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# CHARACTERIZATION OF PYTHAGOREAN FUZZY $\mathfrak{K}$-IDEALS IN $\Gamma$-SEMIRINGS

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Abstract. In this paper, we introduce the algebraic structures of Pythagorean fuzzy set in $\Gamma$ semirings. Moreover we characteristics some interesting properties, results.
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## 1. Introduction

In 1964, Nobusawa extended the concept of rings to introduce $\Gamma$-rings, providing a broader mathematical framework. Sen[14] further advanced this notion in 1981 by introducing $\Gamma$ semigroups as a generalization of $\Gamma$ groups. Expanding upon these developments, Murali Krishna Rao [12] explored the concept of $\Gamma$ semirings in 1995, extending the scope of $\Gamma$ rings. More recently, Yager[17, 18] proposed Pythagorean fuzzy sets as a powerful tool for effectively managing uncertainty or imprecise information in real-world scenarios. These sets enforce a constraint where the sum of squares of membership and non-membership degrees is less than or equal to 1. Pythagorean fuzzy sets have showcased remarkable efficacy in navigating uncertainties, prompting a surge of scholarly exploration across diverse research avenues, resulting in significant progress. The conceptualization of Pythagorean fuzzy sets facilitates a more comprehensive and accurate portrayal of uncertain information when juxtaposed with intuitionistic fuzzy sets. Across various disciplines, academics have meticulously examined the algebraic attributes of Pythagorean fuzzy sets, shedding light on their practical applications and foundational theoretical constructs.

Throughout this paper, we will denote a $\Gamma$-semiring as $\mathfrak{S}$ and a Pythagorean fuzzy set as $\mathfrak{A}$. The paper is structured into four sections. The first and second sections serve as the introduction and cover basic results pertinent to the paper's topic. In the third section, we introduce Pythagorean fuzzy $\mathfrak{K}$-ideals in $\Gamma$-semirings. Finally, the fourth section discusses various properties of Pythagorean fuzzy $\mathfrak{K}$-ideals within $\Gamma$-semirings.

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## 2. Preliminaries

In this section we present the basic concepts related to this paper.

Basic results from [4, 5, 12, 13]

(1) If $(\mathfrak{S}, +)$ and $(\Gamma, +)$ be two commutative semigroups then $\mathfrak{S}$ is called a $\Gamma$ - semiring if there exists a structure $\mathfrak{S} \times \Gamma \times \mathfrak{S}$ denoted by $\sigma\gamma\rho$ for all $\sigma, \rho \in \mathfrak{S}$ and $\gamma \in \Gamma$ satisfying the following properties, $\sigma\gamma(\rho+\nu) = \sigma\gamma\rho+\sigma\gamma\nu$, $(\rho+\nu)\gamma\sigma = \rho\gamma\sigma+\nu\gamma\sigma$, $\sigma(\gamma + \gamma_1)\nu = \sigma\gamma\nu + \sigma\gamma_1\nu$, $\sigma\gamma(\rho\gamma_1\nu) = (\sigma\gamma\rho)\gamma_1\nu$ for all $\sigma, \rho, \nu \in \mathfrak{S}$ and $\gamma, \gamma_1 \in \Gamma$.

(2) Define addition in the following way $A, B \in \mathfrak{R}$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{\sigma\gamma\rho/\sigma, \rho \in \mathfrak{S}\}$. Then $\mathfrak{S}$ is a $\Gamma$ - semiring.

(3) A $\Gamma$ - semiring $\mathfrak{S}$ is said to be commutative if $\sigma\gamma\rho = \rho\gamma\sigma$, for all $\sigma, \rho \in \mathfrak{S}$ and $\gamma \in \Gamma$.

(4) A $\Gamma$ - semiring $\mathfrak{S}$ is said to have a zero element if $0\rho\sigma = 0 = \sigma\rho 0$ and $\sigma + 0 = \sigma = 0 + \sigma$, for all $\sigma \in \mathfrak{S}$ and $\gamma \in \Gamma$.

(5) If $\mathfrak{S}$ is said to have a identity element if there exists $\gamma \in \Gamma$ such that $1\gamma\sigma = \sigma = \sigma\gamma 1$ for all $\sigma \in \mathfrak{S}$.

(6) If $\mathfrak{S}$ is said to have a strong identity element if for all $\sigma \in \mathfrak{S}$, $1\gamma\sigma = \sigma = \sigma\gamma 1$ for all $\gamma \in \Gamma$.

