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# EXTRACTING TRIPLE CONNECTED CERTIFIED DOMINATION NUMBER FOR THE STRONG PRODUCT OF PATHS AND CYCLES

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ABSTRACT. A dominating set $S$ of a graph $G$ is said to be a triple connected certified dominating set (TCCD - set) if for every vertex $v \in S$, $\mid N(v) \cap (V - S) \mid \neq 1$ and $\langle S \rangle$ is triple connected. The minimum cardinality of a TCCD - set is called the triple connected certified domination number (TCCD - number) and is denoted by $\gamma_{TCC}(G)$. The novelty of triple connected certified domination number is which the certified domination holds the triple connected in induced $S$. The upper bound and loweer bound of $\gamma_{TCC}$ for the given graphs is found and then proved that the upper bound and lower bound of $\gamma_{TCC}$ were equal. This article investigates the TCCD number for the strong product of paths and cycles.

Keywords: Domination number, certified domination, triple connected, triple connected certified domination, product graphs, cycle.

AMS Subject Classification: 05C38, 05C69

## 1. Introduction

The graph $H = (V(H), E(H))$ under consideration is finite, simple, non-trivial and undirected. A set [1] $S \subseteq V(H)$ is termed a dominating set if every vertex not in $S$ has a neighbor in $S$. The minimum cardinality of a dominating set is known as the domination number $\gamma(H)$. Following this, G. Mahadevan et al.[2][3] introduced the triple connected domination number. A dominating set $S$ is said to be a triple connected dominating set if the induced subgraph $\langle S \rangle$ $\{$*If $S$ is a subset of $G'$s nodes, then the subgraph of $G$ induced by $S$ is the graph that has $S$ as its set of vertices and contains all the edges of $G$ that have both endpoints in $S$*$\}$ is triple connected, i.e., any three vertices of $\langle S \rangle$ lie on a path. The minimum cardinality of a triple connected dominating set is called the triple connected domination number and is denoted by $\gamma_{tc}(H)$. Recently, M. Detlaff et al. [4] proposed a new parameter, a subset $S \subseteq V(H)$ is said to be certified dominating set if it is a dominating set and any vertex $v \in S$ does not have one neighbor in $V - S$.

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The minimum cardinality taken among all the certified dominating sets is known as the certified domination number and is denoted by $\gamma_{cer}(H)$. Building upon these concepts, the authors in [5] introduced the triple connected certified domination number, and in [6], they obtained the $\gamma_{TCC}$ number for the strong product of a path and a path. A dominating set $S$ is said to be a triple connected certified dominating set (TCCD-set), with the condition that every vertex v∈ S , $\mid N(v) \cap (V - S) \mid \neq 1$ and any three vertices of $S$ lie on a path in $\langle S \rangle$. If $D$ is a TCCD-set of $H$ with minimum cardinality, then the triple connected certified domination number is $\gamma_{TCC} = \mid D \mid$. Let $V(H)$ and $E(H)$ represent the vertex and edge set of $H$. Let $P_n$ denote the path on $n$ vertices and $C_m$ indicate the cycle of length $m$. We introduced the concept of TCCD-number and provided exact values for standard and particular types of graphs[5]. We also derived results for Cartesian, strong, lexicographic, corona product of a path with a path and generalized TCCD numbers. We show that the TCCD-set does not exist for the tensor product of graphs due to the failure to satisfy the triple connected property[6]. Additionally, we examine distances in graphs for various particular types. The power graph of a graph $H^g$ denotes what $H$ has the same vertices as in $H$ and $d_H(u,v) \leq g$, where $H^2$ is the square graph and $H^3$ is the cube graph[7]. Many other researchers have recently focused on certified domination by imposing conditions in induced $S$ or $V - S$ like connected certified domination.[8]-[10]. Our study includes results for the $\gamma_{TCC}$ number concerning the strong product of a path and a cycle and a cycle. We observe that while a path $P_{10}$ and a cycle $C_{10}$ have the same number of vertices, the cycle has one more edge than the path, resulting in a lower $\gamma_{TCC}$ domination number. The innovation of this article lies in the concept of triple connected certified domination, where the certified domination ensures the interconnectedness of the triple within the $\langle S \rangle$. The strong product combines the cartesian product [11] and tensor product[12]. The strong product [13] $G \times H$ is the connected loopless graph with $V(G \times H) = V(G) \times V(H)$ as its vertex set and two vertices $(g,h)$ and $g', h'$ are adjacent in $G \times H$ if and only if $\begin{cases} g = g' \textit{ and } h, h' \textit{ are adjacent}, \\ h = h' \textit{ and } g, g' \textit{ are adjacent}, \\ g \textit{ is adjacent to } h \textit{ and } g' \textit{ is adjacent to } h'. \end{cases}$ This product is also referred to as the AND or Normal product. Section 2 discusses the strong product of a path $P_r$ and a cycle $C_s$, while Section 3 focuses on the strong product of two cycles $C_r$ and $C_s$.

## 2. TCC domination for Strong product of path and cycle

This section covers various outcomes concerning the strong product, culminating in the determination of precise values for $\gamma_{TCC}$ on paths and cycles. Let $V(P_r \times C_s) = \{v_p, v_q : 1 \leq p \leq n, 1 \leq q \leq s\}$ and $E(P_r \times C_s) = \{v_p v_q \ v_{p+1} v_q : 1 \leq p \leq r-1, 1 \leq q \leq s\} \cup \{v_p v_q \ v_p v_{q+1} : 1 \leq p \leq r, 1 \leq q \leq s-1\} \cup \{v_p v_1 \ v_p v_s : 1 \leq p \leq r\} \cup \{v_p v_q \ v_{p+1} v_{q+1} : 1 \leq p \leq r-1, 1 \leq q \leq s-1\} \cup \{v_p v_q \ v_{p+1} v_{q-1} : 1 \leq p \leq r-1, s \leq q \leq 2\} \cup \{v_p v_1 \ v_{p+1} v_s : 1 \leq p \leq r-1\} \cup \{v_p v_s \ v_{p+1} v_1 : 1 \leq p \leq r-1\}$ of $P_r \times C_s$.

**Observation**

(1) Since $P_1 \times C_s$ is disconnected, then $\gamma_{TCC}(P_1 \times C_s)$ is impossible.
(2) If $s \geq 3$, then $\gamma_{TCC}(P_2 \times C_s) = s - 2$.
(3) If $s \geq 3$, then $\gamma_{TCC}(P_3 \times C_s) = s - 2$.
(4) If $s \geq 3$, then $\gamma_{TCC}(P_4 \times C_s) = s$.
(5) If $s = 3l, s \geq 6$, then $\gamma_{TCC}(P_5 \times C_s) = \frac{4s}{3}$.
(6) If $s = 3l + 1, s \geq 7$, then $\gamma_{TCC}(P_5 \times C_s) = 3\lceil \frac{s}{3} \rceil + \lfloor \frac{s}{3} \rfloor - 2$.
(7) If $r = 3l + 2, r \geq 5$, then $\gamma_{TCC}(P_r \times C_5) = 4\lfloor \frac{r}{3} \rfloor + 3$.

(8) If $r \geq 6$, then

$$\gamma_{TCC}(P_r \times C_6) = \begin{cases} 5\frac{r}{4} + 2\lfloor\frac{r}{5}\rfloor & if\ r = 4l, \\ 5\lfloor\frac{r}{4}\rfloor + 2\lceil\frac{r}{5}\rceil & if\ r = 4l+1, \\ 5\lceil\frac{r}{4}\rceil + 2(\lceil\frac{r}{5}\rceil - 1) & if\ r = 4l+2, \\ 5\lceil\frac{r}{4}\rceil + 2\lfloor\frac{r}{5}\rfloor + 1 & if\ r = 4l+3. \end{cases}$$

