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THE NOVEL BESSEL–MAITLAND FUNCTION INVOLVING SOME CHARACTERISTIC PROPERTIES AND INTEGRAL TRANSFORMS

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Abstract. Inspired by certain recent generalizations of the Bessel-Maitland function in this paper, we introduce a new extension of the Bessel-Maitland function associated with the beta function. Some of its characteristic properties including integral representation, recurrence relation and differentiation formula are investigated. Furthermore, we evaluated some integral transforms such as the Mellin transform, K- transform, Euler transform, Laplace transform and Whittaker transform. In addition, we investigated Riemann-Liouville fractional integrals for this Bessel-Maitland function.

Keywords: Bessel–Maitland function, beta function, fractional derivative, Mellin transform, Laplace transform, Whittaker transform.

AMS Subject Classification: 33B15, 33C10, 26A33, 35A22, 33C45, 33C15, 33C20.

1. Introduction

In recent years, various extensions of some well-known special functions have been investigated [2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 19, 23]. The theory of the Bessel function [1] is significant in examining the solutions of differential equations and they are related to a wide scope of problems in numerous regions of mathematical physics, likewise radiophysics, fluid dynamics and material sciences. Watson [27] provides a thorough explanation of the Bessel function's uses in the natural sciences and engineering. Integral transforms involving Bessel-Maitland function play a crucial role in many problems of physics and applied mathematics. We present here certain integral representations, recurrence relation, Mellin transform, K- transform, Euler transform, Laplace transform and Whittaker transform involving Bessel-Maitland function due to the importance of such type of transform. Here and in the following, $\mathbb{C}, \mathbb{R}^+$, $\mathbb{N}$ are the sets of complex numbers, positive real numbers, positive integers and $\Re(z)$ represent the real part of the complex number z respectively.

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The Bessel–Maitland function is a generalization of the Bessel function introduced by Edward Maitland Wright [17] as:

$$\mathcal{J}_s^r(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(rn+s+1)n!}, \tag{1}$$
$$(\Re(r) > 0, \Re(s) > -1; z \in \mathbb{C}).$$

Singh et al. [24] introduced and studied properties of the generalized Bessel–Maitland function, defined as follows:

$$\mathcal{J}_{s,q}^{r,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}(-z)^n}{\Gamma(rn+s+1)n!}, \tag{2}$$
$$(\Re(r) \geq 0, \Re(s) \geq -1, \Re(\delta) \geq 0; q \in \mathbb{N} \cup (0,1); z \in \mathbb{C}).$$

The classical beta function known as Euler's integral [20] defined by

$$\mathcal{B}(x,y) = \int_0^1 t^{x-1}\ (1-t)^{y-1}\ dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3}$$
$$(\Re(x) > 0,\ \Re(y) > 0).$$

Chaudhry et al. [3] first time accelerate the beta function with the help of exponential function defined as:

$$\mathcal{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(\frac{-p}{t(1-t)}\right) dt, \tag{4}$$
$$(\Re(p) > 0;\ \Re(x) > 0, \Re(y) > 0).$$

In 2014, another extension of the beta function was introduced by Choi et al. [5] as follows:

$$\mathcal{B}_{p,q}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{(1-t)}\right) dt, \tag{5}$$
$$(\Re(x) > 0, \Re(y) > 0; \Re(p) > 0, \Re(q) > 0).$$

Recently, Usman et al. [26] introduced the following generalized Bessel-Maitland function and discussed some of its important properties as:

$$\mathcal{J}_{s,q}^{r,\delta,c}(z;p) = \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\delta+qn, c-\delta)(c)_{qn}(-z)^n}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+s+1)n!}, \tag{6}$$
$$(p > 0, q \in \mathbb{N}; \Re(c) > \Re(\delta) > 0).$$

The above equation is obtained by using the relation $\frac{(\delta)_{qn}}{(c)_{qn}} = \frac{\mathcal{B}(\delta+qn,c-\delta)}{\mathcal{B}(\delta,c-\delta)}$, where $(c)_{qn}$ is the generalized Pochhammer symbol and it is expressed as:

$$(c)_{qn} = \frac{\Gamma(c+qn)}{\Gamma(c)}, \tag{7}$$

and they have used the extended beta function (4).

To begin our main results we recall some important notions that we will use in the following sections.