(7) A non-empty subset $R$ of a $\Gamma$ - semiring $\mathfrak{S}$ is said to be a sub$\Gamma$ - semiring of $\mathfrak{S}$ if $(R, +)$ is a sub semigroup of $(\mathfrak{S}, +)$ and $\sigma\gamma\rho \in \mathfrak{S}$ for all $\sigma, \rho \in \mathfrak{S}$ and $\gamma \in \Gamma$.

(8) A non-empty subset $R$ of a $\Gamma$ - semiring $\mathfrak{S}$ is called an ideal if $\sigma, \rho \in R$ implies $\sigma + \rho \in R$ and $a \in R, \sigma \in \mathfrak{S}$ and $\gamma \in \Gamma$ implies $\sigma\gamma a \in R$ and $a\sigma\gamma \in R$.

(9) A left ideal $\mathfrak{A}$ of $\mathfrak{S}$ is called a left $\mathfrak{K}$ - ideal of $\mathfrak{S}$ if $\rho, \nu \in \mathfrak{A}$, $\sigma \in \mathfrak{S}$, and $\sigma + \rho = \nu$ implies $\sigma \in \mathfrak{A}$.

**Definition 2.1.** [17] *Let $X$ be a non-empty set. A Pythagorean Fuzzy Set(PFS) $\mathfrak{A}$ in $X$ is given by $\mathfrak{A} = \{\sigma, \mathfrak{A}_x(\sigma), \mathfrak{A}_y(\sigma)/\sigma \in X\}$ where $\mathfrak{A}_x : X \to [0, 1]$ and $\mathfrak{A}_y : X \to [0, 1]$ represent the degree of membership and degree of non-membership of $\mathfrak{A}$ respectively. Also, $\mathfrak{A}_x$ and $\mathfrak{A}_y$ satisfies the condition $0 \leq (\mathfrak{A}_x)^2 + (\mathfrak{A}_y)^2 \leq 1$ for all $\sigma \in X$.*

**Theorem 2.1.** [15] *Let $\mathfrak{S}$ be a $\Gamma$ - semiring. A fuzzy set $\mathfrak{A}$ of $\mathfrak{S}$ is a fuzzy right (left) ideal if and only if $U(\mathfrak{A}, t)$ is a right (left) ideal of $\mathfrak{S}$ for all $t \in [0, 1]$.*

**Theorem 2.2.** [15] *Let $\mathfrak{S}$ be a $\Gamma$ - semiring. A fuzzy set $\mathfrak{A}$ of $\mathfrak{S}$ is an anti fuzzy right (left) ideal if and only if $L(\mathfrak{A}, t)$ is a right (left) ideal of $\mathfrak{S}$ for all $t \in [0, 1]$.*

**Theorem 2.3.** [15] *Let $\mathfrak{S}$ be a $\Gamma$ - semiring. A fuzzy set(FS) $\mathfrak{A}$ is a fuzzy right (left) ideal of $\mathfrak{S}$ if and only if $1 - \mathfrak{A} = \overline{\mathfrak{A}}$ is an anti fuzzy right (left) ideal of $\mathfrak{S}$.*

**Theorem 2.4.** [16] *Let $\mathfrak{S}$ be a $\Gamma$ - semiring. A fuzzy set $\mathfrak{A}$ of $\mathfrak{S}$ is a fuzzy $\mathfrak{K}$ - ideal(FKI) if and only if $U(\mathfrak{A}, t)$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$ for all $t \in [0, 1]$.*

**Theorem 2.5.** [16] *Let $\mathfrak{S}$ be a $\Gamma$ - semiring. A fuzzy set $\mathfrak{A}$ of $\mathfrak{S}$ is an anti fuzzy $\mathfrak{K}$ - ideal(FKI) if and only if $L(\mathfrak{A}, t)$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$ for all $t \in [0, 1]$.*

## 3. Pythagorean fuzzy $\mathfrak{K}$ - ideal $\Gamma$ - semirings

This section deals with the Pythagorean fuzzy $\mathfrak{K}$ - ideals(PFKI) in $\Gamma$ - semirings. Some important properties are discussed .

