(d) *Minimal subgraphs containing* $H_4 = \{e_1, e_4, e_5, v_1, v_2, v_3, v_4, v_5\}$ *are:* $S_1 = \{e_1, e_3, e_4, e_5\}$, $S_2 = \{e_1, e_2, e_4, e_5\}$ *and* $S_3 = \{e_1, e_2, e_3, e_4, e_5\}$. *And their complements are:*
$(S_1)^c = \{e_2, e_6, e_7, e_8, e_9, e_{10}\}$, $(S_2)^c = \{e_3, e_6, e_7, e_8, e_9, e_{10}\}$ *and* $(S_3)^c = \{e_6, e_7, e_8, e_9, e_{10}\}$. *Then, we have*
$c_{H_4} = \{e_6, e_7, e_8, e_9, e_{10}\}$, $i_{H_4} = \{e_2, e_3, e_6, e_7, e_8, e_9, e_{10}\}$ *and* $b_{H_4} = \{e_2, e_3\}$.

(e) *It is easy to get* c_{H_5}, i_{H_5} *and* b_{H_5} *of* H_5 *in Figure 2 are given as* $c_{H_5} = \phi$ *and* $i_{H_5} = b_{H_5} = G_{k_1}$.

(2) *Consider the graphs* G_2, G_2^c, G_{k_2} *and selected subgraph* H *from a graph* G_2 *are shown in Figure 3.*

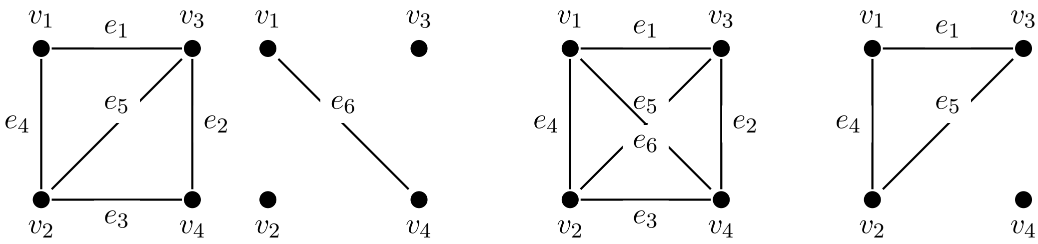

FIGURE 3. Graphs G_2, G_2^c, G_{k_2} and selected subgraph H from G_2.

Minimal subgraphs containing $H = \{e_1, e_4, e_5, v_1, v_2, v_3, v_4\}$ *in Figure 3 are:* $S_1 = \{e_1, e_2, e_4, e_5\}$, $S_2 = \{e_1, e_3, e_4, e_5\}$ *and* $S_3 = \{e_1, e_2, e_3, e_4, e_5\}$. *And their complements are:* $(S_1)^c = \{e_3, e_6\}$, $(S_2)^c = \{e_2, e_6\}$ *and* $(S_3)^c = \{e_6\}$. *Then, we have* $c_H = \{e_6\}$, $i_H = \{e_2, e_3, e_6\}$ *and* $b_H = \{e_2, e_3\}$.

(3) *Consider the graphs* G_3, G_3^c, G_{k_3} *and selected subgraph* H *from a graph* G_3 *are shown in Figure 4.*

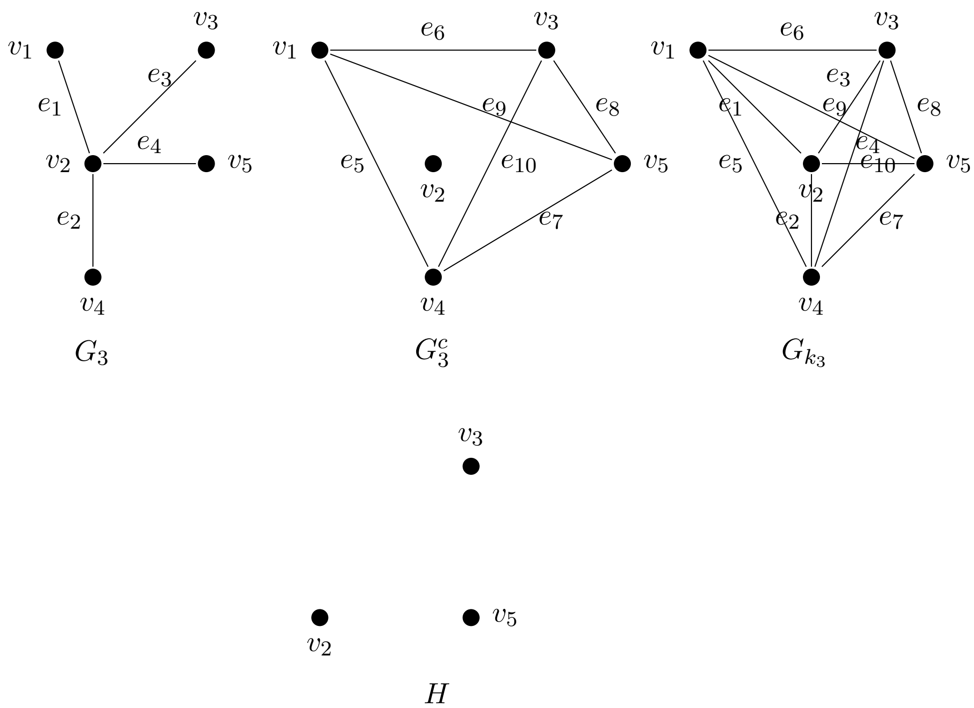

FIGURE 4. Graphs G_3, G_3^c, G_{k_3} and selected subgraph H from G_3.

Minimal subgraphs containing $H = \{v_2, v_3, v_5\}$ *in Figure 4 are:* $S_1 = \{e_3\}$, $S_2 = \{e_4\}$ *and* $S_3 = \{e_3, e_4\}$. *And their complements are:*
$(S_1)^c = \{e_1, e_2, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$, $(S_2)^c = \{e_1, e_2, e_3, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ *and* $(S_3)^c = \{e_1, e_2, e_5, e_6, e_7, e_8, e_9, e_{10}\}$. *Then, we have*
$c_H = \{e_1, e_2, e_5, e_6, e_7, e_8, e_9, e_{10}\}$,
$i_H = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ *and* $b_H = \{e_3, e_4\}$.