(9) $\gamma_{TCC}(P_{10} \times C_{10}) = 29.$

**Theorem 2.1.** *If $r \geq 7$, then $\gamma_{TCC}(P_r \times C_7)$*

$$= \begin{cases} (s-2)\frac{r}{4} + 3\lfloor\frac{r}{9}\rfloor + 3\lceil\frac{r}{9}\rceil + 1 & if\ r = 8l, \\ (s-2)\lfloor\frac{r}{4}\rfloor + 6 + 6\lfloor\frac{r}{9}\rfloor & if\ r = 8l+1, \\ (s-2)\lceil\frac{r}{4}\rceil + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 1 & if\ r = 8l+2, \\ (s-2)\lceil\frac{r}{4}\rceil + 6\lfloor\frac{r}{9}\rfloor & if\ r = 8l+3, \\ (s-2)\frac{r}{4} + 6\lfloor\frac{r}{9}\rfloor + 1 & if\ r = 8l+4, \\ (s-2)\lfloor\frac{r}{4}\rfloor + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 7 & if\ r = 8l+5, \\ (s-2)\lceil\frac{r}{4}\rceil + 6\lfloor\frac{r}{9}\rfloor + 2 & if\ r = 8l+6, \\ (s-2)\lceil\frac{r}{4}\rceil + 3\lceil\frac{r}{9}\rceil + 3(\lceil\frac{r}{9}\rceil - 1) & if\ r = 8l+7. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : p = 8l+0\ or\ 8l+1\ or\ 8l+7, 7 \leq p \leq r-2, q = 1\}, S_2 = \{v_p, v_q : p = 8l+3\ or\ 8l+4\ or\ 8l+5\ , 3 \leq p \leq r, q = s\}, S_3 = \{v_p, v_q : p = 8l+2, j = 2l\}, S_4 = \{v_p, v_q : p = 8l+3, q = 2l+1, 3 \leq q \leq s-1\}, S_5 = \{v_p, v_q : p = 8l+6, 6 \leq p \leq r-3, q = 2l\}, S_6 = \{v_p, v_q : p = 8l+7, 6 \leq p \leq r-3, q = 2l+1, 3 \leq q \leq s-1\}, S_7 = \{v_p, v_q : p = r-2, q = 3, 5, 6\} \cup \{v_p, v_q : p = r-1, q = 2, 4, 6\}, S_8 = \{v_p, v_q : p = 8l+3\ or\ 8l+7, 7 \leq p \leq r-7, q = 2l\} \cup \{v_{r-5}, v_2\}, S_9 = \{v_p, v_q : p = 8l\ or\ 8l+4, q = 3, 8 \leq p \leq r-6\} \cup \{v_{r-6}, v_3\}, S_{10} = \{v_p, v_q : p = 8l+2\ or\ 8l+6, q = 5, 6 \leq pleqr-3\}, S_{11} = \{v_P, v_q : q = s, p = 8l\ or\ 8l+1\ or\ 8l+2, 8 \leq p \leq r-2\}, S_{12} = \{v_p, v_q : p = 8l+4\ or\ 8l+5\ or\ 8l+6, 12 \leq p \leq r-6, q = 1\} \cup \{v_p, v_q : q = 1, p = 2, 6, r-4, r-3, r-2, i \geq r-2\}, S_{13} = \{v_P, v_q : p = r-6, q = 3, 4, 6\} \cup \{v_p, v_q : p = r-7, q = 5\} \cup \{v_p, v_q : i = r-5, q = 1\} \cup \{v_P, v_2 : p = r-1, 5\} \cup \{v_p, v_5 : p = 2, r-1, r-2, r-3\} \cup \{v_p, v_6 : p = 3, r-1\} \cup \{v_2, v_s\} \cup \{v_2, v_1\} \cup \{v_6, v_1\}, S_{14} = \{v_p, v_q : p = 8l+2\ or\ 8l+6, 2 \leq p \leq r-5, q = 2l\} \cup \{v_{r-1}, v_2\}, S_{15} = \{v_p, v_q : p = 8l+3\ or\ 8l+7, 3 \leq p \leq r-5, q = 2l+1, 3 \leq q \leq s-1\} \cup \{v_{r-3}, v_4\}, S_{16} = \{v_p, v_q : q = s, p = 8l+3\ or\ 8l+4\ or\ 8l+5, 3 \leq p \leq r-1\} \cup \{v_{r-2}, v_5\}, S_{17} = \{v_p, v_q : q = 1, p = 8l\ or\ 8l+1\ or\ 8l+7, 7 \leq p \leq r-4\} \cup \{v_p, v_q : q = s, p = r-3, r-2\}, S_{18} = \{v_p, v_q : q = s-1, p = r-1, r-4\} \cup \{v_p, v_q : q = 3, r-4 \leq p \leq r-1\}, S_{19} = \{v_p, v_q : q = 2l, p = 8l+2\ or\ 8l+6, 2 \leq p \leq r-2)\}, S_{20} = \{v_p, v_q : q = 2l+1, p = 8l+3\ or\ 8l+7, 3 \leq p \leq r-2\} \cup \{v_p, v_q : p = r-1, 2 \leq q \leq s-1\}, S_{21} = \{v_p, v_q : q = s, p = 8l+3\ or\ 8l+4\ or\ 8l+5\} \cup \{v_p, v_q : q = 1, p = 8l\ or\ 8l+1\ or\ 8l+7, 7 \leq p \leq r-1\}, S_{22} = \{v_p, v_q : q = 2l, p = 8l+2\ or\ 8l+6, 2 \leq p \leq r\} \cup \{v_p, v_q : q = 2l+1, p = 8l+3\ or\ 8l+7, 3 \leq p \leq r\}, S_{23} = \{v_p, v_q : p = 8l\ or\ 8l+1\ or\ 8l+2, q = s, 8 \leq p \leq r-1\} \cup \{v_p, v_q : q = 1, p = 8l+4\ or\ 8l+5\ or\ 8l+6, 12 \leq p \leq r-2\}, S_{24} = \{v_p, v_q : q = 2l, p = 8l+3\ or\ 8l+7\ (mod\ 8), 3 \leq p \leq r\} \cap \{v_3, v_2\}, S_{25} = \{v_p, v_q : q = 3, p = 8l\ or\ 8l+4, 4 \leq p \leq r-1\} \cup \{v_p, v_q : q = 5, p = 8l+2\ or\ 8l+6, 2 \leq p \leq r-1\}, S_{26} = \{v_p, v_1 : p = 2, 6\} \cup \{v_p, v_2 : p = 5, r-1\} \cup \{v_2, v_q : q = 4, 7\} \cup \{v_{r-2}, v_5\} \cup \{v_{r-1}, v_6\}, S_{27} = \{v_p, v_q : p = 8l+2\ or\ 8l+6, 2 \leq p \leq r-5, q = 2l\} \cup \{v_p, v_q : p = 8l+3\ or\ 8l+7 (mod\ 8), 3 \leq p \leq r-5, q = 3l+1, 3 \leq q \leq s-1\}, S_{28} = \{v_p, v_q : q = s, p = 8l+3\ or\ 8l+4\ or\ 8l+5, 3 \leq p \leq r-4\} \cup \{v_p, v_q : q = 1, p = 8l\ or\ 8l+1\ or\ 8l+7, 7 \leq p \leq r\} \cup \{v_{r-4}, v_q : q = 2, 5, 6\}, S_{29} = \{v_{r-3}, v_q : q = 1, 4, 5\} \cup \{v_{r-2}, v_q : q = 1, 3, 5\} \cup \{v_r - 1, v_q : q = 2, 5, 6\}.$

$$\text{Then } S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 & if\ r = 8l, \\ S_8 \cup S_9 \cup S_{10} \cup S_{11} \cup S_{12} \cup S_{13} & if\ r = 8l+1, \\ S_{14} \cup S_{15} \cup S_{16} \cup S_{17} \cup S_{18} & if\ r = 8l+2, \\ S_{19} \cup S_{20} \cup S_{21} & if\ r = 8l+3 \ or\ 8l+7, \\ S_{21} \cup S_{22} & if\ r = 8l+4, \\ S_{23} \cup S_{24} \cup S_{25} \cup S_{26} & if\ r = 8l+5, \\ S_{27} \cup S_{28} & if\ r = 8l+6, \end{cases}$$

clearly $S$ is a TCCD-set of $P_r \times C_s$ and hence $\gamma_{TCC}(P_r \times C_s) \leq$

$$| S | = \begin{cases} (s-2)\frac{r}{4} + 3\lfloor\frac{r}{9}\rfloor + 3\lceil\frac{r}{9}\rceil + 1 & if\ r = 8l, \\ (s-2)\lfloor\frac{r}{4}\rfloor + 6 + 6\lfloor\frac{r}{9}\rfloor & if\ r = 8l+1, \\ (s-2)\lceil\frac{r}{4}\rceil + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 1 & if\ r = 8l+2, \\ (s-2)\lceil\frac{r}{4}\rceil + 6\lfloor\frac{r}{9}\rfloor & if\ r = 8l+3, \\ (s-2)\frac{r}{4} + 6\lfloor\frac{r}{9}\rfloor + 1 & if\ r = 8l+4, \\ (s-2)\lfloor\frac{r}{4}\rfloor + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 7 & if\ r = 8l+5, \\ (s-2)\lceil\frac{r}{4}\rceil + 6\lfloor\frac{r}{9}\rfloor + 2 & if\ r = 8l+6, \\ (s-2)\lceil\frac{r}{4}\rceil + 3\lceil\frac{r}{9}\rceil + 3(\lceil\frac{r}{9}\rceil - 1) & if\ r = 8l+7. \end{cases}$$

Assume a TCCD-set $D \subseteq P_r \times C_s$ exists of cardinality

$$\text{at most } d = \begin{cases} (s-2)\frac{r}{4} + 3\lfloor\frac{r}{9}\rfloor + 3\lceil\frac{r}{9}\rceil & if\ r = 8l, \\ (s-2)\lfloor\frac{r}{4}\rfloor + 6 + 6\lfloor\frac{r}{9}\rfloor - 1 & if\ r = 8l+1, \\ (s-2)\lceil\frac{r}{4}\rceil + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) & if\ r = 8l+2, \\ (s-2)\lceil\frac{r}{4}\rceil + 6\lfloor\frac{r}{9}\rfloor - 1 & if\ r = 8l+3, \\ (s-2)\frac{r}{4} + 6\lfloor\frac{r}{9}\rfloor & if\ r = 8l+4, \\ (s-2)\lfloor\frac{r}{4}\rfloor + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 6 & if\ r = 8l+5, \\ (s-2)\lceil\frac{r}{4}\rceil + 6\lfloor\frac{r}{9}\rfloor + 1 & if\ r = 8l+6, \\ (s-2)\lceil\frac{r}{4}\rceil + 3\lceil\frac{r}{9}\rceil + 3(\lceil\frac{r}{9}\rceil - 1) - 1 & if\ r = 8l+7. \end{cases}$$

whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(P_r \times C_s) \geq d + 1 \ = \begin{cases} (s-2)\frac{r}{4} + 3\lfloor\frac{r}{9}\rfloor + 3\lceil\frac{r}{9}\rceil + 1 & if\ r = 8l, \\ (s-2)\lfloor\frac{r}{4}\rfloor + 6 + 6\lfloor\frac{r}{9}\rfloor & if\ r = 8l+1, \\ (s-2)\lceil\frac{r}{4}\rceil + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 1 & if\ r = 8l+2, \\ (s-2)\lceil\frac{r}{4}\rceil + 6\lfloor\frac{r}{9}\rfloor & if\ r = 8l+3, \\ (s-2)\frac{r}{4} + 6\lfloor\frac{r}{9}\rfloor + 1 & if\ r = 8l+4, \\ (s-2)\lfloor\frac{r}{4}\rfloor + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 7 & if\ r = 8l+5, \\ (s-2)\lceil\frac{r}{4}\rceil + 6\lfloor\frac{r}{9}\rfloor + 2 & if\ r = 8l+6, \\ (s-2)\lceil\frac{r}{4}\rceil + 3\lceil\frac{r}{9}\rceil + 3(\lceil\frac{r}{9}\rceil - 1) & if\ r = 8l+7. \end{cases}$$