The Mellin transform [25] of the function $f(z)$ is defined as:

$$\mathcal{M}[f(z);\eta] = \int_0^\infty t^{\eta-1} f(t)dt = f^*(\eta), \qquad \Re(\eta) > 0. \tag{8}$$

For the function $f(z)$, the Euler-Beta [25] transform is given by

$$B[f(z);c,d] = \int_0^1 t^{c-1}(1-t)^{d-1} f(t)\ dt. \tag{9}$$

The Laplace transform of a function $f(z)$ is defined by

$$\mathcal{L}[f(z);\eta] = \int_0^\infty e^{-\eta t} f(t)\ dt, \qquad \Re(\eta) > 0 \in \mathbb{C}. \tag{10}$$

The left and right sided Riemann-Liouville fractional integral operators [16] is defined by (11) and (12) respectively

$$[\mathcal{I}^{\eta}_{0^+} f](x) = \frac{1}{\Gamma(\eta)} \int_0^x \frac{f(t)}{(x-t)^{1-\eta}}\ d\eta, \tag{11}$$

$$[\mathcal{I}^{\eta}_{0^-} f](x) = \frac{1}{\Gamma(\eta)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\eta}}\ d\eta. \tag{12}$$

In section 2, we define a novel Bessel-Maitland function using the extended beta function involving seven parameter Mittag-Leffler function. We show that the new one reduces to the original Bessel-Maitland function under certain particular conditions. In section 3, we discuss the integral representation and differential formulae for the new Bessel-Maitland function. Section 4, deals with some recurrence relations of the Bessel-Maitland function. In section 5 and section 6, we investigated some integral transform and Riemann-Liouville fractional integral operator associated with the new Bessel-Maitland function.
The novelty of the paper is that the authors have established a novel type of Bessel-Maitland function that involves extended beta function with seven parameter Mittag-Leffler function.This is a huge variation from the original Bessel-Maitland function,which is clear from remarks 2.1 and remarks 2.2. The manifold generality of the function, its properties and connection with fractional calculus have been clearly discussed in the paper. This function has various applications in the representation of the solutions of various types of engineering and mathematical physics problems.

2. A Novel Type of Bessel-Maitland Function(NBMF)

In this section, we introduce a new type of Bessel-Maitland function using extended beta function involving seven parameter Mittag-Leffler function and discuss some of its particular cases.

The Bessel-Maitland function using extended beta function involving seven parameter Mittag-Leffler function is defined as:

$$\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q) = \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn, c-\delta)(c)_{qn}(-z)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)n!}, \tag{13}$$

$$(s,r,\delta \in \mathbb{C};\ \Re(c) > \Re(\delta) > 0;\ p > 0, q \in \mathbb{N};\ \sigma_1,\lambda_1,\sigma_2,\lambda_2 \geq 0)$$

$$(\alpha,\beta > 0;\ \Re(u),\Re(v),\Re(\tau) > 0)\,,$$

which will be known as the novel Bessel–Maitland function (NBMF) and the beta function $\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(x,y)$ in (13) is defined as [15]:

$$\begin{aligned}&\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(x,y)\\&=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-p}{t^\alpha}\right)\,\mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-q}{(1-t)^\beta}\right)\,dt,\end{aligned} \tag{14}$$
$$\left(\Re(x),\Re(y)>0;\ p,q,\sigma_1,\lambda_1,\sigma_2,\lambda_2\geq 0;\ \alpha,\beta>0;\Re(u),\Re(v),\Re(\tau)>0\right),$$

also the seven-parameters Mittag-Leffler function in the above equation defined as [21]:

$$\mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z)=\sum_{n=0}^{\infty}\frac{(u)_n(v)_n}{(\tau)_n}\frac{z^n}{\Gamma(\sigma_1 n+\lambda_1)\Gamma(\sigma_2 n+\lambda_2)};\qquad z\in\mathbb{C}. \tag{15}$$

Remarks 2.1. (a). If we substitute $u=1,\ \tau=v;\ \alpha=\beta=\sigma_1=\sigma_2=\lambda_1=\lambda_2=1$ in (14), we obtain the beta function (5) defined by Choi et al. [5].

(b). By substituting $u=1,\ \tau=v;\ \alpha=\beta=\sigma_1=\sigma_2=\lambda_1=\lambda_2=1$ and $q=p$ in (14), we get extended beta function (4) introduced by Chaudhry et al. [3].

