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SOLVABILITY OF AN INVERSE COEFFICIENT PROBLEM FOR A TIME-FRACTIONAL DIFFUSION EQUATION WITH PERIODIC BOUNDARY AND INTEGRAL OVERDETERMINATION CONDITIONS

J. J. JUMAEV[1*], D. K. DURDIEV[2], Z. R. BOZOROV[3], §

Abstract. This article studies the inverse problem for time-fractional diffusion equations with periodic boundary and integral overdetermination conditions on the rectangular domain. First, we introduce a definition of a classical solution, and then the direct problem is reduced to an equivalent integral equation by the Fourier method. Existence and uniqueness of the solution of the equivalent problem is proved using estimates of the Mittag-Leffer function and generalized singular Gronwall inequalities. In the second part, the inverse problem is considered. This problem reduces to the equivalent integral equation. For solving this equation the contracted mapping principle is applied. The local existence and uniqueness results are proven.

Keywords: time-fractional diffusion equation, periodic boundary conditions, inverse problem, integral equation.

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1. Introduction

Periodic boundary conditions (PBCs) are a set of boundary conditions which are often chosen for approximating a large (infinite) system by using a small part called a unit cell. PBCs are often used in computer simulations and mathematical models. The topology of two-dimensional PBC is equal to that of a world map of some video games; the geometry of the unit cell satisfies perfect two-dimensional tiling, and when an object passes through one side of the unit cell, it re-appears on the opposite side with the same velocity (see [1, 16, 36]).

The PBCs arise from many important applications in heat transfer, life sciences [2, 4, 20, 26, 27]. In these papers, it was proven the existence, the uniqueness and the continuous dependence on the data of the solution and we will develop the numerical solution of diffusion problem with periodic boundary conditions.

[1,2,3] Bukhara branch of the institute of Mathematics named after V.I. Romanovskiy at the Academy of sciences of the Republic of Uzbekistan; Bukhara State University, Bukhara, Uzbekistan.
[1] e-mail: jonibekjj@mail.ru; ORCID:.0000-0001-8496-1092.
[2] e-mail: d.durdiev@mathinst.uz; ORCID: 0000-0002-6054-2827.
[2] e-mail: zavqiddinbozorov2011@mail.ru; ORCID: 0000-0001-5309-7553.
[*] Corresponding author.
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Various statements of inverse problems on determination of thermal coefficient in one-dimensional heat equation were studied in [20, 21, 39, 40, 24, 33, 35]. It is important to note that in the papers [20, 21] the time-dependent thermal coefficient is determined from nonlocal overdetermination condition data. Besides, in [5, 17, 18] the coefficients of the heat equations are determined in the case of nonlocal boundary conditions

The papers [7, 11, 19, 37, 38] investigated the inverse problem of finding diffusion coefficients in one and multi-dimensional time-fractional equation. Under some assumption on the data, the existence, uniqueness, and continuous dependence on the data of the solution were shown.

The problem of determining the kernel $k(t)$ of the integral term in an integro-differential heat equation were studied in many publications [6, 8, 9, 12, 13, 14, 15, 25, 34], in which both one- and multidimensional inverse problems with classical initial, initial-boundary conditions were investigated. The existence and uniqueness theorems of inverse problem solutions were proved.

In the present work, time-fractional diffusion equation is used with initial, periodic boundary conditions for the determination of coefficients. The existence and uniqueness of the classical solution of the problem (1)-(4) is reduced to fixed point principles by applying the Fourier method.

2. Formulation of problem

We consider the initial-periodic boundary problem for the fractional diffusion equation

$$\partial_t^\alpha u - u_{xx} + a(t)u = f(x,t)g(t), \quad (x,t) \in D_T, \tag{1}$$

$$u(x,0) = \varphi(x), x \in [0,1], \tag{2}$$

$$u(0,t) = u(1,t), \quad u_x(0,t) = u_x(1,t), \varphi(0) = \varphi(1), \quad \varphi'(0) = \varphi'(1), t \in [0,T], \tag{3}$$

where ∂_t^α is the Caputo fractional derivative of order $0 < \alpha \leq 1$ in the time variable (see Theorem 3.1), $a(t), g(t), t > 0$ are the source control terms, $f(x,t)$ is known source term, $\varphi(x)$ is the initial temperature, T is arbitrary positive number and $D_T := \{(x,t) : 0 < x < 1, 0 < t \leq T\})$.

The problem of determining a function $u(x,t), (x,t) \in D_T$, that satisfies (1)-(3) with known functions $a(t), g(t), f(x,t)$ and $\varphi(x)$ will be called the direct problem.

In the inverse problem, it is required to determine the coefficients $a(t), g(t), t > 0$, in (1) using overdetermination conditions about the solution of the direct problem (1)-(3):

$$\int_0^1 \omega_i(x)u(x,t)dx = h_i(t), \; i = 1,2, \quad x \in [0,1], \tag{4}$$

where $\omega_i(x), h_i(t), i = 1,2$ are given functions.

In heat propagation in a thin rod in which the law of variation $h_i(t)$ of the total quantity of heat in the rod is given in [19]. This integral condition in parabolic problems is also called heat moments which are analyzed in [21].

Let $C^{2,\alpha}(D_T)$ be the class of functions that are $2-$ times continuously differentiable with respect to x in D_T for which a continuous derivative ∂_t^α exists.