Hence the result follows. □

**Theorem 2.2.** *If* $r \geq 8$, $r = 4l$ *and* $s \neq 4l, s \leq r-4$ *then*

$$\gamma_{TCC}(P_r \times C_s) = \begin{cases} (s-2)\frac{r}{4} + 6\lfloor\frac{r}{9}\rfloor + 3 & if\ r = 8l, \\ (s-2)\frac{r}{4} + 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 3 & if\ r = 8l+4. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : q = 2l, 4 \leq q \leq s-3, p = 2, 6\} \cup \{v_p, v_q : q = 2l+1, 3 \leq q \leq s-4, p = 3, 7\}$, $S_2 = \{v_p, v_{q-2} : p = 2, 6\} \cup \{v_p, v_{q-1} : p = 3, 7\} \cup \{v_2, v_q\} \cup \{v_p, v_1 : p = 1, 5\} \cup \{v_p, v_2 : p = 4, 6\}$, $S_3 = \{v_p, v_q : q = s, p = 8l+0 \ or\ 8l+1 \ or\ 8l+2, 8 \leq p \leq r-2\} \cup \{v_p, v_q : p = 8l+4 \ or\ 2l+5 \ or\ 8l+6, 12 \leq p \leq r-2, q = 1\}$, $S_4 = \{v_p, v_q : p = 4l+2, 10 \leq p \leq r-2, q = 2l+1, 3 \leq q \leq s-2\} \cup \{v_p, v_q : p = 4l+3, 11 \leq p \leq n-1, q = 2l, 2 \leq q \leq s-1\}$, $S_5 = \{v_p, v_q : q = 2l, 4 \leq q \leq s-2, p = 2, 6\} \cup \{v_p, v_q : q = 2l+1, 3 \leq q \leq s-1, p = 3, 7\} \cup \{v_2, v_s\}$, $S_6 = \{v_p, v_1 : p = 1, 5\} \cup \{v-p, v_2 : p = 4, 6\} \cup \{v_p, v_q : p = 8l+0 \ or\ 8l+1 \ or\ 8l+2, 8 \leq p \leq r-2, q = s\}$, $S_7 = \{v_p, v_q : p = 8l+4 \ or\ 8l+5 \ or\ 8l+6, 12 \leq p \leq r-2, q = 1\} \cup \{v_p, v_q : p = 4l+2, 10 \leq p \leq r-2, q = 2l+1, 3 \leq q \leq s-3\}$,

$S_8 = \{v_p, v_q : p = 4l+3, 11 \le p \le r-1, q = 2l, 2 \le q \le s-4\}$, $S_9 = \{v_p, v_q : p = 4l+2, 10 \le p \le r-2, q = s-2\} \cup \{v_p, v_q : p = 4l+3, 11 \le p \le r-1, q = s-1\}$.

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 & if\ s\ is\ odd, \\ S_5 \cup S_6 \cup S_7 \cup S_8 \cup S_9 & if\ s\ is\ even, \end{cases}$ clearly $S$ is a TCCD-set of $P_r \times C_s$ and hence $\gamma_{TCC}(P_r \times C_s) \le$

$$\mid S \mid = \begin{cases} (s-2)\frac{r}{4} + 6\lfloor \frac{r}{9} \rfloor + 3 & if\ r = 8l, \\ (s-2)\frac{r}{4} + 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 3 & if\ r = 8l+4. \end{cases}$$

Assume a TCCD-set $D \subseteq P_r \times C_s$ exists of cardinality at most $d = \begin{cases} (s-2)\frac{r}{4} + 6\lfloor \frac{r}{9} \rfloor + 2 & if\ r = 8l, \\ (s-2)\frac{r}{4} + 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 2 & if\ r = 8l+4, \end{cases}$
whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(P_r \times C_s) \ge d+1 = \begin{cases} (s-2)\frac{r}{4} + 6\lfloor \frac{r}{9} \rfloor + 3 & if\ r = 8l, \\ (s-2)\frac{r}{4} + 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 3 & if\ r = 8l+4. \end{cases}$$

Hence the result follows. □

**Theorem 2.3.** *If $r \ge 9$, $r = 4l+1$, $s \ne 4l$ and $s = 3l$ then*

$$\gamma_{TCC}(P_r \times C_s) = \begin{cases} 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +2(\lceil \frac{r}{3} \rceil - 3) + 1 & if\ r = 8l+1,\ r \ne 9, \\ 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +2(\lceil \frac{r}{3} \rceil - 3) + 2 & if\ r = 9, \\ 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +3(\lfloor \frac{r}{6} \rfloor - 1) + 3(\lfloor \frac{r}{6} \rfloor - 2) + 1 & if\ r = 8l+5. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : q = 2l, 4 \le q \le s-3, p = 2, 6\} \cup \{v_p, v_q : q = 2l+1, 3 \le q \le s-4, p = 3, 7\}$, $S_2 = \{v_p, v_{s-2} : p = 4l+2, 1 \le p \le r\} \cup \{v_p, v_{s-1} : p = 4l+3, 1 \le p \le r\} \cup \{v_2, v_s\} \cup \{v_p, v_1 : p = 1, 5\} \cup \{v_p, v_2 : p = 4, 6\}$, $S_3 = \{v_p, v_q : p = 4l+2, 10 \le p \le r-4, q = 2l+1, 3 \le q \le s-3\} \cup \{v_p, v_q : p = 4l+3, 11 \le p \le r-4, q = 2l, 2 \le q \le s-4\}$, $S_4 = \{v_p, v_q : q = s, p = 8l+0\ or\ 8l+1\ or\ 8l+2, 8 \le p \le r-2\} \cup \{v_p, v_q : q = 1, p = 8l+4\ or\ 8l+5\ or\ 8l+6, 12 \le p \le r-2\}$, $S_5 = \{v_6, v_1\}$, $S_6 = \{v_p, v_q : q = 3l+2, 2 \le q \le s-1, p = r-1, r-2\} \cup \{v_p, v_q : q = 3l+1, 4 \le q \le s-2, p = r-2\} \cup \{v_p, v_q : q = 3l, 3 \le q \le s-3, p = r-3\}$. Then $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$, clearly $S$ is a TCCD-set of $P_r \times C_s$ and hence $\gamma_{TCC}(P_r \times C_s) \le$

$$\mid S \mid = \begin{cases} 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +2(\lceil \frac{r}{3} \rceil - 3) + 1 & if\ r = 8l+1,\ r \ne 9, \\ 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +2(\lceil \frac{r}{3} \rceil - 3) + 2 & if\ r = 9, \\ 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +3(\lfloor \frac{r}{6} \rfloor - 1) + 3(\lfloor \frac{r}{6} \rfloor - 2) + 1 & if\ r = 8l+5. \end{cases}$$

Assume a TCCD-set $D \subseteq P_r \times C_s$ exists of cardinality at most

$$d = \begin{cases} 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +2(\lceil \frac{r}{3} \rceil - 3) & if\ r = 8l+1,\ r \ne 9, \\ 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +2(\lceil \frac{r}{3} \rceil - 3) + 1 & if\ r = 9, \\ 4\frac{s}{3} + (s-2)\lfloor \frac{r}{5} \rfloor \\ +3(\lfloor \frac{r}{6} \rfloor - 1) + 3(\lfloor \frac{r}{6} \rfloor - 2) & if\ r = 8l+5, \end{cases}$$

whose subgraph induced $\langle D\rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(P_r\times C_s)\geq d+1 \;=\begin{cases} 4\frac{s}{3}+(s-2)\lfloor\frac{r}{5}\rfloor & \\ +2(\lceil\frac{r}{3}\rceil-3)+1 & if\ r=8l+1,\ r\neq 9,\\ 4\frac{s}{3}+(s-2)\lfloor\frac{r}{5}\rfloor & \\ +2(\lceil\frac{r}{3}\rceil-3)+2 & if\ r=9,\\ 4\frac{s}{3}+(s-2)\lfloor\frac{r}{5}\rfloor & \\ +3(\lfloor\frac{r}{6}\rfloor-1)+3(\lfloor\frac{r}{6}\rfloor-2)+1 & if\ r=8l+5.\end{cases}$$

Hence the result follows. □

**Theorem 2.4.** *If* $r\geq 9$, $r=4l+1$, $s\neq 4l$ *and* $s=3l+1$ *then*

$$\gamma_{TCC}(P_r\times C_s)=\begin{cases} (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+3 & if\ r=8l+1,\ r\neq 9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+4 & if\ r=9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+4\lfloor\frac{s}{3}\rfloor+4 & \\ +3(\lfloor\frac{r}{6}\rfloor-1)+3(\lfloor\frac{r}{6}\rfloor-2) & if\ r=8l+5.\end{cases}$$

*Proof.* Let $S_1=\{v_p,v_q: q=2l,4\leq q\leq s-3,p=2,6\}\cup\{v_p,v_q: q=2l+1,3\leq q\leq s-4,p=3,7\}$, $S_2=\{v_p,v_{s-2}: p=4l+2,1\leq p\leq r\}\cup\{v_p,v_{s-1}: p=4l+3,1\leq p\leq r\}\cup\{v_2,v_s\}\cup\{v_p,v_1: p=1,5\}\cup\{v_p,v_2: p=4,6\}$, $S_3=\{v_p,v_q: p=4l+2,10\leq p\leq r-4,q=2l+1,3\leq q\leq s-3\}\cup\{v_p,v_q: p=4l+3,11\leq p\leq r-4,q=2l,2\leq q\leq s-4\}$, $S_4=\{v_p,v_q: q=s,p=8l+0\ or\ 8l+1\ or\ 8l+2,8\leq p\leq r-2\}\cup\{v_p,v_q: q=1,p=8l+4\ or\ 8l+5\ or\ 8l+6,12\leq p\leq r-2\}$, $S_5=\{v_6,v_1\}$, $S_6=\{v_p,v_q: q=3l+2,2\leq q\leq s-2,p=r-1,r-2\}\cup\{v_p,v_q: q=3l+1,4\leq q\leq s-3,p=r-3\}\cup\{v_p,v_q: p=3l,3\leq q\leq s-4,p=r-2\}\cup\{v_{r-3},v_{s-2}\}\cup\{v_{r-1},v_{s-1}\}$, $S_7=\{v_p,v_q: q=3l+2,5\leq q\leq s-2,p=r-2\}\cup\{v_p,v_q: q=3l+1,4\leq q\leq s-3,p=r-3\}\cup\{v_p,v_q: q=3l,3\leq q\leq s-1,p=r-1,r-2\}\cup\{v_{r-1},v_2\}\cup\{v_{r-3},v_3\}$.