(c). Taking $u=1,\ \tau=v;\ \alpha=\beta=\sigma_1=\sigma_2=\lambda_1=\lambda_2=1$ and $p=q=0$ in (14), we obtain the classical beta function (3) introduced by Euler [20].

Remarks 2.2. (a). After substituting the values $u=1,\ \tau=v;\ \alpha=\beta=\sigma_1=\sigma_2=\lambda_1=\lambda_2=1$ and $q=p$ in (13), we can easily obtain (6) the generalized Bessel-Maitland function introduced by Usman et al. [26].

(b). If we more simplify and substitute the values $u=1,\ \tau=v;\ \alpha=\beta=\sigma_1=\sigma_2=\lambda_1=\lambda_2=1$ and $p=q=0$ in (13) and apply the relation $\frac{(\delta)_{qn}}{(c)_{qn}}=\frac{\mathcal{B}(\delta+qn,c-\delta)}{\mathcal{B}(\delta,c-\delta)}$, we obtain Bessel-Maitland function investigated by Singh et al. [24].

3. Characteristic Properties

In this section, we first discuss various integral representations of the novel Bessel-Maitland function (NBMF). After that we also discuss the successive derivative formulae for the NBMF.

Theorem 3.1. *For $p>0,q\in\mathbb{N};\ s,r,\delta\in\mathbb{C};\ \Re(c)>\Re(\delta)>0;\ \sigma_1,\lambda_1,\sigma_2,\lambda_2\geq 0,\alpha,\beta>0;\ \Re(u),\Re(v),\Re(\tau)>0$, the novel Bessel-Maitland function(NBMF) has the following integral representation:*

$$\begin{aligned}&\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q)\\&=\frac{1}{\mathcal{B}(\delta,c-\delta)}\int_0^1 t^{\delta-1}\,(1-t)^{c-\delta-1}\mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-p}{t^\alpha}\right)\\&\times\mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-q}{(1-t)^\beta}\right)\mathcal{J}^{r,c}_{s,q}(t^q z)\,dt.\end{aligned} \tag{16}$$

Proof. Utilizing the definition of NBMF (13) and apply (14), we get

$$\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q)$$

$$= \sum_{n=0}^{\infty} \int_0^1 t^{\delta+qn-1} \ (1-t)^{c-\delta-1} \ \mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2} \left(\frac{-p}{t^\alpha}\right) \mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2} \left(\frac{-q}{(1-t)^\beta}\right) \ dt$$
$$\times \frac{(c)_{qn}(-z)^n}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+s+1)n!} \ .$$

By changing the order of summation and integration (which is guaranteed under the given condition) in the above expression, we obtain

$$\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q)$$
$$= \frac{1}{\mathcal{B}(\delta, c-\delta)} \int_0^1 t^{\delta-1} \ (1-t)^{c-\delta-1} \ \mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2} \left(\frac{-p}{t^\alpha}\right) \mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2} \left(\frac{-q}{(1-t)^\beta}\right) \ dt$$
$$\times \sum_{n=0}^{\infty} \frac{(c)_{qn}(-t^q z)^n}{\Gamma(rn+s+1)n!} \ .$$

Now, with the help of (2), we get our required result from the above expression. □

Theorem 3.2. *Another integral representation of the novel Bessel-Maitland function (NBMF) is as follows:*

$$\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q)$$
$$= \frac{2}{\mathcal{B}(\delta, c-\delta)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\delta-1} (\cos\theta)^{2c-2\delta-1} \mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2} \left(\frac{-p}{\sin^{2\alpha}\theta}\right)$$
$$\times \mathcal{E}^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2} \left(\frac{-q}{\cos^{2\beta}\theta}\right) \mathcal{J}^{r,c}_{s,q}(z \sin^{2q}\theta) \ d\theta \tag{17}$$

Proof. By putting $t = \sin^2\theta$ in (16) and after some simple calculation we get the desired result (17). □