The proof of the following relations among the above graphs, subgraphs and their complements is straightforward.

Remark 3.2. *If H is a subgraph in G and it has minimal subgraph in G, then*

(1) $H \cup G^c = G_k - H^c$.
(2) $H^c \cup G^c = G_k - H$.
(3) $(G_k - (H \cup H^c)) = (G_k - H) \cap (G_k - H^c)$.
(4) $c_H \cup G^c = c_H$.
(5) $i_H \cup G^c = i_H$.
(6) $c_H \cap G^c = G^c$.
(7) $i_H \cap G^c = G^c$.

Proposition 3.1. *If H is any subgraph in G, then*

(1) $c_H \subseteq i_H$.
(2) $b_H \subseteq i_H$.
(3) $c_H \cap i_H = c_H$.
(4) $c_H \cup i_H = c_H \cup b_H = i_H \cup b_H = i_H$.
(5) $c_H \cap b_H = \phi$.
(6) $i_H \cap b_H = b_H$.

Proof. Follows from Definition 3.2. □

Proposition 3.2. (1) *If H does not contain a minimal subgraph in G, then c_H, i_H and b_H does not exist.*
(2) *If H contains only one minimal subgraph in G, then $c_H = G_k - H_k = i_H$ and $b_H = \phi$.*

Proof. (1) Follows from Definitions 3.1 and 3.2.
(2) If H contains only one minimal subgraph S_H in G, that is $S_H = H \cup H^c = H_k$ and H contains only one complement minimal subgraph $(S_H)^c$ in G_k, so we have $\cap\{(S_H)^c\} = \cup\{(S_H)^c\} = (S_H)^c = G_k - S_H = G_k - H_k$, then by Definition 3.2, $c_H = G_k - H_k = i_H$ and $b_H = \phi$.
□

Proposition 3.3. *If H contains more than one minimal subgraph in G, then $c_H = G_k - H_k$, $i_H = G_k - H$ and $b_H = (G_k - H) - (H_k)^c$.*

Proof. Let $\{S_1, S_2, ..., S_m\}$ be the minimal subgraphs containing H and their complements is the set $\{(S_1)^c, (S_2)^c, ..., (S_m)^c\}$. Then by Definition 3.2 (1), we have $S_m = H \cup H^c = H_k$ is one of the minimal subgraph containing H in G. Therefore, $(S_m)^c = G_k - S_m = G_k - H_k$, so $(S_m)^c$ is a subgraph of all other complements of minimal subgraphs in G, then by Definition 3.2 (1), we have $c_H = (S_1)^c \cap (S_2)^c \cap (S_3)^c \cap ... \cap (S_m)^c = (S_m)^c = G_k - H_k$. If e is an edge in H, then e is also an edge in all minimal subgraphs containing H in G, but e is not an edge in complements of all minimal subgraphs containing H, so by Definition 3.2 (2), we have $i_H = G_k - H$. By Definitions 3.2 (3) and 3.3, we have $b_H = (G_k - H) - (G_k - H_k) = (G_k - H) - (H_k)^c$ □

Proposition 3.4. *If H is a subgraph in G, then $i_H = c_H \cup H^c$.*

Proof. From Proposition 3.3, we have $c_H \cup H^c = (G_k - H_k) \cup H^c = (G_k - (H \cup H^c)) \cup H^c = (G_k - H) \cap (G_k - H^c) \cup H^c = [(H^c \cup G^c) \cap (H \cup G^c)] \cup H^c = [(H^c \cap (H \cup G^c)) \cup (G^c \cap (H \cup G^c)] \cup H^c == H^c \cup G^c = G_k - H = i_H$. By Proposition 3.3, the result is hold. □

Proposition 3.5. *Let G, G^c and ϕ be subgraphs in G_k. Then the following statements are true:*

(1) $c_\phi = \phi$ *and* $i_\phi = b_\phi = G_k$.

(2) $c_G = \phi$ *and* $i_G = b_G = G^c$.
(3) $c_{G^c} = \phi$ *and* $i_{G^c} = b_{G^c} = G$.

Proof. Follows from Proposition 3.3. □

Proposition 3.6. (1) *H is an exact subgraph in G if and only if H contains only one minimal subgraph.*
(2) *H is a rough subgraph in G if and only if H has more than one minimal subgraph in G.*

Proof. (1) Follows from Proposition 3.3 and Definition 3.2.
(2) Follows from Proposition 3.3 and Definition 3.2.

□

Proposition 3.7. *Let G_k be a graph and H be a subgraph in G, then the following statements are true:*
(1) $c_{c_H} = c_{i_H} = c_{b_H} = \phi$.
(2) $i_{c_H} = (c_H)^c$.
(3) $b_{c_H} = (c_H)^c$.
(4) $i_{i_H} = (i_H)^c$.
(5) $b_{i_H} = (i_H)^c$.
(6) $i_{b_H} = (b_H)^c$.
(7) $b_{b_H} = (b_H)^c$.

Proof. (1) Let S_j be the family of minimal subgraphs of c_H, i_H and b_H, so one of minimal subgraphs containing all other minimal subgraphs in S_G is G_k and $(G_k)^c = \phi$, then by Definition 3.2 (1, 2, 3), we get the result.
(2) By Proposition 3.3, we have $i_{c_H} = G_k - c_H$ and by Definition 3.3(1), we get $i_{c_H} = G_k - c_H = (c_H)^c$.
(3) By subtracting (1) from (2), we get the result.
(4) By Proposition 3.3, we have $i_{i_H} = G_k - i_H$ and by Definition 3.3(2), we get $i_{i_H} = G_k - i_H = (i_H)^c$.
(5) By subtracting (1) from (4), we get the result.
(6) By Proposition 3.3, we have $i_{b_H} = G_k - b_H$ and by Definition 3.3(3), we get $i_{b_H} = G_k - b_H = (b_H)^c$.
(7) By subtracting (1) from (6), we get the result.