Definition 2.1. *The triple of functions $\{u(x,t), a(t), g(t)\}$ from the class $C^{2,\alpha}(D_T) \cap C^{1,0}(\overline{D}_T) \times C[0,T] \times C[0,T]$ is said to be a classical solution of problem* (1)-(4)*, if the functions $u(x,t), a(t)$ and $g(t)$ satisfy the following conditions:*

(1) the function $u(x,t)$ and its derivatives $\partial_t^\alpha u(x,t), u_{xx}(x,t)$ are continuous in the domain D_T ;

(2) the functions $a(t), g(t)$ are continuous on the interval $[0,T]$;

(3) equation (1) *and conditions* (2)-(4) *are satisfied in the classical sense.*

Throughout this article the functions φ, f, ω_i and h_i $(i := 1, 2)$ are assumed to satisfy the following conditions:

$(A1)$ $\varphi(x) \in C^2(0,1)$; $\varphi^{(3)}(x) \in L_2(0,1)$; $\varphi(0) = \varphi(1)$; $\varphi'(0) = \varphi'(1)$; $\varphi''(0) = \varphi''(1)$; $\varphi^{(3)}(0) = \varphi^{(3)}(1)$;

$(A2)$ $f(x,t) \in C(\overline{D}_T) \cap C^{2,1}(D_T)$; $f^{(3)}_{xxx}(x,t) \in L_2(0,1)$; $f(0,t) = f(1,t)$; $f'_x(0,t) = f'_x(1,t)$; $f''_{xx}(0,t) = f''_{xx}(1,t)$;

$(A3)$ $h_i(t) \in C^1[0,T]$; $\omega_i(x) \in C^2[0,1]$; $\omega^{(3)}_i(x,t) \in L_2(0,1)$; $\int_0^1 \omega_i(x)\varphi(x)dx = h_i(0)$; $\omega_i(0) = \omega_i(1)$; $\omega'_i(0) = \omega'_i(1)$; $\omega''_i(0) = \omega''_i(1), i = 1, 2$.

In the next section, we recall basic definitions and notations from fractional calculus, which will be used in the future.

3. Preliminaries

Theorem 3.1. [30, pp. 93-94]. *Let* $0 < \alpha < 1$. *Also let* $u(x,t) \in C^{0,1}_{x,t}(\overline{D}_T)$. *Then the Caputo fractional derivatives* $\partial^\alpha_{0+,t}u(x,t)$ *is continuous on* $[0,T] : \partial^\alpha_{0+,t}u(x,t) \in C(\overline{D}_T)$. *It has, the form*

$$\partial^\alpha_{0+,t}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u(x,\tau)}{\partial \tau}d\tau, \quad \partial^1_{0+,t}u(x,t) = u_t(x,t).$$

Moreover,

$$\partial^\alpha_{0+,t}u(x,0) = \partial^\alpha_{0+,t}u(x,T) = 0, \quad for \quad \alpha \notin \{0,1\},$$

where $\Gamma(\cdot)$ *is the Euler's Gamma function.*

Two parameter Mittag-Leffler function. [30, pp. 40-42] The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$ denotes the real part of the complex number α .

Proposition 3.1. [30, pp. 40-45]. *Let* $0 < \alpha < 2$ *and* $\beta \in \mathbb{R}$ *be arbitrary. We suppose that* κ *is such that* $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. *Then, there exists a constant* $C = C(\alpha,\beta,\kappa) > 0$ *such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|}, \quad \kappa \le |arg(z)| \le \pi.$$

Proposition 3.2. [30, pp. 40-45] *For* $0 < \alpha < 1$, $t > 0$, *we have* $0 < E_{\alpha,1}(-t) < 1$. *Moreover,* $E_{\alpha,1}(-t)$ *is completely monotonic, that is*

$$(-1)^n \frac{d^n}{dt^n}E_{\alpha,1}(-t) \ge 0, \quad \forall n \in \mathbb{N}.$$

Proposition 3.3. [30, pp. 40-45] *For* $0 < \alpha < 1$, $\eta > 0$, *we have* $0 \le E_{\alpha,\alpha}(-\eta) \le \frac{1}{\Gamma(\alpha)}$. *Moreover,* $E_{\alpha,\alpha}(-\eta)$ *is a monotonic decreasing function with* $\eta > 0$.

Lemma 3.1. *(Gronwall inequality.)* [28],[32, pp. 188-210]. *Let* $m(t) \in C[t_0,T] (t_0 \in \mathbb{R}_+ = [0,\infty), T \le +\infty)$ *and suppose that*

$$m(t) \le m_0 + \frac{L}{\Gamma(\gamma)}\int_{t_0}^t (t-s)^{\gamma-1}m(s)ds, \quad t \in [t_0,T].$$

Then we have

$$m(t) \le m_0E_{\gamma,1}(L(t-t_0)^\gamma), \quad t \in [t_0,T],$$

where m_0 and L are nonnegative constants, $\gamma \in (0,1)$.

Theorem 3.2. [30, pp. 135-144]. The solution $T(t) \in AC[0,T]$ of the linear nonhomogeneous fractional problem

$$\partial^{\alpha}_{0+,t} T(t) + \lambda T(t) = f(t), \quad t \in (0,T], \quad \lambda > 0,$$
$$T(0) = c,$$

where $f \in L^1[0,T]$, is given by the integral expression

$$T(t) = cE_{\alpha,1}\left(-\lambda t^{\alpha}\right) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) f(\tau) d\tau.$$

We will use these facts everywhere in this article.

4. Direct problem

The use of the Fourier method for solving problem (1)-(3) leads to the spectral problem for the operator given by the differential expression and boundary conditions

$$X''_n(x) + \lambda^2 X_n(x) = 0, \ x \in (0,1), \quad X_n(0) = X_n(1), \quad X'_n(0) = X'_n(1), n = 0,1,2,.... \quad (5)$$

In [3], it is known that the system of eigenfunctions

$$1, \cos\lambda_1 x, \sin\lambda_1 x, \cos\lambda_2 x, \sin\lambda_2 x, ..., \cos\lambda_n x, \quad \sin\lambda_n x, ... \quad (6)$$

where $\lambda_n = 2\pi n \ \ (n = 0,1,...)$, is a basis for $L_2(0,1)$. That system is eigenfunctions of spectral problem (5).