Theorem 3.3. *For $k \in \mathbb{N}; p > 0, q \in \mathbb{N}; \Re(c) > \Re(\delta) > 0$, the successive differentiation formula for the novel Bessel-Maitland function(NBMF) is defined as:*

$$\frac{d^k}{dz^k} \left[\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q)\right]$$
$$= (-1)^k (c)_q (c+q)_q ... (c+(k-1)q)_q \ \mathcal{J}^{r,\delta+kq,c+kq,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+kr,\alpha,\beta,u,v,\tau}(z;p,q). \tag{18}$$

Proof. In (13), taking the differentiation w. r. t. 'z', we obtain

$$\frac{d}{dz}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q) = \sum_{n=1}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn, c-\delta)(c)_{qn}(-1)^n(z)^{n-1}}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+s+1)(n-1)!}$$
$$= \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn+q, c-\delta)(c)_{qn+q}(-1)^{n+1}(z)^n}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+r+s+1)(n)!}$$
$$=(-1)(c)_q \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn+q, c-\delta)(c+q)_{qn}(-z)^n}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+r+s+1)(n)!}$$
$$=(-1)(c)_q \mathcal{J}^{r,\delta+q,c+q,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+r,\alpha,\beta,u,v,\tau}(z;p,q). \tag{19}$$

Again taking the derivative w.r.t.'z' in (19), we get

$$\frac{d^2}{dz^2}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q) = (-1)^2(c)_q(c+q)_q\mathcal{J}^{r,\delta+2q,c+2q,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+2r,\alpha,\beta,u,v,\tau}(z;p,q).$$

Repeating this process k times, we get the desired result (18). □

Corollary 3.3. For, $k=1$ in (18), we get the first order derivative formula for the novel Bessel-Maitland function (NBMF) (13):

$$\frac{d}{dz}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q) = (-1)(c)_q\mathcal{J}^{r,\delta+q,c+q,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+r,\alpha,\beta,u,v,\tau}(z;p,q).$$

Theorem 3.4. *Let $p>0,\ q\in\mathbb{N};\Re(\omega)>0;\Re(c)>\Re(\delta)>0$, then the novel Bessel-Maitland function(NBMF) holds the following derivative formula:*

$$\frac{d^k}{dz^k}\left[z^s\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(\omega z^r;p,q)\right] = z^{s-k}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s-k,\alpha,\beta,u,v,\tau}(\omega z^r;p,q). \tag{20}$$

Proof. Replace z by ωz^r in (13) and after simplifying we take the derivative w.r.t. 'z'

$$\frac{d}{dz}\left[z^s\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(\omega z^r;p,q)\right]$$

$$=\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-\omega)^n(rn+s)(z)^{rn+s-1}}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)(n)!}$$

$$= z^{s-1}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-\omega z^r)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s)(n)!}$$

$$= z^{s-1}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s-1,\alpha,\beta,u,v,\tau}(\omega z^r;p,q).$$

Differentiating k times with respect to z completes the proof of the theorem. □

Corollary 3.4. If we substitute $k=1$ in (20), we obtain the following derivative formula for the novel Bessel-Maitland function (NBMF) (13):

$$\frac{d}{dz}\left[z^s\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(\omega z^r;p,q)\right] = z^{s-1}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s-1,\alpha,\beta,u,v,\tau}(\omega z^r;p,q).$$

4. Recurrence Formula

In this section, we investigate recurrence formula for the novel Bessel-Maitland function(NBMF) (13).

Theorem 4.1. *The recurrence relation for the NBMF is given by the following formula:*

$$(s+k)\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+k,\alpha,\beta,u,v,\tau}(z;p,q) + rz\frac{d}{dz}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+k,\alpha,\beta,u,v,\tau}(z;p,q) = \mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+k-1,\alpha,\beta,u,v,\tau}(z;p,q), \tag{21}$$

$$(k\in\mathbb{N};p>0,\ q\in\mathbb{N};\ \Re(c)>\Re(\delta)>0).$$

Proof. Using definition (13) on the left hand side of (21), we get

$$
\begin{aligned}
&(s+k)\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+k,\alpha,\beta,u,v,\tau}(z;p,q)+rz\frac{d}{dz}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+k,\alpha,\beta,u,v,\tau}(z;p,q)\\
=&(s+k)\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-z)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+k+1)n!}\\
&+rz\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-z)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+k+1)n!}\\
=&\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-z)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+k+1)n!}(rn+s+k)\\
=&\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-z)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+k)n!}\\
=&\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+k-1,\alpha,\beta,u,v,\tau}(z;p,q).
\end{aligned}
$$

□

Corollary 4.1. If we substitute $k=1$ in (21), we get

$$
(s+1)\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+1,\alpha,\beta,u,v,\tau}(z;p,q)+rz\frac{d}{dz}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+1,\alpha,\beta,u,v,\tau}(z;p,q)=\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q). \quad (22)
$$

5. Integral Transforms

In this section, we derive several integral transforms for the newly introduced function (13), such as Mellin transform, Beta transform, Laplace transform, Whittaker transform and K-transform, which are asserted by the following theorems.