□

Proposition 3.8. *Let G_k be a graph and H_1, H_2 be two non-null subgraphs in G, if $H_1 \subseteq H_2$, then*
(1) $c_{H_2} \subseteq c_{H_1}$.
(2) $i_{H_2} \subseteq i_{H_1}$.
(3) $b_{H_1} \subseteq b_{H_2}$.

Proof. (1) Let $e \in c_{H_2}$, then $e \in \cap\{S^c_{H_2}\}$ and $e \notin H_2$ since $H_1 \subseteq H_2$ this implies that $e \notin H_1$ but $e \in \cap\{S^c_{H_1}\}$, thus $e \in c_{H_1}$.
(2) Let $e \in i_{H_2}$, then $e \in \cup\{S^c_{H_2}\}$ and $e \notin H_2$ since $H_1 \subseteq H_2$ this implies that $e \notin H_1$ but $e \in \cup\{S^c_{H_1}\}$, thus $e \in i_{H_1}$.
(3) Since $H_1 \subseteq H_2$, then $i_{H_1} - c_{H_1} \subseteq i_{H_2} - c_{H_2}$, so by Definition 3.1(3) implies that $b_{H_1} \subseteq b_{H_2}$.

□

Proposition 3.9. *If H is any non-empty subgraph in G, then $|b_H| \leq |c_H| \leq |i_H|$ and $|c_H| = |i_H|$ if H contains only one minimal subgraph in G.*

Proof. Follows from Propositions 3.1 and 3.2. □

Proposition 3.10. *If H_1 and H_2 are subgraphs in G and $|E(H_1)| \leq |E(H_2)|$, then $|c_{H_2}| \leq |c_{H_1}|$, $|i_{H_2}| \leq |i_{H_1}|$ and $|b_{H_1}| \leq |b_{H_2}|$, equality holds if H_1 and H_2 are isomorphic subgraphs.*

Proof. Follows from Proposition 3.8. □

Proposition 3.11. *Let G_k be a graph and H_1, H_2 be two subgraphs in G, then the following are true:*

(1) $c_{H_1 \cap H_2} = c_{H_1} \cap c_{H_2}$.
(2) $c_{H_1 \cup H_2} \subseteq c_{H_1} \cup c_{H_2}$.
(3) $i_{H_1 \cup H_2} = i_{H_1} \cup i_{H_2}$.
(4) $i_{H_1} \cap i_{H_2} \subseteq i_{H_1 \cap H_2}$.

Proof. (1) Since $H_1 \cap H_2 \subseteq H_1$ and $H_1 \cap H_2 \subseteq H_2$, by Proposition 3.8(1), we have $c_{H_1} \subseteq c_{H_1 \cap H_2}$ and $c_{H_2} \subseteq c_{H_1 \cap H_2}$. Then $c_{H_1} \cap c_{H_2} \subseteq c_{H_1 \cap H_2}$ ... (i)
Let $e \in c_{H_1 \cap H_2}$ implies $e \in \cap\{(S_{H_1 \cap H_2})^c\}$ implies $e \notin H_1 \cap H_2$ implies $e \notin H_1$ and $e \notin H_2$ implies $e \in \cap S_{H_1^c}$ and $e \in \cap S_{H_2^c}$, by Definition 3.2 (1), $e \in c_{H_1 \cap H_2}$ ... (ii)
From (i) and (ii), we get $c_{H_1 \cap H_2} = c_{H_1} \cap c_{H_2}$.
(2) Let $e \in c_{H_1} \cup c_{H_2}$ implies $e \in c_{H_1}$ or $e \in c_{H_2}$ implies $e \in \cap\{(S_{H_1})^c\}$ or $e \in \cap\{(S_{H_2})^c\}$ implies $e \notin H_1$ or $e \notin H_2$ implies $e \notin H_1 \cup H_2$ by Definition 3.2(1), we have $e \in c_{H_1 \cup H_2}$.
(3) Let $e \in i_{H_1 \cup H_2}$ if and only if $e \notin H_1 \cup H_2$ if and only if $e \notin H_1$ or $e \notin H_2$, by Definition 3.2 (2), we have $e \cup \{(S_{H_1})^c\}$ or $e \in \cup\{(S_{H_2})^c\}$ if and only if $e \in i_{H_1}$ or $e \in i_{H_2}$ if and only if $e \in i_{H_1} \cup i_{H_2}$.
(4) Since $H_1 \cap H_2 \subseteq H_1$ and $H_1 \cap H_2 \subseteq H_2$, then by Proposition 3.8(2), we have $i_{H_1} \subseteq i_{H_1 \cap H_2}$ and $i_{H_2} \subseteq i_{H_1 \cap H_2}$, hence $i_{H_1} \cap i_{H_2} \subseteq i_{H_1 \cap H_2}$.
□

Remark 3.3. *The equality and the converse of case (2) in Proposition 3.11 is not true in general. From Example 3.1(1), let H_1 and H_2 be two subgraphs in a graph G_1, we have $H_1 = \{e_1, e_3, v_1, v_2, v_3, v_4\}$, $H_2 = \{e_1, e_5, v_1, v_3, v_4, v_5\}$, $H_1 \cup H_2 = \{e_1, e_3, e_5, v_1, v_2, v_3, v_4, v_5\}$, $c_{H_1} = \{e_5, e_6, e_7, e_8, e_9, e_{10}\}$, $c_{H_2} = \{e_2, e_3, e_6, e_7, e_8, e_9, e_{10}\}$ and $c_{H_1} \cup c_{H_2} = \{e_2, e_3, e_5, e_6, e_7, e_8, e_9, e_{10}\}$, then the minimal subgraphs of $H_1 \cup H_2$ are: $S_1 = \{e_1, e_2, e_3, e_5\}$, $S_2 = \{e_1, e_3, e_4, e_5\}$ and $S_3 = \{e_1, e_2, e_3, e_4, e_5\}$, so their complements are:*
$(S_1)^c = \{e_4, e_6, e_7, e_8, e_9, e_{10}\}$, $(S_2)^c = \{e_2, e_6, e_7, e_8, e_9, e_{10}\}$ and $(S_3)^c = \{e_6, e_7, e_8, e_9, e_{10}\}$, then $c_{H_1 \cup H_2} = \{e_6, e_7, e_8, e_9, e_{10}\}$.