Since the system (6) form a basis in $L_2(0,1)$, we shall seek the $u(x,t)$ of classical solution of the problem (1)-(3) in the form

$$u(x,t) = \sum_{n=0}^{\infty} u_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty} u_{2n}(t)\sin\lambda_n x, \lambda_n = 2\pi n, \quad (7)$$

$$f(x,t) = \sum_{n=0}^{\infty} f_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty} f_{2n}(t)\sin\lambda_n x,$$

where

$$u_{10}(t) = \int_0^1 u(x,t)dx, \quad u_{1n}(t) = 2\int_0^1 u(x,t)\cos\lambda_n x dx, \quad u_{2n}(t) = 2\int_0^1 u(x,t)\sin\lambda_n x dx.$$

$$f_{10}(t) = \int_0^1 f(x,t)dx, \quad f_{1n}(t) = 2\int_0^1 f(x,t)\cos\lambda_n x dx, \quad f_{2n}(t) = 2\int_0^1 f(x,t)\sin\lambda_n x dx.$$

Then, applying the formal scheme of the Fourier method for determining of unknown coefficients $u_{10}(t)$ and $u_{in}(t) \ \ (i := 1,2; n = 1,2,...)$ of function $u(x,t)$ from (1) and (2), we have

$$\partial^{\alpha} u_{10}(t) = -a(t)u_{10}(t) + g(t)f_{10}(t), \quad (8)$$

$$u_{10}(t)|_{t=0} = \varphi_{10}, \quad (9)$$

$$\partial^{\alpha} u_{in}(t) + \lambda_n^2 u_{in}(t) = -a(t)u_{in}(t) + g(t)f_{in}(t), \quad (10)$$

$$u_{in}(t)|_{t=0} = \varphi_{in}, \quad i = 1,2, \quad n = 1,2,..., \quad (11)$$

where

$$\varphi_{10} = \int_0^1 \varphi(x)dx, \quad \varphi_{1n} = 2\int_0^1 \varphi(x)\cos\lambda_n x dx, \quad \varphi_{2n} = 2\int_0^1 \varphi(x)\sin\lambda_n x dx.$$

According to Theorem 3.2, the solutions of problems (8),(9) and (10),(11) satisfy the following integral equations

$$u_{10}(t) = \varphi_{10} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(f_{10}(\tau)g(\tau) - a(\tau)u_{10}(\tau))d\tau, \tag{12}$$

and

$$u_{in}(t) = \varphi_{in}E_\alpha(-\lambda^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda^2(t-\tau)^\alpha)(g(\tau)f_{in}(\tau) - a(\tau)u_{in}(\tau))d\tau. \tag{13}$$

Estimating the functions $u_{10}(t), u_{in}(t)$, we obtain the integral inequalities:

$$|u_{10}(t)| \leq |\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{10}\| + \frac{\|a\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|u_{10}(\tau)|d\tau,$$

$$|u_{in}(t)| \leq |\varphi_{in}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{in}\| +$$

$$+\frac{\|a\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|u_{in}(\tau)|d\tau, \quad (i = 1,2, \quad n = 1,2,...)$$

where $\|g\| = \max_{t\in[0,T]}|g(t)|, \|a\| = \max_{t\in[0,T]}|a(t)|$. Applying Gronwall's Lemma 3.1., from last inequalities we obtain the following estimate

$$|u_{10}(t)| \leq (|\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{10}\|)E_\alpha(\|a\|t^\alpha), \tag{14}$$

$$|u_{in}(t)| \leq \left(|\varphi_{in}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{in}\|\right)E_\alpha(\|a\|t^\alpha). \tag{15}$$

Using equalities (8), (10) and (14), (15) we obtain estimates for $\partial^\alpha u_{10}(t), \partial^\alpha u_{in}(t)$:

$$|\partial^\alpha u_{10}(t)| \leq \|a\|(|\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{10}\|)E_\alpha(\|a\|t^\alpha) + \|g\|\|f_{10}\|,$$

$$|\partial^\alpha u_{in}(t)| \leq (\lambda^2 + \|a\|)\left(|\varphi_{in}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{in}\|\right)E_\alpha(\|a\|t^\alpha) + \|g\|\|f_{in}\|.$$

Thus we have proved the following lemma:

Lemma 4.1. *For any $t \in [0;T]$ the following estimates are valid:*

$$|u_{10}(t)| \leq (|\varphi_{10}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{10}\|)E_\alpha(\|a\|T^\alpha),$$

$$|u_{in}(t)| \leq \left(|\varphi_{in}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{in}\|\right)E_\alpha(\|a\|T^\alpha),$$

$$|\partial^\alpha u_{10}(t)| \leq \|a\|(|\varphi_{10}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{10}\|)E_\alpha(\|a\|T^\alpha) + \|g\|\|f_{10}\|,$$

$$|\partial^\alpha u_{in}(t)| \leq (\lambda^2 + \|a\|)\left(|\varphi_{in}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{in}\|\right)E_\alpha(\|a\|T^\alpha) + \|g\|\|f_{in}\|.$$

Formally, from (7) by term-by-term differentiation we compose the series

$$\partial^{\alpha}_{t,0+}u(x,t)=\sum_{n=0}^{\infty}\partial^{\alpha}_{0+}u_{1n}(t)\cos\lambda_n x+\sum_{n=1}^{\infty}\partial^{\alpha}_{0+}u_{2n}(t)\sin\lambda_n x, \tag{16}$$

$$u_{xx}(x,t)=-\sum_{n=0}^{\infty}\lambda_n^2 u_{1n}(t)\cos\lambda_n x-\sum_{n=1}^{\infty}\lambda_n^2 u_{2n}(t)\sin\lambda_n x. \tag{17}$$

In view of Lemma 4.1, if following series converge then the series (7), (16), and (17) to be converge for any $(x,t)\in D_T$

$$C_4\sum_{n=1}^{\infty}(\lambda_n^2|\varphi_{in}|+\lambda_n^2\|f_{in}\|),$$

where the constant C_4 depends only on $T,\alpha,\|a\|,\|g\|$.