Then $S=\begin{cases} S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_6 & if\ r=8l+1,\\ S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S-7 & if\ r=8l+5,\end{cases}$ clearly $S$ is a TCCD-set of $P_r\times C_s$ and hence $\gamma_{TCC}(P_r\times C_s)\leq$

$$\mid S\mid=\begin{cases} (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+3 & if\ r=8l+1,\ r\neq 9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+4 & if\ r=9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+4\lfloor\frac{s}{3}\rfloor+4 & \\ +3(\lfloor\frac{r}{6}\rfloor-1)+3(\lfloor\frac{r}{6}\rfloor-2) & if\ r=8l+5.\end{cases}$$

Assume a TCCD-set $D\subseteq P_r\times C_s$ exists of cardinality at most

$$d=\begin{cases} (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+2 & if\ r=8l+1,\ r\neq 9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+3 & if\ r=9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+4\lfloor\frac{s}{3}\rfloor+3 & \\ +3(\lfloor\frac{r}{6}\rfloor-1)+3(\lfloor\frac{r}{6}\rfloor-2) & if\ r=8l+5,\end{cases}$$

whose subgraph induced $\langle D\rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(P_r\times C_s)\geq d+1 = \begin{cases} (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+3 & if\ r=8l+1,\ r\neq 9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+4 & if\ r=9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+4\lfloor\frac{s}{3}\rfloor+4 & \\ +3(\lfloor\frac{r}{6}\rfloor-1)+3(\lfloor\frac{r}{6}\rfloor-2) & if\ r=8l+5.\end{cases}$$

Hence the result follows. □

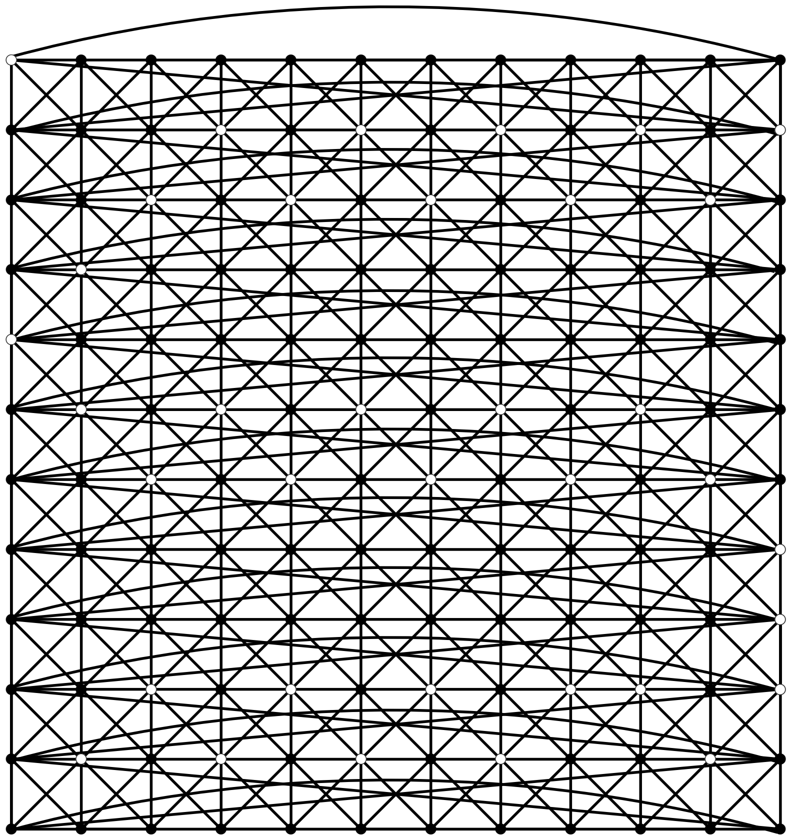

FIGURE 1. The set of lightened vertices denote the TCCD-set and $\gamma_{TCC}(P_{12}\times C_{12}) = 36$.

**Theorem 2.5.** *If $r\geq 9$, $4l+1$, $s\neq 4l$ and $s=3l+2$ then*

$$\gamma_{TCC}(P_r\times C_s)= \begin{cases} (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+5 & if\ r=8l+1,\ r\neq 9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+2(\lceil\frac{r}{3}\rceil-3) & \\ +4\lfloor\frac{s}{3}\rfloor+6 & if\ r=9,\\ (s-2)\lfloor\frac{r}{5}\rfloor+4\lfloor\frac{s}{3}\rfloor+5 & \\ +3(\lfloor\frac{r}{6}\rfloor-1)+3(\lfloor\frac{r}{6}\rfloor-2) & if\ r=8l+5.\end{cases}$$

*Proof.* Let $S_1=\{v_p,v_q : q=2l, 4\leq q\leq s-3, p=2,6\}\cup\{v_p,v_q : q=2l+1, 3\leq q\leq s-4, p=3,7\}$, $S_2=\{v_p,v_{q-2} : p=4l+2, 1\leq p\leq r\}\cup\{v_p,v_{s-1} : p=4l+3, 1\leq p\leq r\}\cup\{v_2,v_s\}\cup\{v_p,v_1 : p=1,5\}\cup\{v_p,v_2 : p=4,6\}$, $S_3=\{v_p,v_q : p=4l+2, 10\leq p\leq$

$r-4, q=2l+1, 3 \leq q \leq s-3\} \cup \{v_p, v_q : p=4l+3, 11 \leq p \leq r-4, q=2l, 2 \leq q \leq s-4\}$, $S_4=\{v_p, v_q : q=s, p=8l+0$ *or* $8l+1$ *or* $8l+2, 8 \leq p \leq r-2\} \cup \{v_p, v_q : q=1, p=8l+4$ *or* $8l+5$ *or* $8l+6, 12 \leq p \leq r-2\}$, $S_5=\{v_6, v_1\}$, $S_6=\{v_p, v_q : q=3l+2, 2 \leq q \leq s-3, p=r-1, r-2\} \cup \{v_p, v_q : q=3l+1, 3 \leq q \leq s-1, p=r-3\} \cup \{v_p, v_q : q=3l, 3 \leq q \leq s-4, p=r-2\} \cup \{v_p, v_{s-1} : p=r-1, r-2\}$,$S_7=\{v_p, v_q : q=3l+2, 2 \leq q \leq s-3, p=r-3\} \cup \{v_p, v_q : q=3l+1, 4 \leq q \leq s-1, p=r-1, r-2\} \cup \{v_p, v_q : q=3l, 3 \leq q \leq s-2, p=r-2\} \cup \{v_p, v_2 : p=r-1, r-2\}$.

Then $S=\begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 & if\ r=8l+1, \\ S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S-7 & if\ r=8l+5, \end{cases}$ clearly $S$ is a TCCD-set of $P_r \times C_s$ and hence $\gamma_{TCC}(P_r \times C_s) \leq$

$$| S | = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +4\lfloor \frac{s}{3} \rfloor + 5 & if\ r=8l+1,\ r \neq 9, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +4\lfloor \frac{s}{3} \rfloor + 6 & if\ r=9, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 4\lfloor \frac{s}{3} \rfloor + 5 & \\ +3(\lfloor \frac{r}{6} \rfloor - 1) + 3(\lfloor \frac{r}{6} \rfloor - 2) & if\ r=8l+5. \end{cases}$$

Assume a TCCD-set $D \subseteq P_r \times C_s$ exists of cardinality at most

$$d = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +4\lfloor \frac{s}{3} \rfloor + 4 & if\ r=8l+1,\ r \neq 9, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +4\lfloor \frac{s}{3} \rfloor + 5 & if\ r=9, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 4\lfloor \frac{s}{3} \rfloor + 4 & \\ +3(\lfloor \frac{r}{6} \rfloor - 1) + 3(\lfloor \frac{r}{6} \rfloor - 2) & if\ r=8l+5, \end{cases}$$

whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(P_r \times C_s) \geq d+1 = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +4\lfloor \frac{s}{3} \rfloor + 5 & if\ r=8l+1,\ r \neq 9, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +4\lfloor \frac{s}{3} \rfloor + 6 & if\ r=9, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 4\lfloor \frac{s}{3} \rfloor + 5 & \\ +3(\lfloor \frac{r}{6} \rfloor - 1) + 3(\lfloor \frac{r}{6} \rfloor - 2) & if\ r=8l+5. \end{cases}$$

Hence the result follows. □

**Theorem 2.6.** *If* $r, s \geq 9$ *and* $r, s = 4l+2$ *and* $r = 8l+2$ *then*

$$\gamma_{TCC}(P_r \times C_s) = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 8 & if\ s=12l+2, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 3 & if\ s=12l+6, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) & \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 5 & if\ s=12l+10. \end{cases}$$

*Proof.* Let $S_1=\{v_p, v_q : q=2l, 4 \leq q \leq s-4, p=2,6\} \cup \{v_2, v_s\} \cup \{v_p, v_q : q=2l+1, 3 \leq q \leq s-3, p=3,7\} \cup \{v_p, v_{s-2} : p=4l+2, 1 \leq p \leq r-5\}$, $S_2=\cup\{v_p, v_{s-1} : p=4l+3, 3 \leq p \leq r-5\} \cup \{v_p, v_1 : p=1,5\} \cup \{v_p, v_2 : p=4,6\} \cup \{v_p, v_q : q=s, p=8l+0$ *or* $8l+1$ *or* $8l+2, 8 \leq p \leq r-4\}$, $S_3=\{v_p, v_q : p=4l+2, 10 \leq p \leq r-5, q=2l+1, 3 \leq q \leq s-3\} \cup \{v_p, v_q : p=4l+3, 11 \leq p \leq r-5, q=2l, 2 \leq q \leq s-4\} \cup \{v_p, v_q : q=1, p=8l+4$ *or* $8l+5$ *or* $8l+6, 12 \leq p \leq r-4\}$, $S_4=\{v_p, v_q : q=3l+2, 2 \leq q \leq s-3, p=r-1, r-2, r-3\} \cup \{v_p, v_q : q=3l+1, 4 \leq q \leq s-1, p=r-3, r-4\} \cup \{v_p, v_q : q=3l, 3 \leq q \leq s-2, p=r-2\} \cup \{v_p, v_{s-1} : p=r-1, r-2\}$, $S_5=\{v_p, v_q : q=3l+2, 2 \leq q \leq s-1, p=r-1, r-2, r-3\} \cup \{v_p, v_q : q=3l+1, 4 \leq$

$q \le s-2, p=r-3, r-4\} \cup \{v_p, v_q : q=3l, 3 \le q \le s-3, p=r-2\}$, $S_6 = \{v_p, v_q : q=3l+2, 2 \le q \le s-2, p=r-1, r-2, r-3\} \cup \{v_p, v_q : q=3l+1, 4 \le q \le s-3, p=r-3, r-4\} \cup \{v_p, v_q : q=3l, 3 \le q \le s-4, p=r-2\} \cup \{v_{r-1}, v_{s-1}\} \cup \{v_{r-4}, v_{s-2}\}$.