Therefore, $c_{H_1} \cup c_{H_2} \subseteq c_{H_1 \cup H_2}$, but $c_{H_1 \cup H_2} \subsetneq c_{H_1} \cup c_{H_2}$.
Also, the equality and the converse of case (4) in Proposition 3.11 is not true in general. From Example 3.1(1), let $H_1 = \{e_1, e_3, v_1, v_2, v_3, v_4\}$, $H_2 = \{e_1, e_5, v_1, v_3, v_4, v_5\}$, $i_{H_1} = \{e_2, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$,
$i_{H_2} = \{e_2, e_3, e_6, e_7, e_8, e_9, e_{10}\}$, $H_1 \cap H_2 = \{e_1, v_1, v_3, v_4\}$, $i_{H_1} \cap i_{H_2} = \{e_2, e_6, e_7, e_8, e_9, e_{10}\}$ and $i_{H_1 \cap H_2} = \{e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ then $i_{H_1} \cap i_{H_2} \subseteq i_{H_1 \cap H_2}$ but $i_{H_1 \cap H_2} \subsetneq i_{H_1} \cap i_{H_2}$.

Proposition 3.12. *Let G_k be a graph and H be a subgraph in G, then the following are true:*

(1) $i_{H^c} = (c_H)^c$ *and* $c_{H^c} = (i_H)^c$.
(2) $(c_H)^c \cup (i_H)^c = (c_H \cap i_H)^c = (c_H)^c$.
(3) $(c_H)^c \cup (b_H)^c = (c_H \cap b_H)^c = G_k$.
(4) $(i_H)^c \cup (b_H)^c = (i_H \cap b_H)^c = (b_H)^c$.
(5) $(c_H)^c \cup (i_H)^c \cup (b_H)^c = G_k$.

(6) $(c_H)^c \cap (i_H)^c = (c_H \cup i_H)^c = (i_H)^c$.
(7) $(c_H)^c \cap (b_H)^c = (c_H \cup b_H)^c$.
(8) $(i_H)^c \cap (b_H)^c = (i_H \cup b_H)^c = (i_H)^c$.
(9) $(c_H)^c \cap (i_H)^c \cap (b_{H_1})^c = (i_H)^c$.

Proof. (1) Let $e \in i_{H^c}$, then $e \notin H^c$ so $e \in H$, therefore $e \notin \cap\{(S_H)^c\}$, then $e \in (\cap S_H^c)^c$, hence $e \in (c_H)^c$.

(2) Let $e \in (c_H)^c \cup (i_H)^c$ implies that $e \in (c_H)^c \cup e \in (i_H)^c$, by Definition 3.3, we have $e \in (G_k - c_H) \cup e \in (G_k - i_H)$ implies $e \in (G_k - c_H) \cup (G_k - i_H)$ implies $e \in (G_k - (c_H \cap i_H))$ implies $e \in (c_H \cap i_H)^c$ and $c_H \cap i_H = c_H$. Then, $e \in (c_H)^c$.

(3) Let $e \in (c_H)^c \cup (b_H)^c$ implies that $e \in (c_H)^c \cup e \in (b_H)^c$, by Definition 3.3, we have $e \in (G_k - c_H) \cup e \in (G_k - b_H)$ implies $e \in G_k - c_H) \cup (G_k - b_H)$ implies $e \in (G_k - (c_H \cap b_H))$ implies $e \in (c_H \cap b_H)^c$ and $c_H \cap b_H = \phi_H$. Then, $e \in G_k$.

(4) Let $e \in (i_H)^c \cup (b_H)^c$ implies that $e \in (i_H)^c \cup e \in (b_H)^c$, by Definition 3.3, we have $e \in (G_k - i_H) \cup e \in (G_k - b_H)$
imlpies $e \in (G_k - i_H) \cup (G_k - b_H)$
implies $e \in (G_k - (i_H \cap b_H))$
implies $e \in (i_H \cap b_H)^c$.
And $i_H \cap b_H = b_H$. Then, $e \in (b_H)^c$.

(5) Let $e \in (c_H)^c \cup (i_H)^c \cup (b_H)^c$ implies that $e \in (c_H)^c \cup e \in (i_H)^c \cup e \in (b_H)^c$, by Definition 3.3, we have $e \in (G_k - c_H) \cup e \in (G_k - i_H) \cup e \in (G_k - b_H)$
implies $e \in (G_k - c_H) \cup (G_k - i_H) \cup e \in (G_k - b_H)$
implies $e \in (G_k - (c_H \cap i_H)) \cup e \in (G_k - b_H)$
implies $e \in (G_k - (c_H \cap i_H)) \cup (G_k - b_H)$
implies $e \in (G_k - (c_H \cap i_H \cap b_H))$
implies $e \in (c_H \cap i_H \cap b_H)^c$.
And $c_H \cap i_H \cap b_H = \phi_H$. Then, $e \in G_k$.

The proofs of cases (6, 7, 8, 9) are the complement of the proofs of cases (2, 3, 4, 5), respectively. □

Proposition 3.13. *Let G_k be a graph and H_1, H_2 be two subgraphs in G, then the following are true:*

(1) $(c_{H_1} \cup c_{H_2})^c = (c_{H_1})^c \cap (c_{H_2})^c$.
(2) $(c_{H_1} \cap c_{H_2})^c = (c_{H_1})^c \cup (c_{H_2})^c$.
(3) $(i_{H_1} \cup i_{H_2})^c = (i_{H_1})^c \cap (i_{H_2})^c$.
(4) $(i_{H_1} \cap i)^c = (i_{H_1}))^c \cup (i_{H_2})^c$.
(5) $(b_{H_1} \cup b_{H_2})^c = (b_{H_1})^c \cap (b_{H_2})^c$.
(6) $(b_{H_1} \cap b_{H_2})^c = (b_{H_1})^c \cup (b_{H_2})^c$.