Theorem 5.1. *The Mellin transform of the novel Bessel-Maitland function (NBMF) defined by*

$$
\mathcal{M}\left\{e^{-az}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q);\eta\right\}
$$

$$
=\frac{1}{a^{\eta}\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{\Gamma(\eta+n)}{a^n}, \quad (23)
$$

$$
(p>0, q\in\mathbb{N};\ s,r,\delta\in\mathbb{C};\ \Re(c)>\Re(\delta)>0;\ \sigma_1,\lambda_1,\sigma_2,\lambda_2\geq 0)
$$

$$
(\alpha,\beta>0;\ \Re(a)>0,\Re(u),\Re(v),\Re(\tau)>0).
$$

Proof. Utilize the definition of the Mellin transform (8) on novel Bessel-Maitland function (13), we obtain

$$
\mathcal{M}\left\{e^{-az}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q);\eta\right\}=\int_0^{\infty}t^{\eta-1}\ e^{-at}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(t;p,q)dt,
$$

now, by applying the definition of NBMF (13) in the above expression, we get

$$\mathcal{M}\left\{e^{-az}\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q);\eta\right\}$$
$$=\int_0^\infty t^{\eta-1}\ e^{-at}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-t)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)n!}dt$$
$$=\frac{1}{\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\int_0^\infty e^{-at}\ t^{\eta+n-1}\ dt$$
$$=\frac{1}{\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{\Gamma(\eta+n)}{a^{\eta+n}}$$
$$=\frac{1}{a^\eta\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{\Gamma(\eta+n)}{a^n}.$$

□

Theorem 5.2. *Let $r,\delta,s\in\mathbb{C};p>0,q\in\mathbb{N};\Re(c)>\Re(\delta)>0,\Re(s)>0$, the Euler-Beta transform of the novel Bessel Maitland function(NBMF) is:*

$$B\left\{\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z^r;p,q);s+1,1\right\}=\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s+1,\alpha,\beta,u,v,\tau}(1;p,q). \tag{24}$$

Proof. By applying the definition of Euler transform (9) in (13), we get

$$B\left\{\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z^r;p,q);s+1,1\right\}$$
$$=\int_0^1 t^{s+1-1}(1-t)^{1-1}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-t^r)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)n!}dt,$$

interchanging the order of summation and integration which is guaranteed under the convergence condition, we get from the above expression

$$=\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)n!}\int_0^1 t^{rn+s+1-1}(1-t)^{1-1}dt,$$

with the help of (3) and (13), we get the desired result (24) from the above expression. □

Theorem 5.3. *The Laplace transform for the novel Bessel-Maitland function(NBMF) is specified through:*

$$\mathcal{L}\left\{\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q);\eta\right\}$$
$$=\frac{1}{\eta\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)\eta^n}, \tag{25}$$

$$(p>0,q\in\mathbb{N};\ s,r,\delta\in\mathbb{C};\ \Re(c)>\Re(\delta)>0;\ \sigma_1,\lambda_1,\sigma_2,\lambda_2\geq 0)$$
$$(\alpha,\beta>0;\Re(u),\Re(v),\Re(\tau)>0).$$

Proof. Using Laplace transform (10) on the novel Bessel-Maitland function (13), we obtain

$$\mathcal{L}\left\{\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(z;p,q);\eta\right\}$$
$$=\int_0^\infty e^{-\eta t}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-t)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)n!}\ dt$$
$$=\frac{1}{\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\int_0^\infty e^{-\eta t}t^n\ dt$$
$$=\frac{1}{\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{\Gamma(n+1)}{\eta^{n+1}}$$
$$=\frac{1}{\eta\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{n!}{\eta^n}$$
$$=\frac{1}{\eta\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)\eta^n}.$$