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 & if\ s=12l+2, \\ S_1 \cup S_2 \cup S_3 \cup S_5 & if\ s=12l+6, \\ S_1 \cup S_2 \cup S_3 \cup S_6 & if\ s=12l+10, \end{cases}$ clearly $S$ is a TCCD-set of $P_r \times C_s$ and hence $\gamma_{TCC}(P_r \times C_s) \le$

$$\mid S \mid = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 8 & if\ s=12l+2, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 3 & if\ s=12l+6, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 5 & if\ s=12l+10. \end{cases}$$

Assume a TCCD-set $D \subseteq P_r \times C_s$ exists of cardinality

at most $d = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 7 & if\ s=12l+2, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 2 & if\ s=12l+6, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 4 & if\ s=12l+10, \end{cases}$

whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(P_r \times C_s) \ge d+1 = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 8 & if\ s=12l+2, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 3 & if\ s=12l+6, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 2(\lceil \frac{r}{3} \rceil - 3) \\ +3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) + 5 & if\ s=12l+10. \end{cases}$$

Hence the result follows. □

**Theorem 2.7.** *If $r, s \ge 9$ and $r, s = 4l+2$ and $r = 8l+6$ then*

$$\gamma_{TCC}(P_r \times C_s) = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 8 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s=12l+2, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 3 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s=12l+6, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 5 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s=12l+10. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : q=2l, 4 \le q \le s-4, =2,6\} \cup \{v_2, v_s\} \cup \{v_p, v_q : q=2l+1, 3 \le q \le s-3, p=3,7\} \cup \{v_p, v_{s-2} : p=4l+2, 1 \le p \le r-5\}$, $S_2 = \cup \{v_p, v_{s-1} : p=4l+3, 3 \le p \le r-5\} \cup \{v_p, v_1 : p=1,5\} \cup \{v_p, v_2 : p=4,6\} \cup \{v_p, v_q : q=s, p=8l+0\ or\ 8l+1\ or\ 8l+2, 8 \le p \le r-4\}$, $S_3 = \{v_p, v_q : p=4l+2, 10 \le p \le r-5, q=2l+1, 3 \le q \le s-3\} \cup \{v_p, v_q : p=4l+3, 11 \le p \le r-5, q=2l, 2 \le q \le s-4\} \cup \{v_p, v_q : q=1, p=8l+4\ or\ 8l+5\ or\ 8l+6, 12 \le p \le r-4\}$,$S_4 = \{v_p, v_q : q=3l+2, 2 \le q \le s-3, p=r-3, r-4\} \cup \{v_p, v_q : q=3l+1, 4 \le q \le s-1, p=r-1, r-2, r-3\} \cup \{v_p, v_q : q=3l, 3 \le q \le s-2, p=r-2\} \cup \{v_p, v_2 : p=r-1, r-2\}$, $S_5 = \{v_p, v_q : q=3l+2, 2 \le q \le s-1, p=r-1, r-2, r-3\} \cup \{v_p, v_q : q=3l+1, 4 \le q \le s-2, p=r-2\} \cup \{v_p, v_q : q=3l, 3 \le q \le s-3, p=r-3, r-4\}$, $S_6 = \{v_p, v_q : q=3l+2, 5 \le q \le s-2, p=r-2\} \cup \{v_p, v_q : q=3l+1, 4 \le q \le s-3, p=r-3, r-4\} \cup \{v_p, v_q : q=3l, 3 \le q \le s-1, p=r-1, r-2, r-3\} \cup \{v_{r-1}, v_2\} \cup \{v_{r-4}, v_3\}$.

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 & if\ s = 12l+2, \\ S_1 \cup S_2 \cup S_3 \cup S_5 & if\ s = 12l+6, \\ S_1 \cup S_2 \cup S_3 \cup S_6 & if\ s = 12l+10, \end{cases}$ clearly $S$ is a TCCD-set of $P_r \times C_s$ and hence $\gamma_{TCC}(P_r \times C_s) \leq$

$$| S | = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 8 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+2, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 3 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+6, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 5 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+10. \end{cases}$$

Assume a TCCD-set $D \subseteq P_r \times C_s$ exists of cardinality

at most $d = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 7 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+2, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 2 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+6, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 4 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+10, \end{cases}$

whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(P_r \times C_s) \geq d+1 \ = \begin{cases} (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 8 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+2, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 3 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+6, \\ (s-2)\lfloor \frac{r}{5} \rfloor + 3\lfloor \frac{r}{8} \rfloor + 5 \\ +3(\lfloor \frac{r}{8} \rfloor - 1) + 3\lfloor \frac{s}{3} \rfloor + 3(\lfloor \frac{s}{3} \rfloor - 1) & if\ s = 12l+10. \end{cases}$$

Hence the result follows. □

**Theorem 2.8.** *If* $r, s \geq 9$ *and* $r = 4l+3$, $r = 4l+2$ *or* 3 *and* $s \neq 4l+3, s \geq r-4$ *then*

$$\gamma_{TCC}(P_r \times C_s) = \begin{cases} (s-2)\lfloor \frac{r}{4} \rfloor + 2(\lfloor \frac{r}{3} \rfloor - 3) + (s-2) + 5 & if\ r = 8l+3, \\ (s-2)\lfloor \frac{r}{4} \rfloor + 2(\lfloor \frac{r}{3} \rfloor - 2) + (s-2) + 2 & if\ r = 8l+7. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : q = 2l, 4 \leq q \leq s-3, p = 2, 6\} \cup \{v_2, v_s\} \cup \{v_p, v_q : q = 2l+1, 3 \leq q \leq s-4, p = 3, 7\}$, $S_2 = \{v_p, v_q : q = s-2, p = 4l+2, 2 \leq p \leq r-2\} \cup \{v_p, v_q : q = s-1, p = 4l+3, 3 \leq p \leq r-2\}$, $S_3 = \{v_p, v_q : p = 4l+2, 10 \leq p \leq r-5, q = 2l+1, 3 \leq q \leq s-4\} \cup \{v_p, v_q : p = 4l+3, 11 \leq p \leq r-4, q = 2l, 2 \leq q \leq s-3\}$, $S_4 = \{v_p, v_q : p = 8l+0$ *or* $8l+1$ *or* $8l+2, 8 \leq p \leq r-2, q = s\} \cup \{v_p, v_q : p = 8l+4$ *or* $8l+5$ *or* $8l+6, 12 \leq p \leq r-2\}$, $S_5 = \{v_{r-1}, v_q : 2 \leq q \leq s-1\} \cup \{v_p, v_1 : p = 1, 5\} \cup \{v_p, v_2 : p = 4, 6\}$
Then $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, clearly $S$ is a TCCD-set of $P_r \times C_s$ and hence

$$\gamma_{TCC}(P_r \times C_s) \leq | S | = \begin{cases} (s-2)\lfloor \frac{r}{4} \rfloor + 2(\lfloor \frac{r}{3} \rfloor - 3) + (s-2) + 5 & if\ r = 8l+3, \\ (s-2)\lfloor \frac{r}{4} \rfloor + 2(\lfloor \frac{r}{3} \rfloor - 2) + (s-2) + 2 & if\ r = 8l+7. \end{cases}$$

Assume a TCCD-set $D \subseteq P_r \times C_s$ exists of cardinality at most

$d = \begin{cases} (s-2)\lfloor \frac{r}{4} \rfloor + 2(\lfloor \frac{r}{3} \rfloor - 3) + (s-2) + 4 & if\ r = 8l+3, \\ (s-2)\lfloor \frac{r}{4} \rfloor + 2(\lfloor \frac{r}{3} \rfloor - 2) + (s-2) + 1 & if\ r = 8l+7, \end{cases}$

whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(P_r \times C_s) \geq d+1 \ = \begin{cases} (s-2)\lfloor \frac{r}{4} \rfloor + 2(\lfloor \frac{r}{3} \rfloor - 3) + (s-2) + 5 & if\ r = 8l+3, \\ (s-2)\lfloor \frac{r}{4} \rfloor + 2(\lfloor \frac{r}{3} \rfloor - 2) + (s-2) + 2 & if\ r = 8l+7. \end{cases}$$

Hence the result follows. □

## 3. TCC domination for Strong product of cycle and cycle

We commence this section with several observations. The subsequent results will provide the precise values of the strong product of two cycles.

Let $V(C_r \times C_s) = \{v_p v_q : 1 \le p \le r, 1 \le q \le s\}$ and $E(C_r \times C_s) = \{v_p v_q\ v_{p+1} v_q : 1 \le p \le r-1, 1 \le q \le s\} \cup \{v_p v_q\ v_p v_{q+1} : 1 \le p \le r, 1 \le q \le s-1\} \cup \{v_p v_1\ v_p v_m : 1 \le p \le r\} \cup \{v_p v_q\ v_{p+1} v_{q+1} : 1 \le p \le r-1, 1 \le q \le s-1\} \cup \{v_p v_q\ v_{p+1} v_{q-1} : 1 \le p \le r-1, s \le q \le 2\} \cup \{v_p v_1\ v_{p+1} v_s : 1 \le p \le r-1\} \cup \{v_p v_q\ v_{p+1} v_1 : 1 \le p \le r-1\} \cup \{v_1 v_q\ v_r v_q : 1 \le q \le s\} \cup \{v_1 v_1\ v_r v_q : q = 2, s\} \cup \{v_1 v_s\ v_r v_q : q = 1, 4\} \cup \{v_r v_1\ v_1 v_2\} \cup \{v_r v_s\ v_1\ v_{s-1}\}$ of $C_r \times C_s$.

**Observation**

(1) If $r \ge 3$, then $\gamma_{TCC}(C_r \times C_3) = r-2$.
(2) If $r \ge 3$, then $\gamma_{TCC}(C_r \times C_4) = r-1$.
(3) If $r \ge 3$, then $\gamma_{TCC}(C_r \times C_5) = r+4$.
(4) If $r \ge 3$, then $\gamma_{TCC}(C_r \times C_6) = 2r-4$.
(5) If $r \ge 3$, then $\gamma_{TCC}(C_r \times C_7) = 2r-2$.
(6) If $r \ge 3$, then $\gamma_{TCC}(C_r \times C_8) = 2r-1$.
(7) $\gamma_{TCC}(C_9 \times C_9) = 23$, $\gamma_{TCC}(C_9 \times C_{10}) = 26$, $\gamma_{TCC}(C_{10} \times C_{10}) = 27$, $\gamma_{TCC}(C_{11} \times C_{10}) = 29$, $\gamma_{TCC}(C_{10} \times C_{13}) = 37$, $\gamma_{TCC}(C_{10} \times C_{14}) = 39$.