Proof. (1) Let $e \in (c_{H_1} \cup c_{H_2})^c$, by Definition 3.3, we have
if and only if $e \in (G_k - (c_{H_1} \cup c_{H_2}))$
if and only if $e \in (G_k - c_{H_1}) \cap (G_k - c_{H_2})$
if and only if $e \in (G_k - c_{H_1})$ and $e \in (G_k - c_{H_1})$, by Definition 3.3, we have
if and only if $e \in (c_{H_1})^c$ and $e \in (c_{H_2})^c$
if and only if $e \in (c_{H_1})^c \cap (c_{H_2})^c$.

(2) Let $e \in (c_{H_1} \cap c_{H_2})^c$, by Definition 3.3, we have
if and only if $e \in G_k - (c_{H_1} \cap c_{H_2})$
if and only if $e \in (G_k - c_{H_1}) \cup (G_k - c_{H_2})$
if and only if $e \in (G_k - c_{H_1})$ or $e \in (G_k - c_{H_1})$, by Definition 3.3, we have
if and only if $e \in (c_{H_1})^c$ or $e \in (c_{H_1})^c$
if and only if $e \in (c_{H_1})^c \cup (c_{H_2})^c$.

The proofs of cases (3) and (5) are similar to the proof of case (1) and proofs of cases (4) and (6) are similar to the proof of case (2). □

4. Nano Graph Topological Space

In this section, we introduce a new type of nano topological space called nano graph topological space in terms of new types of subgraphs indicated in Definition 3.2. The nano closure and nano interior subgraphs and their characterizations are investigate.

Definition 4.1. *Let $G = (V(G), E(G))$ be a finite non-empty graph and $H = (V(H), E(H))$ be a subgraph from G. Let $\tau_{G_k}(H) = \{\phi, G_k, c_H, i_H, b_H\}$ is a topology on G_k called nano graph topology with respect to H, if satisfies the following axioms:*

(1) $G_k, \phi \in \tau_{G_k}(H)$.
(2) *The union of subgraphs of any subcollection in $\tau_{G_k}(H)$ is in $\tau_{G_k}(H)$.*
(3) *The intersection of subgraphs of any finite subcollection in $\tau_{G_k}(H)$ is in $\tau_{G_k}(H)$.*

That is, $(G_k, \tau_{G_k}(H))$ is called nano graph topological space and ϕ is represented the null graph of G_k. The subgraphs of the nano graph topology $\tau_{G_k}(H)$ are called nano−open subgraphs in $\tau_{G_k}(H)$ and the complement of each nano-open subgraph in $\tau_{G_k}(H)$ is called nano−closed subgraph in $\tau_{G_k}(H)$.

Example 4.1. *Consider the graph and subgraphs in Example 3.1(1), then the nano graph topologies with respect to the subgraph H_i for $i = 1, 2, 3, 4$ are given as*

(1) $\tau_{G_k}(H_1) = \{\phi, G_k, c_{H_1}, i_{H_1}, b_{H_1}\}$.
(2) $\tau_{G_k}(H_2) = \{\phi, G_k, c_{H_2}\}$.
(3) $\tau_{G_k}(H_3) = \{\phi, G_k, c_{H_3}\}$.
(4) $\tau_{G_k}(H_4) = \{\phi, G_k, c_{H_4}, i_{H_4}, b_{H_4}\}$.

Theorem 4.1. *If $\tau_{G_k}(H)$ is a nano graph topology with respect to a subgraph H from G, then the collection $\beta_{G_k}(H) = \{G_k, c_H, b_H\}$ form a nano graph basis for $\tau_{G_k}(H)$ with respect to H.*

Proof. (1) Let U be a family of subgraphs in $\beta_{G_k}(H)$, then $\cup U = G_k$.

(2) (a) For G_k and c_H, let $W = c_H$, since $G_k \cap c_H = c_H$, then $W \subset G_k \cap c_H$ and every vertices with incident edges in $G_k \cap c_H$ belongs to W.
(b) For G_k and b_H, let $W = b_H$, since $W \subset G_k \cap b_H$ and every vertices with incident edges in $G_k \cap b_H$ belongs to W, so $G_k \cap b_H = b_H$.
(c) For c_H and b_H, we have $c_H \cap b_H = \phi$.

Hence, $\beta_{G_k}(H)$ form a base for $\tau_{G_k}(H)$. □

Remark 4.1. *Let $\tau_{G_k}(H)$ be a nano graph topology with respect to a subgraph H from G, then $(\tau_{G_k}(H))^c$ is a topology on G_k and is called the dual nano graph topology of $\tau_{G_k}(H)$, members of $(\tau_{G_k}(H))^c$ are called nano closed subgraphs. Let K be subgraph in G_k is a nano closed subgraph in $\tau_{G_k}(H)$ if and only if $G_k - K$ is nano open subgraph in $\tau_{G_k}(H)$.*

Example 4.2. *The dual nano graph topologies of nano graph topologies in Example 4.1 are given as*

(1) $(\tau_{G_k}(H_1))^c = \{\phi, G_k, \{e_1, e_2, e_3, e_4\}, \{e_1, e_3\}, \{e_1, e_3, e_5, e_6, e_7, e_8, e_9, e_{10}\}\}$.
(2) $(\tau_{G_k}(H_2))^c = \{\phi, G_k, \{e_1, e_4, e_5\}\}$.
(3) $(\tau_{G_k}(H_3))^c = \{\phi, G_k, \{e_1, e_4\}\}$.
(4) $(\tau_{G_k}(H_4))^c = \{\phi, G_k, \{e_1, e_2, e_3, e_4, e_5\}, \{e_1, e_4, e_5\}, \{e_1, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}\}$.

In the following proposition, we introduce the types of nano graph topological space.