□

Theorem 5.4. *Let* $r,\delta,s,\rho,c\in\mathbb{C}; p>0, q\in\mathbb{N}; \Re(c)>\Re(\delta)>0; \Re(s)>0, \Re(\rho)>0, \Re(\mu\pm\lambda)>0$, *the Whittaker-transform of the novel Bessel-Maitland function(NBMF) defined as:*

$$\int_0^\infty u^{\mu-1}e^{(\frac{-\rho u}{2})}\mathcal{W}_{\lambda,\nu}(\rho u)\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(u;p,q)du$$
$$=\frac{1}{\rho^\mu\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-\rho)^{-n}}{\Gamma(rn+s+1)n!}\frac{\Gamma(\frac{1}{2}\pm\nu+\mu+n)}{\Gamma(1-\lambda+\mu+n)}. \tag{26}$$

Proof. Applying Whittaker-transform in the definition of novel Bessel-Maitland function (13), we have

$$\int_0^\infty u^{\mu-1}e^{(\frac{-\rho u}{2})}\mathcal{W}_{\lambda,\nu}(\rho u)\mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(u;p,q)du$$
$$=\int_0^\infty u^{\mu-1}e^{(\frac{-\rho u}{2})}\mathcal{W}_{\lambda,\nu}(\rho u)\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-u)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)n!}du$$
$$=\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)n!}\int_0^\infty u^{\mu+n-1}e^{\frac{-\rho u}{2}}\mathcal{W}_{\lambda,\nu}(\rho u)du,$$

putting $\rho u=t$ in the above expression, we get

$$=\frac{1}{\rho^\mu\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn,c-\delta)(c)_{qn}(-\rho)^{-n}}{\Gamma(rn+s+1)n!}\int_0^\infty t^{\mu+n-1}e^{\frac{-t}{2}}\mathcal{W}_{\lambda,\nu}(t)dt. \tag{27}$$

The well-known integral formula [18]

$$\int_0^\infty a^{\mu-1}e^{-\frac{a}{2}}\mathcal{W}_{\lambda,\nu}(a)da=\frac{\Gamma(\frac{1}{2}\pm\nu+\mu)}{\Gamma(1-\lambda+\mu)}, \tag{28}$$

using (28) in (27), we obtain

$$= \frac{1}{\rho^{\mu}\mathcal{B}(\delta, c-\delta)} \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn, c-\delta)(c)_{qn}(-\rho)^{-n}}{\Gamma(rn+s+1)n!} \frac{\Gamma(\frac{1}{2}\pm\nu+\mu+n)}{\Gamma(1-\lambda+\mu+n)}.$$

□

Theorem 5.5. *The K-transform of the novel Bessel-Maitland function(NBMF) is specified through:*

$$\int_0^{\infty} u^{\mu-1} K_{\lambda}(\rho u) \mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(u;p,q)du$$

$$= \frac{2^{\mu-2}}{\rho^{\mu}\mathcal{B}(\delta, c-\delta)} \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn, c-\delta)(c)_{qn}(-2\rho^{-1})^{n}}{\Gamma(rn+s+1)n!} \frac{\Gamma(\mu+n\pm\lambda)}{2}, \tag{29}$$

$$(r,\delta,s,\rho,c \in \mathbb{C}; p>0, q\in\mathbb{N}; \Re(c)>\Re(\delta)>0; \Re(s)>0, \Re(\rho)>0, \Re(\mu\pm\lambda)>0).$$

Proof. Use the definition of K-transform in (13), we get

$$\int_0^{\infty} u^{\mu-1} K_{\lambda}(\rho u) \mathcal{J}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}_{s,\alpha,\beta,u,v,\tau}(u;p,q)du$$
$$= \int_0^{\infty} u^{\mu-1} K_{\lambda}(\rho u) \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn, c-\delta)(c)_{qn}(-u)^{n}}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+s+1)n!} du$$
$$= \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn, c-\delta)(c)_{qn}(-1)^{n}}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+s+1)n!} \int_0^{\infty} u^{\mu+n-1} K_{\lambda}(\rho u)du,$$

putting $\rho u = t$ in the above expression, we have

$$= \frac{1}{\rho^{\mu}} \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn, c-\delta)(c)_{qn}(-\rho)^{-n}}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+s+1)n!} \int_0^{\infty} t^{\mu+n-1} K_{\lambda}(t)\, dt.$$