**Theorem 3.1.** *If $r \ge 11$, then*

$$\gamma_{TCC}(C_r \times C_9) = \begin{cases} 6\lfloor \frac{r}{9} \rfloor + 7\lfloor \frac{r}{4} \rfloor + 10 & if\ r = 8l+1, \\ 6(\lceil \frac{r}{9} \rceil - 1) + 7\lfloor \frac{r}{4} \rfloor + 11 & if\ r = 8l+2, \\ 6(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 12 & if\ r = 8l+3, \\ 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 10 & if\ r = 8l+5, \\ 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 11 & if\ r = 8l+6, \\ 6\lfloor \frac{r}{9} \rfloor + 7\lfloor \frac{r}{4} \rfloor + 9 & if\ r = 8l+7. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : p = 4l+2, q = 2l, 6 \le p \le r-3, 2 \le q \le s-1\}$, $S_2 = \{v_p, v_q : p = 4l+3, q = 2l+1, 7 \le p \le r-2, 3 \le q \le s-2\}$, $S_3 = \{v_2, v_q : q = 4, 6, 7, 9\}$, $S_4 = \{v_3, v_q : q = 3, 5, 8\}$, $S_5 = \{v_p, v_1 : p = 1, 5\} \cup \{v_4, v_2\}$, $S_6 = \{v_p, v_s : p = 8l+0\ or\ 8l+1\ or\ 8l+7, 7 \le p \le r-3\}$, $S_7 = \{v_p, v_1 : p = 8l+3\ or\ 8l+4\ or\ 8l+5, 11 \le p \le r-3\}$, $S_8 = \{v_{r-2}, v_{s-1}\} \cup \{v_{r-1}, v_q : 2 \le q \le s-2\}$, $S_9 = \{v_{r-2}, v_2\} \cup \{v_{r-1}, v_q : 3 \le q \le s-1\}$.
Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8 & if\ r = 8l+1\ or\ 8l+2\ or\ 8l+3, \\ S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_9 & if\ r = 8l+5\ or\ 8l+6\ or\ 8l+7, \end{cases}$
clearly $S$ is a TCCD-set of $C_r \times C_9$ and hence $\gamma_{TCC}(C_r \times C_9) \le$

$$\gamma_{TCC}(C_r \times C_9) \le |\ S\ | = \begin{cases} 6\lfloor \frac{r}{9} \rfloor + 7\lfloor \frac{r}{4} \rfloor + 10 & if\ r = 8l+1, \\ 6(\lceil \frac{r}{9} \rceil - 1) + 7\lfloor \frac{r}{4} \rfloor + 11 & if\ r = 8l+2, \\ 6(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 12 & if\ r = 8l+3, \\ 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 10 & if\ r = 8l+5, \\ 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 11 & if\ r = 8l+6, \\ 6\lfloor \frac{r}{9} \rfloor + 7\lfloor \frac{r}{4} \rfloor + 9 & if\ r = 8l+7. \end{cases}$$

Assume a TCCD-set $D \subseteq C_r \times C_9$ exists of cardinality at most

$$d = \begin{cases} 6\lfloor \frac{r}{9} \rfloor + 7\lfloor \frac{r}{4} \rfloor + 9 & if\ r = 8l+1, \\ 6(\lceil \frac{r}{9} \rceil - 1) + 7\lfloor \frac{r}{4} \rfloor + 10 & if\ r = 8l+2, \\ 6(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 11 & if\ r = 8l+3, \\ 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 9 & if\ r = 8l+5, \\ 3\lfloor \frac{r}{9} \rfloor + 3(\lfloor \frac{r}{9} \rfloor - 1) + 7\lfloor \frac{r}{4} \rfloor + 10 & if\ r = 8l+6, \\ 6\lfloor \frac{r}{9} \rfloor + 7\lfloor \frac{r}{4} \rfloor + 8 & if\ r = 8l+7, \end{cases}$$

whose subgraph induced $\langle D\rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(C_r \times C_9) \geq d+1 \ = \begin{cases} 6\lfloor\frac{r}{9}\rfloor + 7\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+1, \\ 6(\lceil\frac{r}{9}\rceil - 1) + 7\lfloor\frac{r}{4}\rfloor + 11 & if\ r = 8l+2, \\ 6(\lfloor\frac{r}{9}\rfloor - 1) + 7\lfloor\frac{r}{4}\rfloor + 12 & if\ r = 8l+3, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 7\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+5, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 7\lfloor\frac{r}{4}\rfloor + 11 & if\ r = 8l+6, \\ 6\lfloor\frac{r}{9}\rfloor + 7\lfloor\frac{r}{4}\rfloor + 9 & if\ r = 8l+7. \end{cases}$$

Hence the result follows. □

**Theorem 3.2.** *If* $r \geq 15$, *then*

$$\gamma_{TCC}(C_r \times C_{10}) = \begin{cases} 6\lfloor\frac{r}{9}\rfloor + 8\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+1\ or\ 8l+7, \\ 6(\lceil\frac{r}{9}\rceil - 1) + 8\lfloor\frac{r}{4}\rfloor + 11 & if\ r = 8l+2, \\ 6(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 12 & if\ r = 8l+3, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+5, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 12 & if\ r = 8l+6. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : p = 4l+2, q = 2l, 2 \leq p \leq r-3, 2 \leq q \leq s-2\} \cap \{v_2, v_2\}$, $S_2 = \{v_p, v_q : p = 4l+3, q = 2l+1, 3 \leq p \leq r-2, 3 \leq q \leq s-1\} \cup \{v_2, v_s\}$, $S_3 = \{v_p, v_s : p = 8l+0\ or\ 8l+1\ or\ 8l+2, 8 \leq p \leq r-3\}$, $S_4 = \{v_p, v_1 : p = 8l+3\ or\ 8l+4\ or\ 8l+5, 11 \leq p \leq r-3\}$, $S_5 = \{v_p, v_1 : p = 1, 5\} \cup \{v_4, v_2\}$, $S_6 = \{v_{r-1}, v_q : 2 \leq q \leq s-2\} \cup \{v_{r-2}, v_2\}$, $S_7 = \{v_{r-1}, v_q : 3 \leq q \leq s-1\} \cup \{v_{r-2}, v_2\}$, $S_8 = \{v_{r-1}, v_q : 2 \leq q \leq s-2\}$, $S_9 = \{v_{r-1}, v_q : 3 \leq q \leq s-2\} \cup \{v_{r-2}, v_2\}$.

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_8 & if\ r = 8l+1, \\ S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 & if\ r = 8l+2\ or\ 8l+3, \\ S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_9 & if\ r = 8l+5, \\ S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_7 & if\ r = 8l+6\ or\ 8l+7, \end{cases}$ clearly $S$ is a TCCD-set of $C_r \times C_{10}$ and hence $\gamma_{TCC}(C_r \times C_{10}) \leq$

$$|\ S\ | = \begin{cases} 6\lfloor\frac{r}{9}\rfloor + 8\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+1\ or\ 8l+7, \\ 6(\lceil\frac{r}{9}\rceil - 1) + 8\lfloor\frac{r}{4}\rfloor + 11 & if\ r = 8l+2, \\ 6(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 12 & if\ r = 8l+3, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+5, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 12 & if\ r = 8l+6. \end{cases}$$

Assume a TCCD-set $D \subseteq C_r \times C_s$ exists of cardinality

at most $d = \begin{cases} 6\lfloor\frac{r}{9}\rfloor + 8\lfloor\frac{r}{4}\rfloor + 9 & if\ r = 8l+1\ or\ 8l+7, \\ 6(\lceil\frac{r}{9}\rceil - 1) + 8\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+2, \\ 6(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 11 & if\ r = 8l+3, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 9 & if\ r = 8l+5, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 11 & if\ r = 8l+6, \end{cases}$

whose subgraph induced $\langle D\rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(C_r \times C_{10}) \geq d+1 \ = \begin{cases} 6\lfloor\frac{r}{9}\rfloor + 8\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+1\ or\ 8l+7, \\ 6(\lceil\frac{r}{9}\rceil - 1) + 8\lfloor\frac{r}{4}\rfloor + 11 & if\ r = 8l+2, \\ 6(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 12 & if\ r = 8l+3, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 10 & if\ r = 8l+5, \\ 3\lfloor\frac{r}{9}\rfloor + 3(\lfloor\frac{r}{9}\rfloor - 1) + 8\lfloor\frac{r}{4}\rfloor + 12 & if\ r = 8l+6. \end{cases}$$

Hence the result follows. □

**Theorem 3.3.** *If* $r,\ s \geq 9$, $r = 8l$ *and* $s \neq 4l, s \leq r-4$ *then*

$$\gamma_{TCC}(C_r \times C_s) = \begin{cases} s-2+(s-2)(\frac{r}{4}-1) + 6\lfloor\frac{r}{10}\rfloor + 3 & if\ s\ is\ odd, \\ \frac{r}{4}(s-2) + 6\lfloor\frac{r}{10}\rfloor + 3 & if\ s\ is\ even. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : p = 4l+2, q = 2l, 6 \le p \le r-2, 2 \le q \le s-1\}$, $S_2 = \{v_p, v_q : p = 4l+3, q = 2l+1, 7 \le p \le r-1, 3 \le q \le s-2\}$, $S_3 = \{v_p, v_q : q = s, p = 8l+0 \text{ or } 8l+1 \text{ or } 8l+7, 7 \le p \le r-2\}$, $S_4 = \{v_p, v_q : q = 1, p = 8l+3 \text{ or } 8l+4 \text{ or } 8l+5, 11 \le p \le r-2\}$, $S_5 = \{v_p, v_1 : p = 1, 5\} \cup \{v_4, v_2\}$, $S_6 = \{v_2, v_q : q = 2l, 4 \le q \le s-3\} \cup \{v_2, v_q : q = s, s-2\}$, $S_7 = \{v_p, v_q : q = 2l+1, 3 \le q \le s-4\} \cup \{v_3, v_{s-1}\}$, $S_8 = \{v_p, v_q : p = 4l+2, q = 2l, 2 \le p \le r-2, 2 \le q \le s-2\} \cap \{v_2, v_2\}$, $S_9 = \{v_p, v_q : p = 4l+3, q = 2l+1, 3 \le p \le r-1, 3 \le q \le s-1\}$, $S_{10} = \{v_p, v_s : p = 8l+0 \text{ or } 8l+1 \text{ or } 8l+2, 8 \le p \le r-1\}$, $S_{11} = \{v_p, v_1 : p = 8l+3 \text{ or } 8l+4 \text{ or } 8l+5, 11 \le p \le r-1\}$, $S_{12} = \{v_4, v_2\} \cup \{v_2, v_s\} \cup \{v_p, v_1 : p = 1, 5\}$.