We hold the following auxiliary lemma:

Lemma 4.2. *If the conditions (A1)(A2) are valid then, there are equalities*

$$\varphi_{in}=\frac{1}{\lambda_n^3}\varphi_{in}^{(3)},\quad f_{in}(t)=\frac{1}{\lambda_n^3}f_{in}^{(3)},(i=1,2) \tag{18}$$

where

$$\varphi_{1n}^{(3)}=2\int_0^1\varphi^{(3)}(x)\sin\lambda_n xdx,\quad \varphi_{2n}^{(3)}=2\int_0^1\varphi^{(3)}(x)\cos\lambda_n xdx,$$

$$f_{1n}^{(3)}(t)=2\int_0^1 f_{xxx}^{(3)}(x,t)\sin\lambda_n xdx,\quad f_{2n}^{(3)}(t)=2\int_0^1 f_{xxx}^{(3)}(x,t)\cos\lambda_n xdx$$

with the following estimate:

$$\sum_{n=1}^{\infty}|\varphi_{in}^{(3)}|^2\le\|\varphi^{(3)}\|_{L_2[0,1]},\qquad \sum_{n=1}^{\infty}|f_{in}^{(3)}(t)|^2\le\|f^{(3)}\|_{L_2[0,l]\times C[0,T]},(i=1,2). \tag{19}$$

If the functions $\varphi(x),f(x,t)$ satisfy the conditions of Lemma 4.2, then due to representations (18) and (19) series (7), (16) and (17) converge uniformly in the rectangle D_T, therefore, function $u(x,t)$ satisfies relations (1)-(3).

Using the above results, we obtain the following assertion.

Lemma 4.3. *Let* $\{g(t),a(t)\}\in C[0,T],(A1),(A2)$ *are satisfied, then there exists a unique solution of the direct problem (1)-(3)* $u(x,t)\in C^{2,\alpha}(D_T)\cap C^{1,0}(\overline{D}_T)$.

Let us derive an estimate for the norm of the difference between the solution of the original integral equations (12), (13) and the solution of this equation with perturbed functions $\widetilde{a},\widetilde{g},\widetilde{\varphi}_{in},\widetilde{f}_{in}$. Let $\widetilde{u}_{in}(t),(i:=0,1,2)$ be solutions of the integral equation (12), (13) corresponding to the functions $\widetilde{a},\widetilde{g},\widetilde{\varphi}_{in},\widetilde{f}_{in}$; i.e.,

$$\widetilde{u}_{10}(t)=\widetilde{\varphi}_{10}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(\widetilde{f}_{10}(\tau)\widetilde{g}(\tau)-\widetilde{a}(\tau)\widetilde{u}_{10}(\tau))d\tau, \tag{20}$$

$$\widetilde{u}_{in}(t)=\widetilde{\varphi}_{in}E_{\alpha}(-\lambda^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda^2(t-\tau)^{\alpha})(\widetilde{a}(\tau)\widetilde{u}_{in}(\tau)+\widetilde{g}(\tau)\widetilde{f}_{in}(\tau))d\tau. \tag{21}$$

Composing the difference $u_{in}-\widetilde{u}_{in}$ with the help of the equations (12), (20), (13) (21) and introducing the notations $u_{in}-\widetilde{u}_{in}=\overline{u}_{in}, a-\widetilde{a}=\overline{a}, g-\widetilde{g}=\overline{g}, f_{in}-\widetilde{f}_{in}=\overline{f}_{in}$, we obtain the integral equation

$$\overline{u}_{10}(t)=\overline{\varphi}_{10}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(f_{10}(\tau)\overline{g}(\tau)+\widetilde{g}(\tau)\overline{f}_{10}(\tau))d\tau-$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(\overline{a}(\tau)u_{10}(\tau)+\widetilde{a}(\tau)\overline{u}_{10}(\tau))d\tau,$$

$$\overline{u}_{in}(t)=\overline{\varphi}_nE_{\alpha,1}(-\lambda_n^2t^\alpha)+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n^2(t-\tau)^\alpha\right)\overline{g}(\tau)f_{in}(\tau)d\tau+$$

$$+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n^2(t-\tau)^\alpha\right)\widetilde{g}(\tau)\overline{f}_{in}(\tau)d\tau-$$

$$-\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n^2(t-\tau)^\alpha\right)\overline{a}(\tau)u_{in}(\tau)d\tau-$$

$$-\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n^2(t-\tau)^\alpha\right)\widetilde{a}(\tau)\overline{u}_{in}(\tau)d\tau \tag{22}$$

from which, are derived the following linear integral inequalities for $|\overline{u}_{10}(t)|, |\overline{u}_{in}(t)|$:

$$|\overline{u}_{10}(t)|\le|\overline{\varphi}_{10}|+\frac{t^\alpha\|\overline{f}_{10}\|\|\widetilde{g}\|}{\Gamma(\alpha+1)}+\frac{t^\alpha\|f_{10}\|\|\overline{g}\|}{\Gamma(\alpha+1)}+$$

$$+\frac{\|\overline{a}\|t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{10}\|\Big)E_\alpha(\|a\|t^\alpha)+\frac{\|\widetilde{a}\|}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}|\overline{u}_{10}(\tau)|d\tau.$$

$$|\overline{u}_{in}(t)|\le|\overline{\varphi}_{in}|+\frac{t^\alpha\|\overline{f}_{in}\|\|\widetilde{g}\|}{\Gamma(\alpha+1)}+\frac{t^\alpha\|f_{in}\|\|\overline{g}\|}{\Gamma(\alpha+1)}+$$

$$+\frac{\|\overline{a}\|t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{in}\|\Big)E_\alpha(\|a\|t^\alpha)+\frac{\|\widetilde{a}\|}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}|\overline{u}_{in}(\tau)|d\tau.$$