Proposition 4.1. *Let G be a finite non-empty graph and H be a subgraph from G, then*

(1) *If $c_H = i_H \neq G_k \neq \phi$, that is, H has exactly one minimal subgraph, then $\tau_{G_k}(H) = \{\phi, G_k, c_H\}$ and $(\tau_{G_k}(H))^c = \{\phi, G_k, S_1\}$, where S_1 is the only minimal subgraph containing H.*
(2) *If $c_H = \phi$ and $i_H = G_k$, then $\tau_{G_k}(H) = \{\phi, G_k\}$ is the indiscrete nano graph topology with respect to H and $(\tau_{G_k}(H))^c = \{\phi, G_k\}$.*
(3) *If $c_H = \phi$ and $i_H \neq G_k$, in this case we have if $H = G$, then $\tau_{G_k}(H) = \{\phi, G_k, i_H\}$ and $(\tau_{G_k}(H))^c = \{\phi, G_k, (i_H)^c = G\}$, if $H = G^c$, then $\tau_{G_k}(H) = \{\phi, G_k, i_H = G\}$ and $(\tau_{G_k}(H))^c = \{\phi, G_k, (i_H)^c = G^c\}$.*
(4) *If $c_H \neq \phi$ and $i_H = G_k$, then $\tau_{G_k}(H) = \{\phi, G_k, c_H, b_H\}$ and $(\tau_{G_k}(H))^c = \{\phi, G_k, (c_H)^c = b_H, (b_H)^c = c_H\} = \{\phi, G_k, c_H, b_H\}$.*
(5) *If $c_H \neq i_H$, $c_H \neq \phi$ and $i_H \neq G_k$, then $\tau_{G_k}(H) = \{\phi, G_k, c_H, i_H, b_H\}$ and $(\tau_{G_k}(H))^c = \{\phi, G_k, (c_H)^c = H \cup b_H, (i_H)^c = H, (b_H)^c = H \cup c_H\}$.*

In the following example, we illustrate the types of nano graph topological space indicated in the above proposition.

Example 4.3. *Let G be a non-empty finite graph with H and G, G^c are subgraph in G and G_k, respectively.*

(1) *For type (1), see Example 3.1(2).*
(2) *For type (2), see Example 3.1(1)(e).*
(3) *For type (3), if $H = G$ or $H = G^c$ having more than one minimal subgraph in G_k.*
(4) *For type (4), see Example 3.1(3).*
(5) *For type (5), see Example 3.1(1), if $H = H_1$ or $H = H_4$.*

Remark 4.2. *A nano-open subgraph in $(G_k, \tau_{G_k}(H))$ is said to be nano-clopen subgraph if it is both nano-open and nano-closed subgraph in $(G_k, \tau_{G_k}(H))$. In Proposition 4.1, we have only two types of nano graph topological space each nano-open subgraph is nano-clopen subgraph, first type, if $c_H = \phi$ and $i_H = G_k$ and second type, if $c_H \neq \phi$ and $i_H = G_k$.*

Let $(G_k, \tau_{G_k}(H))$ be a nano graph topological space with respect to a subgraph H in G and W is a subgraph in G_k. We define the following:

Definition 4.2. *The nano interior subgraph of W is defined as the union of all nano-open subgraphs contained in W and it is denoted by $Nint_{G_k}(W)$. That is, $Nint_{G_k}(W)$ is the maximal nano-open subgraph of W.*

Definition 4.3. *The nano closure subgraph of W is defined as the intersection of all nano-closed subgraphs containing W and it is denoted by $Ncl_{G_k}(W)$. That is, $Ncl_{G_k}(W)$ is the minimal nano-closed subgraph containing W.*

Theorem 4.2. *Let $(G_k, \tau_{G_k}(H))$ be a nano graph topological space with respect to a subgraph H in G. Let W be a subgraph in G_k. Then*

(1) $G_k - Nint_{G_k}(W) = Ncl_{G_k}(G_k - W)$.
(2) $G_k - Ncl_{G_k}(W) = Nint_{G_k}(G_k - W)$.

Proof. (1) Let $e \in G_k - Nint_{G_k}(W)$, then $e \notin Nint_{G_k}(W)$, then any nano open subgraph in $\tau_{G_k}(H)$, say C containing e is not a subgraph in W, that is $C \cap (G_k - W) \neq \phi$, for every nano subgraph C in $\tau_{G_k}(H)$ containing e, therefore $e \in Ncl_{G_k}(G_k - W)$, then $G_k - Nint_{G_k}(W) \subseteq Ncl_{G_k}(G_k - W)$. Conversely, Let $e \in Ncl_{G_k}(G_k - W)$, then $C \cap (G_k - W) \neq \phi$, for every nano subgraph C in $\tau_{G_k}(H)$ containing e, that is $C \nsubseteq W$, then $e \notin Nint_{G_k}(W)$, therefore $e \in G_k - Nint_{G_k}(W)$, thus, $Ncl_{G_k}(G_k - W) \subseteq G_k - Nint_{G_k}(W)$. Hence $G_k - Nint_{G_k}(W) = Ncl_{G_k}(G_k - W)$.

(2) Proof is similar to (1).

□

Remark 4.3. *It is easy to calculate the nano interior subgraph of subgraphs c_H, i_H and b_H and their complements in a graph G_k in the five types of nano graph topology, while the nano closure subgraph of our subgraphs in each type of nano graph topology is present in the following corollary.*

Corollary 4.1. *Let $(G_k, \tau_{G_k}(H))$ be a nano graph topological space with respect to a subgraph H from G. Then the following statements are true:*

(1) *If H has only one minimal subgraph in G ($c_H = i_H \neq G_k \neq \phi$). Then*
   (a) $Ncl_{G_k}(c_H) = G_k$.
   (b) $Ncl_{G_k}((c_H)^c) = (c_H)^c$.
(2) *If $c_H = \phi$ and $i_H = G_k$. Then*
   (a) $Ncl_{G_k}(i_H) = G_k$.
   (b) $Ncl_{G_k}((i_H)^c) = \phi$.
(3) *If $c_H = \phi$ and $i_H \neq G_k$. Then*
   (a) $Ncl_{G_k}(i_H) = G_k$.
   (b) $Ncl_{G_k}((i_H)^c) = (i_H)^c$.
(4) *If $c_H \neq \phi$ and $i_H = G_k$. Then*
   (a) $Ncl_{G_k}(c_H) = (b_H)^c$.
   (b) $Ncl_{G_k}(b_H) = (c_H)^c$.
(5) *If $c_H \neq i_H$, $c_H \neq \phi$ and $i_H \neq G_k$. Then*
   (a) $Ncl_{G_k}(c_H) = (b_H)^c$.
   (b) $Ncl_{G_k}(i_H) = G_k$.
   (c) $Ncl_{G_k}(b_H) = (c_H)^c$.