The integral formula for the K-transform defined as [18]:

$$\int_0^{\infty} a^{\mu-1} K_b(a)da = 2^{\mu-2}\ \frac{\Gamma(\mu\pm b)}{2},$$

from the above expression, we obtain

$$= \frac{2^{\mu-2}}{\rho^{\mu}\mathcal{B}(\delta, c-\delta)} \sum_{n=0}^{\infty} \frac{\mathcal{B}^{u,v,\tau,\alpha,\beta,p,q}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\delta+qn, c-\delta)(c)_{qn}(-2\rho^{-1})^{n}}{\Gamma(rn+s+1)n!} \frac{\Gamma(\mu+n\pm\lambda)}{2}.$$

□

6. Riemann-Liouville Fractional Integrals

In this section, we deals with the fractional integral properties of the novel Bessel-Maitland function(NBMF) and we present some important theorems involving left and right sided Riemann-Liouville fractional integral operators.

Theorem 6.1. *Let $p > 0, q \in \mathbb{N};\ s, r, \delta \in \mathbb{C};\ \Re(c) > \Re(\delta) > 0;\ \sigma_1, \lambda_1, \sigma_2, \lambda_2 \geq 0; \alpha, \beta > 0;\ \Re(a) > 0, \Re(u), \Re(v), \Re(\tau) > 0$, the fractional integral of novel Bessel-Maitland function(NBMF) is specified through:*

$$\left(I_{0+}^{\mu}[z^a \mathcal{J}_{s,\alpha,\beta,u,v,\tau}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z;p,q)]\right)(m)$$

$$= \frac{m^{\mu+a}}{\Gamma(\mu)\mathcal{B}(\delta, c-\delta)} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn, c-\delta)(c)_{qn}(-m)^n}{\Gamma(rn+s+1)n!} \mathcal{B}(a+n+1, \mu). \tag{30}$$

Proof. Utilizing (11) in (13), we obtain

$$\begin{aligned}
&\left(I_{0+}^{\mu}[z^a \mathcal{J}_{s,\alpha,\beta,u,v,\tau}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z;p,q)]\right)(m) \\
=&\frac{1}{\Gamma(\mu)} \int_0^m (m-u)^{\mu-1} u^a \mathcal{J}_{s,\alpha,\beta,u,v,\tau}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(u;p,q)du \\
=&\frac{1}{\Gamma(\mu)} \int_0^m (m-u)^{\mu-1} u^a \sum_{n=0}^{\infty} \frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn, c-\delta)(c)_{qn}(-u)^n}{\mathcal{B}(\delta, c-\delta)\Gamma(rn+s+1)n!} du \\
=&\frac{1}{\Gamma(\mu)\mathcal{B}(\delta, c-\delta)} \int_0^m (m-u)^{\mu-1} u^a \sum_{n=0}^{\infty} \frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn, c-\delta)(c)_{qn}(-u)^n}{\Gamma(rn+s+1)n!} du.
\end{aligned}$$

Now, using the convergence condition to reverse the order of the integration and the summation in the above expression, we have

$$\begin{aligned}
=&\frac{1}{\Gamma(\mu)\mathcal{B}(\delta, c-\delta)} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn, c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!} \int_0^m (m-u)^{\mu-1} u^{a+n} du \\
=&\frac{m^{\mu+a}}{\Gamma(\mu)\mathcal{B}(\delta, c-\delta)} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn, c-\delta)(c)_{qn}(-m)^n}{\Gamma(rn+s+1)n!} \int_0^1 (1-t)^{\mu-1} t^{a+n}\ dt,
\end{aligned}$$

using (3) in the above expression, we get

$$= \frac{m^{\mu+a}}{\Gamma(\mu)\mathcal{B}(\delta, c-\delta)} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn, c-\delta)(c)_{qn}(-m)^n}{\Gamma(rn+s+1)n!} \mathcal{B}(a+n+1, \mu).$$