Using the Lemma 3.1 from last inequality, we arrive at the estimate:

$$|\overline{u}_{10}(t)|\le\Big\{\overline{\varphi}_{10}|+\frac{t^\alpha\|\overline{f}_{10}\|\|\widetilde{g}\|}{\Gamma(\alpha+1)}+\frac{t^\alpha\|f_{10}\|\|\overline{g}\|}{\Gamma(\alpha+1)}+$$

$$+\frac{\|\overline{a}\|t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{10}\|\Big)E_\alpha(\|a\|t^\alpha)\Big\}E_\alpha(\|\widetilde{a}\|t^\alpha). \tag{23}$$

$$|\overline{u}_{in}(t)|\le\Big\{\overline{\varphi}_{in}|+\frac{t^\alpha\|\overline{f}_{in}\|\|\widetilde{g}\|}{\Gamma(\alpha+1)}+\frac{t^\alpha\|f_{in}\|\|\overline{g}\|}{\Gamma(\alpha+1)}+$$

$$+\frac{\|\overline{a}\|t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{t^\alpha}{\Gamma(\alpha+1)}\|g\|\|f_{in}\|\Big)E_\alpha(\|a\|t^\alpha)\Big\}E_\alpha(\|\widetilde{a}\|t^\alpha). \tag{24}$$

In next section it is studied the inverse problem as the problem of determining of functions $a(t), g(t)$ from relations (1)-(4), using the contraction mapping principle.

5. Solvability of inverse problem

Let us multiply (1) by $\omega_i(x), (i = 1, 2)$ and integrate over x from 0 to l:

$$\int_0^l \omega_i(x)\partial_t^\alpha u(x,t)dx - \int_0^l \omega_i(x)u_{xx}dx + a(t)\int_0^l \omega_i(x)u(x,t)dx =$$
$$= g(t)\int_0^l \omega_i(x)f(x,t)dx,\ i = 1, 2,\ (x,t) \in D_T.$$

After integrating by parts, in view of conditions (2)-(4) and (A4), we obtain the equality

$$\partial_{0+,t}^\alpha h_i(t) - \int_0^l \omega_i''(x)u(x,t)dx + a(t)h_i(t) = g(t)\int_0^l \omega_i(x)f(x,t)dx,\ i = 1, 2. \quad (25)$$

Solving the system (25) with respect to the unknown functions $a(t)$ and $g(t)$, we obtain the following integral equations with respect to the unknowns:

$$g(t) = \frac{1}{\Delta(t)}\sum_{\substack{k,j=1\\k\neq j}}^{2}(-1)^j h_j(t)\Big[\partial_{0+,t}^\alpha h_k(t) - \sum_{n=0}^{\infty} u_{1n}(t;a;g)\lambda_n^2\omega_{k1n}^{(2)} - \sum_{n=1}^{\infty} u_{2n}(t;a;g)\lambda_n^2\omega_{k2n}^{(2)}\Big], \quad (26)$$

$$a(t) = \frac{1}{\Delta(t)}\sum_{\substack{k,j=1\\k\neq j}}^{2}(-1)^j F_j(t)\Big[\partial_{0+,t}^\alpha h_k(t) - \sum_{n=0}^{\infty} u_{1n}(t;a;g)\lambda_n^2\omega_{k1n}^{(2)} - \sum_{n=1}^{\infty} u_{2n}(t;a;g)\lambda_n^2\omega_{k2n}^{(2)}\Big], \quad (27)$$

where

$$\Delta(t) = h_1(t)F_2(t) - h_2(t)F_1(t);\quad F_k(t) = \sum_{n=0}^{\infty} f_{1n}(t)\omega_{k1n} + \sum_{n=1}^{\infty} f_{2n}(t)\omega_{k2n};\quad k = 1, 2;$$

$$\omega_{k1n} = 2\int_0^1 \omega_k(x)\cos(\lambda_n x)dx;\quad \omega_{k1n}^{(2)} = 2\int_0^1 \omega_k''(x)\cos(\lambda_n x)dx.$$
$$\omega_{k2n} = 2\int_0^1 \omega_k(x)\sin(\lambda_n x)dx;\quad \omega_{k2n}^{(2)} = 2\int_0^1 \omega_k''(x)\sin(\lambda_n x)dx.$$

The main result of this work is presented as follows:

Theorem 5.1. *Let (A1)-(A3) are satisfied. Then there exists a number $T^* \in (0, T)$, such that there exists a unique solution $a(t),\ g(t) \in C[0, T^*]$ of the inverse problem* (1)-(4).