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 & if\ s\ is\ odd, \\ S_8 \cup S_9 \cup S_{10} \cup S_{11} \cup S_{12} & if\ s\ is\ even, \end{cases}$ clearly $S$ is a TCCD-set of $C_r \times C_s$ and hence $\gamma_{TCC}(C_r \times C_s) \le$

$$| S | = \begin{cases} s-2+(s-2)(\frac{r}{4}-1)+6\lfloor\frac{r}{10}\rfloor+3 & if\ s\ is\ odd, \\ \frac{r}{4}(s-2)+6\lfloor\frac{r}{10}\rfloor+3 & if\ s\ is\ even. \end{cases}$$

Assume a TCCD-set $D \subseteq C_r \times C_s$ exists of cardinality at most
$d = \begin{cases} s-2+(s-2)(\frac{r}{4}-1)+6\lfloor\frac{r}{10}\rfloor+2 & if\ s\ is\ odd, \\ \frac{r}{4}(s-2)+6\lfloor\frac{r}{10}\rfloor+2 & if\ s\ is\ even, \end{cases}$
whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(C_r \times C_s) \ge d+1 = \begin{cases} s-2+(s-2)(\frac{r}{4}-1)+6\lfloor\frac{r}{10}\rfloor+3 & if\ s\ is\ odd, \\ \frac{r}{4}(s-2)+6\lfloor\frac{r}{10}\rfloor+3 & if\ s\ is\ even. \end{cases}$$

Hence the result follows. □

**Theorem 3.4.** *If* $r,\ s \ge 9,\ r = 8l+4$ *and* $s \ne 4l, s \le r-4$ *then*

$$\gamma_{TCC}(C_r \times C_s) = \begin{cases} s-2+(s-2)(\frac{r}{4}-1)+3(\lfloor\frac{r}{10}\rfloor-1)+ & \\ 3\lfloor\frac{r}{10}\rfloor+3 & if\ s\ is\ odd, \\ s-2+(s-2)(\frac{r}{4}-1)+3(\lfloor\frac{r}{10}\rfloor-1)+ & \\ 3\lfloor\frac{r}{10}\rfloor+2 & if\ s\ is\ even. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : p = 4l+2, q = 2l, 6 \le p \le r-2, 2 \le q \le s-1\}$, $S_2 = \{v_p, v_q : p = 4l+3, q = 2l+1, 7 \le p \le r-1, 3 \le q \le s-2\}$, $S_3 = \{v_p, v_q : q = s, p = 8l+0 \text{ or } 8l+1 \text{ or } 8l+7, 7 \le p \le r-2\}$, $S_4 = \{v_p, v_q : q = 1, p = 8l+3 \text{ or } 8l+4 \text{ or } 8l+5, 11 \le p \le r-2\}$, $S_5 = \{v_p, v_1 : p = 1, 5\} \cup \{v_4, v_2\}$, $S_6 = \{v_2, v_q : q = 2l, 4 \le q \le s-3\} \cup \{v_2, v_q : q = s, s-2\}$, $S_7 = \{v_p, v_q : q = 2l+1, 3 \le q \le s-4\} \cup \{v_3, v_{s-1}\}$, $S_8 = \{v_p, v_q : p = 4l+2, q = 2l, 2 \le p \le r-3, 2 \le q \le s-2\} \cap \{v_2, v_2\}$, $S_9 = \{v_p, v_q : p = 4l+3, q = 2l+1, 3 \le p \le r-2, 3 \le q \le s-1\}$, $S_10 = \{v_p, v_s : p = 8l+0 \text{ or } 8l+1 \text{ or } 8l+2, 8 \le p \le r-3\}$, $S_{11} = \{v_p, v_1 : p = 8l+3 \text{ or } 8l+4 \text{ or } 8l+5, 11 \le p \le r-2\}$, $S_{12} = \{v_4, v_2\} \cup \{v_2, v_s\} \cup \{v_p, v_1 : p = 1, 5\}$, $S_{13} = \{v_{r-2}, v_q : q = 2l+1, 3 \le q \le s-1\}$, $S_{14} = \{v_{r-1}, v_q : q = 2l, 2 \le q \le s-2\}$.

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 & if\ s\ is\ odd, \\ S_8 \cup S_9 \cup S_{10} \cup S_{11} \cup S_{12} \cup S_{13} \cup S_{14} & if\ s\ is\ even, \end{cases}$ clearly $S$ is a TCCD-set of $C_r \times C_s$ and hence $\gamma_{TCC}(C_r \times C_s) \le$

$$| S | = \begin{cases} s-2+(s-2)(\frac{r}{4}-1)+3(\lfloor\frac{r}{10}\rfloor-1)+ & \\ 3\lfloor\frac{r}{10}\rfloor+3 & if\ s\ is\ odd, \\ s-2+(s-2)(\frac{r}{4}-1)+3(\lfloor\frac{r}{10}\rfloor-1)+ & \\ 3\lfloor\frac{r}{10}\rfloor+2 & if\ s\ is\ even. \end{cases}$$

Assume a TCCD-set $D \subseteq C_r \times C_s$ exists of cardinality at most

$$d = \begin{cases} s-2+(s-2)(\frac{r}{4}-1)+3(\lfloor\frac{r}{10}\rfloor-1)+ & \\ 3\lfloor\frac{r}{10}\rfloor+2 & if\ s\ is\ odd, \\ s-2+(s-2)(\frac{r}{4}-1)+3(\lfloor\frac{r}{10}\rfloor-1)+ & \\ 3\lfloor\frac{r}{10}\rfloor+1 & if\ s\ is\ even, \end{cases}$$

whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(C_r \times C_s) \geq d+1 = \begin{cases} s-2+(s-2)(\frac{r}{4}-1)+3(\lfloor\frac{r}{10}\rfloor-1)+ \\ 3\lfloor\frac{r}{10}\rfloor+3 & if\ s\ is\ odd, \\ s-2+(s-2)(\frac{r}{4}-1)+3(\lfloor\frac{r}{10}\rfloor-1)+ \\ 3\lfloor\frac{r}{10}\rfloor+2 & if\ s\ is\ even. \end{cases}$$

Hence the result follows. □

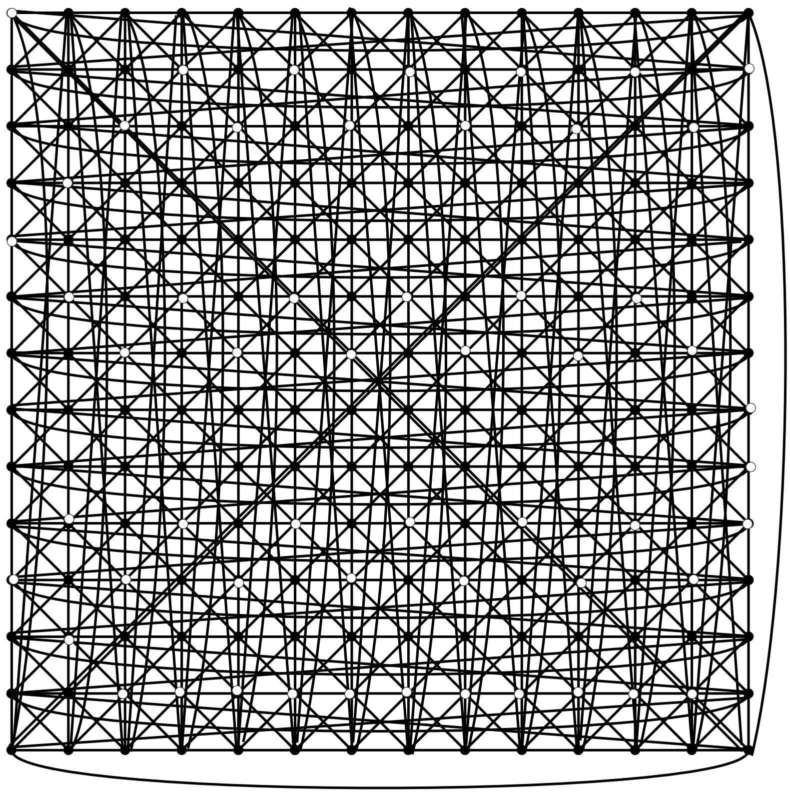

FIGURE 2. The set of lightened vertices denote the TCCD-set and $\gamma_{TCC}(C_{14} \times C_{14}) = 55$.