□

Theorem 6.2. *Let $p > 0, q \in \mathbb{N};\ s, r, \delta \in \mathbb{C};\ \Re(c) > \Re(\delta) > 0;\ \sigma_1, \lambda_1, \sigma_2, \lambda_2 \geq 0; \alpha, \beta > 0;\ \Re(a) > 0, \Re(u), \Re(v), \Re(\tau) > 0$, the fractional integral of novel Bessel-Maitland function(NBMF) is specified through:*

$$\left(I_{0-}^{\mu}[\left(\frac{1}{z}\right)^a \mathcal{J}_{s,\alpha,\beta,u,v,\tau}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z;p,q)]\right)(m)$$

$$= \frac{m^{\mu-a}}{\Gamma(\mu)\mathcal{B}(\delta, c-\delta)} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn, c-\delta)(c)_{qn}(-m)^n}{\Gamma(rn+s+1)n!} \mathcal{B}(a-\mu-n, \mu). \tag{31}$$

Proof. Utilizing (12) in (13), we obtain

$$\left(I_{0-}^{\mu}[\left(\frac{1}{z}\right)^{a}\mathcal{J}_{s,\alpha,\beta,u,v,\tau}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z;p,q)]\right)(m)$$
$$=\frac{1}{\Gamma(\mu)}\int_m^{\infty}(u-m)^{\mu-1}u^{-a}\mathcal{J}_{s,\alpha,\beta,u,v,\tau}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(u;p,q)du$$
$$=\frac{1}{\Gamma(\mu)}\int_m^{\infty}(u-m)^{\mu-1}u^{-a}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn,c-\delta)(c)_{qn}(-u)^n}{\mathcal{B}(\delta,c-\delta)\Gamma(rn+s+1)n!}du$$
$$=\frac{1}{\Gamma(\mu)\mathcal{B}(\delta,c-\delta)}\int_m^{\infty}(u-m)^{\mu-1}u^{-a}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn,c-\delta)(c)_{qn}(-u)^n}{\Gamma(rn+s+1)n!}du.$$

$$=\frac{1}{\Gamma(\mu)\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn,c-\delta)(c)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\int_m^{\infty}(u-m)^{\mu-1}u^{-a+n}du$$
$$=\frac{m^{\mu-a}}{\Gamma(\mu)\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn,c-\delta)(c)_{qn}(-m)^n}{\Gamma(rn+s+1)n!}\int_0^{1}(1-t)^{\mu-1}t^{a-\mu-n-1}\,dt,$$

using (3) in the above expression, we get

$$=\frac{m^{\mu-a}}{\Gamma(\mu)\mathcal{B}(\delta,c-\delta)}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(\delta+qn,c-\delta)(c)_{qn}(-m)^n}{\Gamma(rn+s+1)n!}\mathcal{B}(a-\mu-n,\mu).$$

□

7. Conclusions

In the present investigation, we have attempted to introduce a novel type of Bessel-Maitland function(NBMF) involving the extended beta function (14). We have discussed its certain important properties such as integral representations, recurrence relation and derivative formulae. Further, we investigated some integral transform such as Mellin transform, Euler-Beta transform, Laplace transform, Whittaker transform and K-transform. At the end of our investigation, we have presented the Riemann-Liouville fractional integral formulae related to the novel Bessel-Maitland function.

The Bessel-Maitland function's flexibility makes it an excellent tool for solving generalized differential equations in mathematical physics. By incorporating various parameters, it provides a powerful method to tackle complex problems, offering insights and precise solutions where classical functions fall short. Let's, consider the generalized differential equation where the parameters $p,q,s,\alpha,\beta,u,v,\tau,r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2$ from the Bessel-Maitland function come into play:

$$z^2\frac{d^2y}{dz^2}+(1+\tau z)z\frac{dy}{dz}+\left[(\beta z^2+\alpha z+\delta)-\frac{\nu^2-\frac{(\sigma_1+\lambda_1 z)}{(\sigma_2+\lambda_2 z)}}{(1+\tau z)^2}\right]y=0, \tag{32}$$

the solution to this differential equation can be expressed using the Bessel-Maitland function: $y(z)=z^s\mathcal{J}_{s,\alpha,\beta,u,v,\tau}^{r,\delta,c,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z;p,q)$, where the parameters are chosen to match the coefficients in the differential equation.

This function has various applications in the representation of the solutions of various types of engineering and mathematical physics problems. In future, researchers can produce some other important integral transform like Fourier transform and Hankel transform related to this function or also they can generalize our presented results to produce some important applications in the field of mathematical and engineering science.

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