Proof. We consider the operator equation

$$\psi = \Lambda[\psi]. \quad (28)$$

where $\psi = (\psi_1, \psi_2) := (a(t); g(t))$ is unknown vector function. The components (Λ_1, Λ_2) of Λ are defined by the right hand sides of equations (26), (27):

$$\Lambda_1[\psi](t) = \psi_{01}(t) + \frac{1}{\Delta(t)}\sum_{\substack{k,j=1\\k\neq j}}^{2}(-1)^j F_j(t)\Big[\sum_{n=0}^{\infty} u_{1n}(t;a;g)\omega_{k1n}^{(2)} + \sum_{n=1}^{\infty} u_{2n}(t;a;g)\omega_{k2n}^{(2)}\Big], \quad (29)$$

$$\Lambda_2[\psi](t) = \psi_{02}(t) + \frac{1}{\Delta(t)}\sum_{\substack{k,j=1\\k\neq j}}^{2}(-1)^j h_j(t)\Big[\sum_{n=0}^{\infty} u_{1n}(t;a;g)\omega_{k1n}^{(2)} + \sum_{n=1}^{\infty} u_{2n}(t;a;g)\omega_{k2n}^{(2)}\Big]. \quad (30)$$

Let $\psi_0 := (\psi_{01}, g_{02})$, where

$$\psi_{01}(t) = \frac{1}{\Delta(t)} \sum_{\substack{k,j=1 \\ k\neq j}}^{2} (-1)^j F_j(t)\partial^{\alpha}_{0+,t} h_k(t),$$

$$\psi_{02}(t) = \frac{1}{\Delta(t)} \sum_{\substack{k,j=1 \\ k\neq j}}^{2} (-1)^j h_j(t)\partial^{\alpha}_{0+,t} h_k(t).$$

Consider the functional space of vector functions $\psi \in C(D_T)$ with the norm given by the relation

$$\|\psi\| = \max\{\max_{t\in[0,T]} |\psi_1(t)|, \max_{t\in[0,T]} |\psi_2(t)|\}.$$

Fix a number $\rho > 0$ and consider the ball

$$\Phi^T(g_0, \rho) := \{\psi : \|\psi - \psi_0\|_{C[0,T]} \leq \rho\}.$$

Let us prove for an enough small $T > 0$ the operator Λ maps the ball $\Phi^T(\psi_0, \rho)$ into itself. For this purpose, using the estimates (14),(15) for u_{1n}, u_{2n} respectively, we find following estimates

$$\|\Lambda_1[\psi](t) - \psi_{01}(t)\| \leq \frac{2F_0\omega_0}{\delta}\left|\sum_{n=0}^{\infty} u_{1n}(T;\psi_1;\psi_2) + \sum_{n=1}^{\infty} u_{2n}(T;\psi_1;\psi_2)\right|$$

$$\leq \frac{2\omega_0 F_0}{\delta}\Big[(|\varphi_{10}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_1\|\|f_{10}\|) + \sum_{i=1}^{2}\left(|\varphi_{in}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_1\|\|f_{in}\|\right)\Big]E_\alpha(\|\psi_2\|T^\alpha),$$

$$\|\Lambda_2[\psi](t) - \psi_{02}(t)\| \leq \frac{2\omega_0 h_0}{\delta}\left|\sum_{n=0}^{\infty} u_{1n}(T;\psi_1;\psi_2) + \sum_{n=1}^{\infty} u_{2n}(T;\psi_1;\psi_2)\right|$$

$$\leq \frac{2\omega_0 h_0}{\delta}\Big[(|\varphi_{10}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_1\|\|f_{10}\|) + \sum_{i=1}^{2}\sum_{n=1}^{\infty}\left(|\varphi_{in}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_1\|\|f_{in}\|\right)\Big]E_\alpha(\|\psi_2\|T^\alpha),$$

where $\delta = \min\{|\Delta|\} > 0$, $\delta = const.$, $F_0 = \|F_k(t)\|_{C[0,T]}$, $h_0 = \max\{\|h_i\|_{C^1[0,l]}\}, i = 1,2, f_0 = \|f_{in}\|_{C(\overline{D}_T)}, \omega_0 = \|\omega_i\|_{C^2[0,1]}$.

According to Lemmas 4.1 and 4.2, the above series is a convergent series. Note that the functions occurring on the right-hand side in these inequalities are monotone increasing with T, and the fact that the function $\psi(t)$ belongs to the ball $\Phi^T(\psi_0, \rho)$ implies the inequality

$$\|\psi\| \leq \rho + \|\psi_0\|. \tag{31}$$

Therefore, we only strengthen the inequality if we replace $\|\psi\|$ in these inequalities with the relation $\rho + \|\psi_0\|$. Performing these replacements, we obtain the estimate

$$\|\Lambda_1[\psi](t) - \psi_{01}(t)\| \leq \frac{2F_0\omega_0}{\delta}\Big[(|\varphi_{10}| + \frac{T^\alpha}{\Gamma(\alpha+1)}(\rho + \|\psi_0\|)\|f_{10}\|) +$$

$$+ \sum_{i=1}^{2}\sum_{n=1}^{\infty}\left(|\varphi_{in}| + \frac{T^\alpha}{\Gamma(\alpha+1)}(\rho + \|\psi_0\|)\|f_{in}\|\right)\Big]E_\alpha((\rho + \|\psi_0\|)T^\alpha),$$

$$\|\Lambda_2[\psi](t) - \psi_{02}(t)\| \leq \frac{2\omega_0 h_0}{\delta}\Big[(|\varphi_{10}| + \frac{T^\alpha}{\Gamma(\alpha+1)}(\rho + \|\psi_0\|)\|f_{10}\|) +$$

$$+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\left(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{in}\|\right)\Big]E_\alpha((\rho+\|\psi_0\|)T^\alpha).$$

These relations together with (23) and (24), (25) imply the estimates

$$\|\Lambda[\psi](t)-\psi_0(t)\|=\max\{\|\Lambda_1[\psi](t)-\psi_{01}(t)\|,\ \|\Lambda_2[\psi](t)-\psi_{02}(t)\|\}\le$$

$$\le\max\left\{\frac{2F_0\omega_0}{\delta},\ \frac{2\omega_0 h_0}{\delta}\right\}\Big[(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{10}\|)+$$

$$+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\left(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{in}\|\right)\Big]E_\alpha((\rho+\|\psi_0\|)T^\alpha).$$

Let T_1 be a positive root of the equation

$$\max\left\{\frac{2F_0\omega_0}{\delta},\ \frac{2\omega_0 h_0}{\delta}\right\}\Big[(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{10}\|)+$$

$$+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\left(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{in}\|\right)\Big]E_\alpha((\rho+\|\psi_0\|)T^\alpha)=\rho.$$

Then for $T\in[0,T_1]$ we have $\Lambda[\psi](t)\in\Phi^T(\psi_0,\rho)$.