Proof. Follows from Proposition 4.1 and Definitions 4.2 and 4.3. □

In the following theorems, we examine the nano interior subgraph and nano closure subgraph of any subgraph in a graph G_k.

Theorem 4.3. *Let $(G_k, \tau_{G_k}(H))$ be a nano graph topological space with respect to a subgraph H from G, if H has only one minimal subgraph in G and W is a subgraph of G_k, then*

(1)

$$Nint_{G_k}(W) = \begin{cases} c_H, & if \quad c_H \subseteq W, \\ \phi, & otherwise. \end{cases}$$

(2)

$$Ncl_{G_k}(W) = \begin{cases} (c_H)^c, & if \quad W \subseteq (c_H)^c, \\ G_k, & otherwise. \end{cases}$$

Proof. (1) If $c_H \subseteq W$, since $\tau_{G_k}(H)$ has only one nano open subgraph c_H contained in W, so $Nint_{G_k}(W) = c_H$.
If $c_H \nsubseteq W$, then $\tau_{G_k}(H)$ has no nano open subgraph contained in W, so $Nint_{G_k}(W) = \phi$.
(2) If $W \subseteq (c_H)^c$, then the dual nano graph topology $(\tau_{G_k}(H))^c$ has nano closed subgraphs $(c_H)^c$ and G_k containing W, so $(c_H)^c \cap G_k = (c_H)^c$, hence $Ncl_{G_k}(W) = (c_H)^c$.

If $W \nsubseteq (c_H)^c$, then the dual nano graph topology $(\tau_{G_k}(H)^c$ has only one nano closed subgraph G_k containing W, so $Ncl_{G_k}(W) = G_k$.

□

Theorem 4.4. *Let $(G_k, \tau_{G_k}(H))$ be a nano graph topological space with respect to a subgraph H from G, if $c_H = \phi$ and $i_H = G_k$ and W is a subgraph of G_k, then*

(1)
$$Nint_{G_k}(W) = \begin{cases} G_k, & if \quad i_H \subseteq W, \\ \phi, & otherwise. \end{cases}$$

(2) $Ncl_{G_k}(W) = G_k$.

Proof. (1) Since $c_H = \phi$ and $i_H = G_k$, then $\tau_{G_k}(H) = \{\phi, G_k\}$, then the only open subgraph containing W is G_k, then $Nint_{G_k}(W) = G_k$, otherwise $Nint_{G_k}(W) = \phi$.

(2) The dual nano graph topology $(\tau_{G_k}(H))^c$ has the only nano closed subgraph G_k containing W, hence $Ncl_{G_k}(W) = G_k$.

□

Theorem 4.5. *Let $(G_k, \tau_{G_k}(H))$ be a nano graph topological space with respect to a subgraph H from G, if $c_H = \phi$ and $i_H \neq G_k$ and W is a subgraph of G_k with more than one edge, then*

(1)
$$Nint_{G_k}(W) = \begin{cases} i_H, & if \quad i_H \subseteq W, \\ \phi, & otherwise. \end{cases}$$

(2) *If $H = G$, then*
$$Ncl_{G_k}(W) = \begin{cases} G, & if \quad W \subseteq G, \\ G_k, & otherwise. \end{cases}$$

(3) *If $H = G^c$, then*
$$Ncl_{G_k}(W) = \begin{cases} G^c, & if \quad W \subseteq G^c, \\ G_k, & otherwise. \end{cases}$$

Proof. In case $c_H = \phi$ and $i_H \neq G_k$, we have $H = G$ or $H = G^c$ and $\tau_{G_k}(H) = \{\phi, G_k, i_H\}$ and $i_H = b_H$.

(1) Similar to the proof of Theorem 4.4(1).
(2) In case $c_H = \phi$ and $i_H \neq G_k$, we have $H = G$ or $H = G^c$. if $H = G$, then $\tau_{G_k}(H) = \{\phi, G_k, i_H\} = \{\phi, G_k, G^c\}$ and the dual nano graph topology is of the form $(\tau_{G_k}(H))^c = \{G_k, \phi, G\}$.
(3) If $H = G^c$, then $\tau_{G_k}(H) = \{\phi, G_k, i_H\} = \{\phi, G_k, G\}$ and the dual nano graph topology is of the form $(\tau_{G_k}(H))^c = \{\phi, G_k, G^c\}$. Hence the proof is complete.

□

Theorem 4.6. *Let $(G_k, \tau_{G_k}(H))$ be a nano graph topological space with respect a subgraph H from G, if $c_H \neq \phi$ and $i_H = G_k$ and W is a subgraph of G_k with more than one edge, then*

(1)

$$Nint_{G_k}(W) = \begin{cases} c_H, & if \quad c_H \subseteq W, \\ b_H, & if \quad b_H \subseteq W, \\ G_k, & if \quad W = G_k. \end{cases}$$

(2)

$$Ncl_{G_k}(W) = \begin{cases} (b_H)^c, & if \quad W \subseteq c_H, \\ (c_H)^c, & if \quad W \subseteq b_H, \\ G_k, & otherwise. \end{cases}$$

Proof. Since $c_H \neq \phi$ and $i_H = G_k$, then $\tau_{G_k}(H) = \{\phi, G_k, c_H, b_H\}$ and the dual nano graph topology is of the form $(\tau_{G_k}(H))^c = \{\phi, G_k, (b_H)^c, (c_H)^c\}$. Then by Definitions 4.2 and 4.3, the proof of (1) and (2) it is obvious. □

Theorem 4.7. *Let $(G_k, \tau_{G_k}(H))$ be a nano graph topological space with respect to a subgraph H from G, if $c_H \neq i_H \neq G_k \neq \phi$ and W is a subgraph of G_k, then*

(1)

$$Nint_{G_k}(W) = \begin{cases} c_H, & if \quad c_H \subseteq W, \\ b_H, & if \quad b_H \subseteq W, \\ i_H, & if \quad i_H \subseteq W, \\ \phi, & otherwise. \end{cases}$$

(2) *If W is the set of the edges not in c_H, i_H and b_H, then*
  (a) $Ncl_{G_k}(W \cup c_H) = (b_H)^c$.
  (b) $Ncl_{G_k}(W \cup i_H) = G_k$.
  (c) $Ncl_{G_k}(W \cup b_H) = (c_H)^c$.