**Theorem 3.5.** *If* $r,\ s \geq 11,\ s = 4l+1, r \neq 4l$ *and* $r \neq 4l+1, r \leq s-4$ *then*

$$\gamma_{TCC}(C_r \times C_s) = \begin{cases} s-3+(s-2)\lfloor\frac{r}{4}\rfloor+6\lfloor\frac{r}{9}\rfloor+4 & if\ r=8l+1, \\ s-3+(s-2)\lfloor\frac{r}{4}\rfloor+6(\lceil\frac{r}{9}\rceil-1)+5 & if\ r=8l+2, \\ s-3+(s-2)\lfloor\frac{r}{4}\rfloor+3\lfloor\frac{r}{9}\rfloor+ \\ 3(\lfloor\frac{r}{9}\rfloor-1)+3 & if\ r=8l+3, \\ s-3+(s-2)\lfloor\frac{r}{4}\rfloor+3\lfloor\frac{r}{9}\rfloor+ \\ 3(\lfloor\frac{r}{9}\rfloor-1)+4 & if\ r=8l+5, \\ s-3+(s-2)\lfloor\frac{r}{4}\rfloor+3\lfloor\frac{r}{9}\rfloor+ \\ 3(\lfloor\frac{r}{9}\rfloor-1)+5 & if\ r=8l+6, \\ s-3+(s-2)\lfloor\frac{r}{4}\rfloor+6(\lfloor\frac{r}{9}\rfloor)-1+4 & if\ r=8l+7. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : p = 4l+2, q = 2l, 6 \leq p \leq r-3, 2 \leq q \leq s-1\}$, $S_2 = \{v_p, v_q : p = 4l+3, q = 2l+1, 7 \leq p \leq r-2, 3 \leq q \leq s-2\}$, $S_3 = \{v_p, v_q : q = s, p = 8l+0\ or\ 8l+1\ or\ 8l+7, 7 \leq p \leq r-3\}$, $S_4 = \{v_p, v_q : q = 1, p = 8l+3\ or\ 8l+4\ or\ 8l+5, 11 \leq p \leq r-3\}$,

$S_5 = \{v_p, v_q : p = 2, 4 \le q \le s-3\} \cup \{v_p, v_q : p = 3, 3 \le q \le s-4\}$, $S_6 = \{v_2, v_q : q = s, s-2\} \cup \{v_3, v_{s-1}\} \cup \{v_p, v_1 : p = 1, 5\}$, $S_7 = \{v_{r-2}, v_{s-1}\} \cup \{v_{r-1}, v_q : 2 \le q \le s-2\}$, $S_8 = \{v_{r-2}, v_2\} \cup \{v_{r-1}, v_q : 3 \le q \le s-1\}$.

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 & if\ r = 8l+1\ or\ 8l+2\ or\ 8l+3, \\ S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_8 & if\ r = 8l+5\ or\ 8l+6\ or\ 8l+7, \end{cases}$ clearly $S$ is a TCCD-set of $C_r \times C_s$ and hence $\gamma_{TCC}(C_r \times C_s) \le$

$$| S | = \begin{cases} s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6\lfloor \frac{r}{9} \rfloor + 4 & if\ r = 8l+1, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lceil \frac{r}{9} \rceil - 1) + 5 & if\ r = 8l+2, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 3 & if\ r = 8l+3, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 4 & if\ r = 8l+5, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 5 & if\ r = 8l+6, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 1 + 4 & if\ r = 8l+7. \end{cases}$$

Assume a TCCD-set $D \subseteq C_r \times C_s$ exists of cardinality

at most $d = \begin{cases} s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6\lfloor \frac{r}{9} \rfloor + 3 & if\ r = 8l+1, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lceil \frac{r}{9} \rceil - 1) + 4 & if\ r = 8l+2, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 2 & if\ r = 8l+3, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 3 & if\ r = 8l+5, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 4 & if\ r = 8l+6, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 1 + 4 & if\ r = 8l+7, \end{cases}$

whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(C_r \times C_s) \ge d+1 = \begin{cases} s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6\lfloor \frac{r}{9} \rfloor + 4 & if\ r = 8l+1, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lceil \frac{r}{9} \rceil - 1) + 5 & if\ r = 8l+2, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 3 & if\ r = 8l+3, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 4 & if\ r = 8l+5, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 3\lfloor \frac{r}{9} \rfloor + & \\ 3(\lfloor \frac{r}{9} \rfloor - 1) + 5 & if\ r = 8l+6, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 1 + 4 & if\ r = 8l+7. \end{cases}$$

Hence the result follows. □

**Theorem 3.6.** *If* $r,\ s \ge 11,\ s = 4l+2, r \ne 4l+0\ or\ 1\ and\ r \ne\ 4l+2, r \le s-4$ *then*

$$\gamma_{TCC}(C_r \times C_s) = \begin{cases} s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lceil \frac{r}{9} \rceil - 1) + 4 & if\ r = 8l+2, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor - 1) + 5 & if\ r = 8l+3, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 2 + 4 & if\ r = 8l+6, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 1 + 4 & if\ r = 8l+7. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : p = 4l+2, q = 2l, 2 \le p \le r-3, 2 \le q \le s-2\} \cap \{v_2, v_2\}$ , $S_2 = \{v_p, v_q : p = 4l+3, q = 2l+1, 3 \le p \le r-2, 3 \le q \le s-1\} \cup \{v_2, v_s\}$, $S_3 = \{v_p, v_q : p = 8l+0\ or\ 8l+1\ or\ 8l+2, 8 \le p \le r-3\}$, $S_4 = \{v_p, v_q : p = 8l+3\ or\ 8l+4\ or\ 8l+5, 11 \le p \le r-3\}$, $S_5 = \{v_1, v_p : p = 1, 5\} \cup \{v_4, v_2\}$, $S_6 = \{v_{r-2}, v_2\} \cup \{v_{r-1}, v_q : 3 \le q \le s-1\}$,

$S_7 = \{v_{r-2}, v_{s-1}\} \cup \{v_{r-1}, v_q : 2 \leq q \leq s-2\}$

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_7 & if\ r = 8l+2\ or\ 8l+3, \\ S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 & if\ r = 8l+6\ or\ 8l+7, \end{cases}$

clearly $S$ is a TCCD-set of $C_r \times C_s$ and hence $\gamma_{TCC}(C_r \times C_s) \leq$

$$| S | = \begin{cases} s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lceil \frac{r}{9} \rceil - 1) + 4 & if\ r = 8l+2, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor - 1) + 5 & if\ r = 8l+3, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 2 + 4 & if\ r = 8l+6, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 1 + 4 & if\ r = 8l+7. \end{cases}$$

Assume a TCCD-set $D \subseteq C_r \times C_s$ exists of cardinality

at most $d = \begin{cases} s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lceil \frac{r}{9} \rceil - 1) + 3 & if\ r = 8l+2, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor - 1) + 4 & if\ r = 8l+3, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 2 + 3 & if\ r = 8l+6, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 1 + 3 & if\ r = 8l+7, \end{cases}$

whose subgraph induced $\langle D \rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(C_r \times C_s) \geq d+1 = \begin{cases} s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lceil \frac{r}{9} \rceil - 1) + 4 & if\ r = 8l+2, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor - 1) + 5 & if\ r = 8l+3, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 2 + 4 & if\ r = 8l+6, \\ s-3+(s-2)\lfloor \frac{r}{4} \rfloor + 6(\lfloor \frac{r}{9} \rfloor) - 1 + 4 & if\ r = 8l+7. \end{cases}$$

Hence the result follows. □

**Theorem 3.7.** *If* $r,\ s \geq 11,\ r, s = 4l+3$ *and* $s \neq\ 4l+3, s \geq r$ *then*

$$\gamma_{TCC}(C_r \times C_s) = \begin{cases} s-3+(s-2)2(\lfloor \frac{r}{9} \rfloor)+ & \\ 2(\lfloor \frac{r}{3} \rfloor - 3) + 6 & if\ r = 8l+3, \\ s-3+(s-2)(\lfloor \frac{r}{9} \rfloor + 1)+ & \\ 2(\lceil \frac{r}{3} \rceil - 2) + 3 & if\ r = 8l+7. \end{cases}$$

*Proof.* Let $S_1 = \{v_p, v_q : p = 4l+2, q = 2l, 6 \leq p \leq r-2, 2 \leq q \leq s-1\}$ , $S_2 = \{v_p, v_q : p = 4l+3, q = 2l+1, 7 \leq p \leq r-1, 3 \leq q \leq s-2\}$, $S_3 = \{v_p, v_s : p = 8l+0\ or\ 8l+1\ or\ 8l+7, 7 \leq p \leq r-3\}$, $S_4 = \{v_p, v_1 : p = 8l+3\ or\ 8l+4\ or\ 8l+5, 11 \leq p \leq r-3\}$, $S_5 = \{v_2, v_q : q = 2l, 4 \leq q \leq s-3\}$, $S_6 = \{v_3, v_q : q = 2l+1, 3 \leq q \leq s-4\} \cup \{v_p, v_1 : p = 1, 5\}$, $S_7 = \{v_2, v_q : q = s, s-2\} \cup \{v_3, v_{s-1}\}$, $S_8 = \{v_{r-1}, v_q : 2 \leq q \leq s-2\} \cup \{v_{r-2}, v_{s-1}\}$, $S_9 = \{v_{r-1}, v_q : 3 \leq q \leq s-1\} \cup \{v_{r-2} v_2\}$.

Then $S = \begin{cases} S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8 & if\ s = 8l+3, \\ S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_9 & if\ s = 8l+7, \end{cases}$

clearly $S$ is a TCCD-set of $C_r \times C_s$ and hence $\gamma_{TCC}(C_r \times C_s) \leq$

$$| S | = \begin{cases} s-3+(s-2)2(\lfloor \frac{r}{9} \rfloor)+ & \\ 2(\lfloor \frac{r}{3} \rfloor - 3) + 6 & if\ r = 8l+3, \\ s-3+(s-2)(\lfloor \frac{r}{9} \rfloor + 1)+ & \\ 2(\lceil \frac{r}{3} \rceil - 2) + 3 & if\ r = 8l+7. \end{cases}$$

Assume a TCCD-set $D \subseteq C_r \times C_s$ exists of cardinality

at most $d = \begin{cases} s-3+(s-2)2(\lfloor \frac{r}{9} \rfloor)+ & \\ 2(\lfloor \frac{r}{3} \rfloor - 3) + 5 & if\ r = 8l+3, \\ s-3+(s-2)(\lfloor \frac{r}{9} \rfloor + 1)+ & \\ 2(\lceil \frac{r}{3} \rceil - 2) + 2 & if\ r = 8l+7, \end{cases}$

whose subgraph induced $\langle D\rangle$ is not either triple connected or certified. Then we have

$$\gamma_{TCC}(C_r \times C_s) \geq d+1 \ = \begin{cases} s-3+(s-2)2(\lfloor \frac{r}{9} \rfloor)+ & \\ 2(\lfloor \frac{r}{3} \rfloor - 3)+6 & if\ r = 8l+3, \\ s-3+(s-2)(\lfloor \frac{r}{9} \rfloor + 1)+ & \\ 2(\lceil \frac{r}{3} \rceil - 2)+3 & if\ r = 8l+7. \end{cases}$$

Hence the result follows. □

## 4. Conclusions

In this paper, following the exploration of triple connected certified domination, we obtained the general findings regarding the strong product involving cycles and paths. A comparison between the strong product of two paths, two cycles, and paths with cycles will be presented in forthcoming articles.

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