**Definition 3.1.** *A Pythagorean fuzzy ideal(PFI) $\mathfrak{A}$ in $\mathfrak{S}$ is called a PFKI in $\mathfrak{S}$ if the following conditions are hold for all $\sigma, \rho \in \mathfrak{S}$.*

(i) $\mathfrak{A}_x(\sigma) \geq \mathfrak{A}_x(\sigma + \rho) \wedge \mathfrak{A}_x(\rho)$

(ii) $\mathfrak{A}_y(\sigma) \leq \mathfrak{A}_y(\sigma + \rho) \vee \mathfrak{A}_y(\rho)$

**Definition 3.2.** *A Pythagorean fuzzy set (PFS) $\mathfrak{A}$ in $\mathfrak{S}$. Let $s, t \in [0,1]$. Then the set $\mathfrak{R}_{\mathfrak{A}}^{(s,t)} = \{\sigma \in \mathfrak{S} | \mathfrak{A}_x(\sigma) \geq s, \mathfrak{A}_y(\sigma) \leq t\}$ is called $(s,t)-$ level set of $\mathfrak{A}$.*

**Example 3.1.** *Let $\mathfrak{S} = \{0, 1, 2, 3\}$ be a $\Gamma$ - semiring with the following multiplication table. Let $\mathfrak{A}$ be a PFI of $\mathfrak{S}$ defined by, $\mathfrak{A}_x(i) = \{0.8, 0.6, 0.3, 0.3\}$,*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 2 |

| • | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 |
| 2 | 0 | 1 | 1 | 1 |
| 3 | 0 | 1 | 1 | 1 |

$\mathfrak{A}_y(i) = \{0.2, 0.3, 0.7, 0.8\}$. *Then $\mathfrak{A}$ is a PFS $\mathfrak{A}$ of $\mathfrak{S}$.*

**Theorem 3.1.** *A PFS $\mathfrak{A} = (\mathfrak{A}_x, \mathfrak{A}_y)$ of $\mathfrak{S}$ is a PFKI of $\mathfrak{S}$ if and only if $\mathbb{U}(\mathfrak{A}_x, t)$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$ for all $t \in [0,1]$.*

*Proof.* Let $\mathfrak{A}_x$ be a PFKI in $\mathfrak{S}$ and $t \in [0,1]$. If there exists $\sigma, \rho \in \mathfrak{S}$ such that $\sigma + \rho, \rho \in U(\mathfrak{A}_x, t)$ and $\sigma \notin U(\mathfrak{A}_x, t)$.
Implies $\mathfrak{A}_x(\sigma + \rho) \wedge \mathfrak{A}_x(\rho) \geq t > \mathfrak{A}_x(\sigma)$, a contradiction. By thm2.4 $U(\mathfrak{A}_x, t)$ is a $\mathfrak{K}$ - ideal in $\mathfrak{S}$.
Conversely let us assume that $\sigma, \rho \in \mathfrak{S}$ such that $\mathfrak{A}_x(\sigma) < \mathfrak{A}_x(\sigma + \rho) \wedge \mathfrak{A}_x(\rho)$.
Then we get, $\sigma + \rho, \rho \in U(\mathfrak{A}_x, t)$ and $\sigma \notin U(\mathfrak{A}_x, t)$ where $t = \mathfrak{A}_x(\sigma + \rho) \wedge \mathfrak{A}_x(\rho)$, a contradiction. By thm2.4 $\mathfrak{A}_x$ be a PFKI in $\mathfrak{S}$. □

**Theorem 3.2.** *A PFS $\mathfrak{A} = (\mathfrak{A}_x, \mathfrak{A}_y)$ of $\mathfrak{S}$ is a Pythagorean anti fuzzy $\mathfrak{K}$ - ideal(PAFKI) of $\mathfrak{S}$ if and only if $L(\mathfrak{A}_y, t)$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$ for all $t \in [0,1]$.*

*Proof.* Let $\mathfrak{A}_y$ be a PAFKI in $\mathfrak{S}$ and $t \in [0,1]$. If there exists $\sigma, \rho \in \mathfrak{S}$ such that $\sigma + \rho, \rho \in L(\mathfrak{A}_y, t)$ and $\rho \notin U(\mathfrak{A}_y, t)$.
Implies $\mathfrak{A}_x(\sigma + \rho) \vee \mathfrak{A}_x(\rho) \leq t \leq \mathfrak{A}_x(\sigma)$, a contradiction. By thm2.5 $L(\mathfrak{A}_x, t)$ is a $\mathfrak{K}$ - ideal in $\mathfrak{S}$. Conversely let us assume that $\sigma, \rho \in \mathfrak{S}$ such that $\mathfrak{A}_y(\sigma) > \mathfrak{A}_y(\sigma + \rho) \vee \mathfrak{A}_y(\rho)$.
Then we get, $\sigma + \rho, \rho \in L(\mathfrak{A}_y, t)$ and $\sigma \notin L(\mathfrak{A}_y, t)$ where $t = \mathfrak{A}_y(\sigma + \rho) \wedge \mathfrak{A}_y(\rho)$, a contradiction. By thm2.5 $\mathfrak{A}_y$ be a PAFKI in $\mathfrak{S}$. □