Now consider two functions $\psi(t)$ and $\widetilde{\psi}(t)$ belonging $\Phi^T(\psi_0,\rho)$ and estimate the distance between their images $\Lambda[\psi](t)$ and $\Lambda[\widetilde{\psi}](t)$ in the space $C[0,T]$. The function $\widetilde{u}_n(t)$ corresponding to $\widetilde{\psi}(t)$ satisfies the integral equation (20),(21) with the functions $\varphi_n=\widetilde{\varphi}_n$ and $f_n=\widetilde{f}_n$. Composing the difference $\Lambda[\psi](t)-\Lambda[\widetilde{\psi}](t)$ with the help of equations (12),(13), (20),(21) and then estimating its norm, we get

$$\|\Lambda_1[\psi](t)-\Lambda_1[\widetilde{\psi}](t)\|\le$$

$$\le\frac{2F_0\omega_0}{\delta}\left(\sum_{n=0}^{\infty}\left\|u_{1n}(T;\psi_1;\psi_2)-\widetilde{u}_{1n}(T;\widetilde{\psi}_1;\widetilde{\psi}_2)\right\|+\sum_{n=1}^{\infty}\left\|u_{2n}(T;\psi_1;\psi_2)-\widetilde{u}_{2n}(T;\widetilde{\psi}_1;\widetilde{\psi}_2)\right\|\right)\le$$

$$\le\frac{2F_0\omega_0}{\delta}\Big\{|\overline{\varphi}_{10}|+\frac{T^\alpha\|\overline{f}_{10}\|\|\widetilde{\psi}_2\|}{\Gamma(\alpha+1)}+\frac{T^\alpha\|f_{10}\|\|\overline{\psi}_2\|}{\Gamma(\alpha+1)}+$$

$$+\frac{\|\overline{\psi}_2\|t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_2\|\|f_{10}\|\Big)E_\alpha(\|\psi_1\|T^\alpha)+$$

$$+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Big\{|\overline{\varphi}_{in}|+\frac{T^\alpha\|\overline{f}_{in}\|\|\widetilde{\psi}_2\|}{\Gamma(\alpha+1)}+\frac{T^\alpha\|f_{in}\|\|\overline{\psi}_2\|}{\Gamma(\alpha+1)}+$$

$$+\frac{\|\overline{\psi}_1\|T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_2\|\|f_{in}\|\Big)E_\alpha(\|\psi_1\|T^\alpha)\Big\}\Big\}E_\alpha(\|\widetilde{\psi}_1\|T^\alpha).\tag{32}$$

$$\|\Lambda_2[\psi](t)-\Lambda_2[\widetilde{\psi}](t)\|\le$$

$$\le\frac{2h_0\omega_0}{\delta}\left(\sum_{n=0}^{\infty}\left\|u_{1n}(T;\psi_1;\psi_2)-\widetilde{u}_{1n}(T;\widetilde{\psi}_1;\widetilde{\psi}_2)\right\|+\sum_{n=1}^{\infty}\left\|u_{2n}(T;\psi_1;\psi_2)-\widetilde{u}_{2n}(T;\widetilde{\psi}_1;\widetilde{\psi}_2)\right\|\right)\le$$

$$\le\frac{2h_0\omega_0}{\delta}\Big\{|\overline{\varphi}_{10}|+\frac{T^\alpha\|\overline{f}_{10}\|\|\widetilde{\psi}_2\|}{\Gamma(\alpha+1)}+\frac{T^\alpha\|f_{10}\|\|\overline{\psi}_2\|}{\Gamma(\alpha+1)}+$$

$$+\frac{\|\overline{\psi}_2\|T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{t^\alpha}{\Gamma(\alpha+1)}\|\psi_2\|\|f_{10}\|\Big)E_\alpha(\|\psi_1\|T^\alpha)+$$

$$+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Big\{|\overline{\varphi}_{in}|+\frac{T^\alpha\|\overline{f}_{in}\|\|\widetilde{\psi}_2\|}{\Gamma(\alpha+1)}+\frac{T^\alpha\|f_{in}\|\|\overline{\psi}_2\|}{\Gamma(\alpha+1)}+$$
$$+\frac{\|\overline{\psi}_1\|T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_2\|\|f_{in}\|\Big)E_\alpha(\|\psi_1\|T^\alpha)\Big\}\Big\}E_\alpha(\|\widetilde{\psi}_1\|T^\alpha). \tag{33}$$

Using inequality (14),(15) and the estimate (23) with $\|\overline{\psi}\|=\|\psi-\widetilde{\psi}\|=\max\{\|\psi_1-\widetilde{\psi}_1\|,\|\psi_2-\widetilde{\psi}_2\|\}$, $\varphi_n=\widetilde{\varphi}_n$ and $f_n=\widetilde{f}_n$, we continue the previous inequality in following form:

$$\|\Lambda_1[\psi](t)-\Lambda_1[\widetilde{\psi}](t)\|\le\frac{2F_0\omega_0}{\delta}\Big[\frac{T^\alpha\|f_{10}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_2\|\|f_{10}\|\Big)E_\alpha(\|\psi_1\|T^\alpha)+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Big\{\frac{T^\alpha\|f_{in}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_2\|\|f_{in}\|\Big)E_\alpha(\|\psi_1\|T^\alpha)\Big\}\Big]E_\alpha(\|\widetilde{\psi}_1\|T^\alpha)\|\psi-\widetilde{\psi}\|. \tag{34}$$

$$\|\Lambda_2[\psi](t)-\Lambda_2[\widetilde{\psi}](t)\|\le\frac{2h_0\omega_0}{\delta}\Big[\frac{T^\alpha\|f_{10}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_2\|\|f_{10}\|\Big)E_\alpha(\|\psi_1\|T^\alpha)+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Big\{\frac{T^\alpha\|f_{in}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}\|\psi_2\|\|f_{in}\|\Big)E_\alpha(\|\psi_1\|T^\alpha)\Big\}\Big]E_\alpha(\|\widetilde{\psi}_1\|T^\alpha)\|\psi-\widetilde{\psi}\|. \tag{35}$$