(3)

$$Ncl_{G_k}(W) = \begin{cases} (b_H)^c, & if \quad W \subseteq c_H, \\ G_k, & if \quad W \subseteq i_H, \\ (c_H)^c, & if \quad W \subseteq b_H, \\ (c_H)^c, & if \quad W = (c_H)^c, \\ (i_H)^c, & if \quad W = (i_H)^c, \\ (b_H)^c, & if \quad W = (b_H)^c, \\ (c_H)^c, & if \quad \exists \quad e \in W, e \in b_H \quad \& \quad e \notin c_H, \\ (i_H)^c, & otherwise. \end{cases}$$

Proof. Proof of (1) follows from Definition 4.2.

Proof of (2), if W is the set of edges not in c_H, i_H and b_H, then we have the following cases:

(1) $W \cup c_H = (b_H)^c$, then $Ncl_{G_k}(W \cup c_H) = (b_H)^c$.
(2) $W \cup i_H = G_k$, then $Ncl_{G_k}(W \cup i_H) = G_k$.
(3) $W \cup b_H = (c_H)^c$, then $Ncl_{G_k}(W \cup b_H) = (c_H)^c$.

Proof of (3), we have the following cases:

(1) If $W \subseteq c_H$ and W, c_H are subgraphs in G_k then $G_k - W \subseteq G_k - c_H$, take the nano interior subgraph of both sides we have $Nint_{G_k}(G_k - W) \subseteq Nint_{G_k}(G_k - c_H)$, by Theorem 4.2 this implies that $G_k - Ncl_{G_k}(W) = G_k - Ncl_{G_k}(c_H)$, by Corollary 4.1, we have $G_k - Ncl_{G_k}(W) = G_k - (b_H)^c$, then $Ncl_{G_k}(W) = (b_H)^c$.

(2) If $W \subseteq i_H$, then $G_k - W \subseteq G_k - i_H$, take the nano interior subgraph of both sides we have $Nint_{G_k}(G_k - W) \subseteq Nint_{G_k}(G_k - i_H)$ by Theorem 4.2 this implies that $G_k - Ncl_{G_k}(W) = G_k - Ncl_{G_k}(i_H)$, by Corollary 4.1, we have $G_k - Ncl_{G_k}(W) = G_k - G_k$, then $Ncl_{G_k}(W) = G_k$.
(3) If $W \subseteq b_H$, then $G_k - W \subseteq G_k - b_H$, take the nano interior subgraph of both sides we have $Nint_{G_k}(G_k - W) \subseteq Nint_{G_k}(G_k - b_H)$ by Theorem 4.2 this implies that $G_k - Ncl_{G_k}(W) = G_k - Ncl_{G_k}(b_H)$, by Corollary 4.1, we have $G_k - Ncl_{G_k}(W) = G_k - (c_H)^c$, then $Ncl_{G_k}(W) = (c_H)^c$.
(4) For $W = (c_H)^c$, $W = (i_H)^c$ and $W = (b_H)^c$, the proof it is obvious.
(5) If at least one of the edges in W is in b_H but not in c_H, directly from case 2(c), we get the result.
(6) If W is not in the above cases, it is clear $Ncl_{G_k}(W) = (i_H)^c$.

□

In the following example, the cases in Theorem 4.7 are illustrated.

Example 4.4. *Consider the graph in Example 3.1 (2), we have* $c_H = \{e_6\}$, $i_H = \{e_2, e_3, e_6\}$ *and* $b_H = \{e_2, e_3\}$, *then the nano graph topology is given by*
$\tau_{G_k}(H) = \{\phi, G_k, \{e_6\}, \{e_2, e_3, e_6\}, \{e_2, e_3\}\}$ *and the dual nano graph topology is given by* $(\tau_{G_k}(H))^c = \{\phi, G_k, \{e_1, e_2, e_3, e_4, e_5\}, \{e_1, e_4, e_5\},$
$\{e_1, e_4, e_5, e_6\}\}$.
To illustrate cases in (2), let $W = \{e_1, e_4\}$, *then* $Ncl_{G_k}(W \cup c_H) = Ncl_{G_k}(\{e_1, e_4, e_6\}) = \{e_1, e_4, e_5, e_6\} = (b_H)^c$,
$Ncl_{G_k}(W \cup i_H) = Ncl_{G_k}(\{e_1, e_2, e_3, e_4, e_6\}) = \{e_1, e_2, e_3, e_4, e_5, e_6\} = G_k$ *and* $Ncl_{G_k}(W \cup b_H) = Ncl_{G_k}(\{e_1, e_2, e_3, e_4\}) = \{e_1, e_2, e_3, e_4, e_5\} = (c_H)^c$.
To illustrate cases in (3), the first six cases are obvious, for other cases let $W = \{e_1, e_3\}$, *then* $Ncl_{G_k}(W) = Ncl_{G_k}(\{e_1, e_3\}) = \{e_1, e_2, e_3, e_4, e_5\} = (c_H)^c$, *let* $W = \{e_1, e_5\}$, *then* $Ncl_{G_k}(W) = Ncl_{G_k}(\{e_1, e_5\}) = \{e_1, e_4, e_5\} = (i_H)^c$ *and let* $W = \{e_4, e_5\}$, *then* $Ncl_{G_k}(W) = Ncl_{G_k}(\{e_4, e_5\}) = \{e_1, e_4, e_5\} = (i_H)^c$.

5. Conclusions

In this paper, we have introduced new types of subgraphs of a subgraph H from a graph G. We have also studied various topological properties and results related to these newly defined subgraphs. Furthermore, we have presented a new type of nano topology generated by these subgraphs and explored results related to nano closure and nano interior subgraphs. Additionally, this study demonstrates the usefulness of investigating subgraphs for creating various types of topological structures.

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