**Theorem 3.3.** *A PFS $\mathfrak{A} = (\mathfrak{A}_x, \mathfrak{A}_y)$ of $\mathfrak{S}$ is a PFKI of $\mathfrak{S}$ if and only if $\mathfrak{R}_{\mathfrak{A}}^{(s,t)}$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$ for all $s, t \in [0,1]$.*

*Proof.* Let $\mathfrak{A}$ be a PFKI in $\mathfrak{S}$ then clearly.
$\mathfrak{R}_{\mathfrak{A}}^{(s,t)} = U(\mathfrak{A}_x, t) \cap L(\mathfrak{A}_y, t)$.
Then by thm 2.4 and thm 2.5 $\mathfrak{R}_{\mathfrak{A}}^{(s,t)}$ is a $\mathfrak{K}$ - ideal in $\mathfrak{S}$ for all $s, t \in [0,1]$.
Conversely by thm2.1 $\mathfrak{A}$ is a PFI in $\mathfrak{S}$.
Let $\sigma, \rho \in S$ , $\mathfrak{A}_x(\sigma + \rho) \wedge \mathfrak{A}_x(\rho) = s$ and $\mathfrak{A}_y(\sigma + \rho) \vee \mathfrak{A}_y(\rho) = t$
Then $\sigma + \rho, \rho \in \mathfrak{R}_{\mathfrak{A}}^{(s,t)}$ implies $\sigma \in \mathfrak{R}_{\mathfrak{A}}^{(s,t)}$.

Hence $\mathfrak{A}_x(\sigma) \geq \mathfrak{A}_x(\sigma+\rho) \wedge \mathfrak{A}_x(\rho)$ and $\mathfrak{A}_y(\sigma) \geq \mathfrak{A}_y(\sigma+\rho) \vee \mathfrak{A}_y(\rho)$.
Thus $\mathfrak{A}$ is a PFKI in $\mathfrak{S}$. □

**Corollary 3.1.** *A PFS $\mathfrak{A} = (\mathfrak{A}_x, \mathfrak{A}_y)$ of $\mathfrak{S}$ is a PFKI of $\mathfrak{S}$ if and only if $\mathfrak{A}_x$ is a FKI of $\mathfrak{S}$ and $\mathfrak{A}_y$ is a anti FKI of $\mathfrak{S}$.*

**Corollary 3.2.** *A PFS $\mathfrak{A} = \{\mathfrak{A}_x(\sigma), (1-\mathfrak{A}_x)(\sigma)/\sigma \in \mathfrak{S}\}$ of $\mathfrak{S}$ is a PFKI of $\mathfrak{S}$ if and only if $\mathfrak{A}_x$ is a FKI of $\mathfrak{S}$.*

**Corollary 3.3.** *A PFS $\mathfrak{A} = \{(1-\mathfrak{A}_y)(\sigma), \mathfrak{A}_y(\sigma)/\sigma \in \mathfrak{S}\}$ of $\mathfrak{S}$ is a PFKI of $\mathfrak{S}$ if and only if $\mathfrak{A}_y$ is an anti FKI of $\mathfrak{S}$.*

**Theorem 3.4.** *Let $\mathfrak{A}$ be a PFS of $\mathfrak{S}$ such that $\mathfrak{A}_x$ is a PFKI of $\mathfrak{S}$ then $\mathfrak{A} = (\mathfrak{A}_x, \overline{\mathfrak{A}_x})$ is a PFKI of $\mathfrak{S}$.*

*Proof.* Let $\sigma, \rho \in \mathfrak{S}$. Since $\mathfrak{A}_x$ is a PFKI of $\mathfrak{S}$. Then $\mathfrak{A}_x$ is a PFI of $\mathfrak{S}$.

$$\mathfrak{A}_x(\sigma+\rho) \geq \mathfrak{A}_x(\sigma) \wedge \mathfrak{A}_x(\rho),$$
$$\mathfrak{A}_x(\sigma\gamma\rho) \geq \mathfrak{A}_x(\sigma),$$
$$\mathfrak{A}_x(\sigma\gamma\rho) \geq \mathfrak{A}_x(\rho) \text{ for all } \sigma, \rho \in \mathfrak{S} \text{ and } \gamma \in \Gamma.$$