Because of $\psi(t)$ and $\widetilde{\psi}(t)$ belong to the ball $\Phi^T(g_0,\rho)$, then for these functions takes place the inequality (31). Note that the functions on the right-hand side in inequality (28) at the factor $\|\overline{\psi}\|$ is monotone increasing with $\|\psi\|,\|\widetilde{\psi}\|$, and T. Consequently, replacing $\|\psi\|$ and $\|\widetilde{\psi}\|$ in inequality (28) with $\rho+\|\psi_0\|$ will only strengthen the inequality. In this way, we obtain

$$\|\Lambda_1[\psi](t)-\Lambda_1[\widetilde{\psi}](t)\|\le\frac{2F_0\omega_0}{\delta}\Big[\frac{T^\alpha\|f_{10}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{10}\|\Big)E_\alpha((\rho+\|\psi_0\|)T^\alpha)+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Big\{\frac{T^\alpha\|f_{in}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{in}\|\Big)E_\alpha((\rho+\|\psi_0\|)T^\alpha)\Big\}\Big]E_\alpha((\|\rho+\|\psi_0\|)T^\alpha)\|\overline{\psi}\|.$$
$$\|\Lambda_2[\psi](t)-\Lambda_2[\widetilde{\psi}](t)\|\le\frac{2h_0\omega_0}{\delta}\Big[\frac{T^\alpha\|f_{10}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{10}\|\Big)E_\alpha((\rho+\|\psi_0\|)T^\alpha)+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Big\{\frac{T^\alpha\|f_{in}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{in}\|\Big)E_\alpha((\rho+\|\psi_0\|)T^\alpha)\Big\}\Big]E_\alpha((\|\rho+\|\psi_0\|)T^\alpha)\|\overline{\psi}\|.$$

Hence,

$$\|\Lambda[\psi](t)-\Lambda[\widetilde{\psi}](t)\|\le\max\left\{\frac{2\omega_0F_0}{\delta},\ \frac{2\omega_0h_0}{\delta}\right\}\Big[\frac{T^\alpha\|f_{10}\|}{\Gamma(\alpha+1)}+$$
$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi0\|)\|f_{10}\|\Big)E_\alpha((\rho+\|\psi_0\|)T^\alpha)+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Big\{\frac{T^\alpha\|f_{in}\|}{\Gamma(\alpha+1)}+$$

$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{in}\|\Big)E_\alpha((\rho+\|\psi_0\|)T^\alpha)\Big\}\Big]E_\alpha((\|\rho+\|\psi_0\|)T^\alpha)\|\overline{\psi}\|.$$

Let T_2 be a positive root of the equation

$$\max\left\{\frac{2\omega_0 F_0}{\delta},\ \frac{2\omega_0 h_0}{\delta}\right\}\Big[\frac{T^\alpha\|f_{10}\|}{\Gamma(\alpha+1)}+$$

$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{10}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{10}\|\Big)E_\alpha((\rho+\|\psi_0\|)T^\alpha)+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Big\{\frac{T^\alpha\|f_{in}\|}{\Gamma(\alpha+1)}+$$

$$+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{in}|+\frac{T^\alpha}{\Gamma(\alpha+1)}(\rho+\|\psi_0\|)\|f_{in}\|\Big)E_\alpha((\rho+\|\psi_0\|)T^\alpha)\Big\}\Big]E_\alpha((\|\rho+\|\psi_0\|)T^\alpha)=1.$$

Because, it is a transcendental equation and on the right side the function is a monotonic increasing function, besides (0,0) point belongs to the function. So, the above equation has positive roots.

Then for $T\in[0,T_2)$ the operator Λ contracts the distance between the elements $\psi(t),\ \widetilde{\psi}(t)\in\Phi^T(\psi_0,\rho)$. Consequently, if we choose $T^*<\min(T_1,T_2)$ then the operator Λ is a contraction in the ball $\Phi^T(\psi_0,\rho)$. However, in accordance with the Banach theorem (see [[28], pp. 87-97]), the operator Λ has unique fixed point in the ball $\Phi^T(\psi_0,\rho)$; i.e., there exists a unique solution of equation (29). □

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Jonibek Jumaev Jamolovich completed his PhD in Differential equations and mathematical physics at Bukhara State University. Currently, he is a post-doctor student at the Institute of Mathematics named after V.I. Romanovskiy and an assistant professor in the Differential Equations Department at Bukhara State University. His primary research focuses on direct and inverse problems for diffusion equations.

Durdimurod Durdiev Kalandarovich received a Doctor of Science degree at the Institute of Mathematics and Information Technologies, Academy of Sciences of Uzbekistan in 2010. His research interests are the inverse problems for whole and fractional differential, integro-differential equations. Currently, he works as a Professor at Bukhara State University and is the Head of the Bukhara branch of the Institute of Mathematics named after V.I. Romanovskiy.

Zavqiddin Bozorov Ravshanovich completed his PhD in Differential equations and mathematical physics at Bukhara State University. He is a senior researcher at the Bukhara branch of the Institute of Mathematics named after V.I. Romanovskiy and an assistant professor in the Differential Equations Department at Bukhara State University. His primary research focuses on direct and inverse problems for wave-diffusion equations.