Now

$$\begin{aligned}\overline{\mathfrak{A}_x}(\sigma+\rho) &= 1 - \mathfrak{A}_x(\sigma+\rho)\\ &\leq 1 - (\mathfrak{A}_x(\sigma) \wedge \mathfrak{A}_x(\rho))\\ &= (1-\mathfrak{A}_x(\sigma)) \vee (1-\mathfrak{A}_x(\rho))\\ &= \overline{\mathfrak{A}_x}(\sigma) \vee \overline{\mathfrak{A}_x}(\rho),\end{aligned}$$

and

$$\begin{aligned}\overline{\mathfrak{A}_x}(\sigma\gamma\rho) &= 1 - \mathfrak{A}_x(\sigma\gamma\rho)\\ &\leq 1 - \mathfrak{A}_x(\rho))\\ &= \overline{\mathfrak{A}_x}(\rho).\end{aligned}$$

Similarly

$$\begin{aligned}\overline{\mathfrak{A}_x}(\sigma\gamma\rho) &= 1 - \mathfrak{A}_x(\sigma\gamma\rho)\\ &\leq 1 - \mathfrak{A}_x(\sigma))\\ &= \overline{\mathfrak{A}_x}(\sigma).\end{aligned}$$

$\mathfrak{A}$ is a PFI of $\mathfrak{S}$.
Let $\mathfrak{A}_x(\sigma) \geq \mathfrak{A}_x(\sigma+\rho) \wedge \mathfrak{A}_x(\rho)$. Then

$$\begin{aligned}\overline{\mathfrak{A}_x}(\sigma) &= 1 - \mathfrak{A}_x(\sigma)\\ &\leq 1 - (\mathfrak{A}_x(\sigma+\rho) \wedge \mathfrak{A}_x(\rho))\\ &= (1-\mathfrak{A}_x(\sigma+\rho)) \vee (1-\mathfrak{A}_x(\rho))\\ &= \overline{\mathfrak{A}_x}(\sigma+\rho) \vee \overline{\mathfrak{A}_x}(\rho).\end{aligned}$$

Hence $\mathfrak{A}$ is a PFKI of $\mathfrak{S}$. □

**Theorem 3.5.** *Let $\mathfrak{A}$ be a PFS of $\mathfrak{S}$ if and only if the FS $\mathfrak{A}_x$ and $\overline{\mathfrak{A}_y}$ are FKI of $\mathfrak{S}$.*

*Proof.* Let $\mathfrak{A} = (\mathfrak{A}_x, \mathfrak{A}_y)$ be a PFKI of $\mathfrak{S}$. Then $\mathfrak{A}_x$ is FKI of $\mathfrak{S}$.
We prove that
$\mathfrak{A}_y(\sigma+\rho) \leq \mathfrak{A}_y(\sigma) \wedge \mathfrak{A}_y(\rho)$,
$\mathfrak{A}_y(\sigma\gamma\rho) \leq \mathfrak{A}_y(\sigma)$ and
$\mathfrak{A}_y(\sigma\gamma\rho) \leq \mathfrak{A}_y(\rho)$. For all $\sigma, \rho \in \mathfrak{S}$ and $\gamma \in \Gamma$.
Now

$$\begin{aligned}\overline{\mathfrak{A}_y}(\sigma+\rho) &= 1 - \mathfrak{A}_y(\sigma+\rho)\\ &\geq 1 - (\mathfrak{A}_y(\sigma) \vee \mathfrak{A}_y(\rho))\\ &= (1-\mathfrak{A}_y(\sigma)) \wedge (1-\mathfrak{A}_y(\rho))\\ &= \overline{\mathfrak{A}_y}(\sigma) \wedge \overline{\mathfrak{A}_y}(\rho),\end{aligned}$$

$$\overline{\mathfrak{A}_y}(\sigma\gamma\rho) = 1 - \mathfrak{A}_y(\sigma\gamma\rho)$$

$\geq 1 - \mathfrak{A}_y(\rho))$
$= \overline{\mathfrak{A}_y}(\rho).$

Similarly

$\overline{\mathfrak{A}_y}(\sigma\gamma\rho) = 1 - \mathfrak{A}_y(\sigma\gamma\rho)$
$\geq 1 - \mathfrak{A}_y(\sigma))$
$= \overline{\mathfrak{A}_y}(\sigma).$

Then

$\overline{\mathfrak{A}_y}(\sigma) = 1 - \mathfrak{A}_y(\sigma)$
$\geq 1 - (\mathfrak{A}_y(\sigma+\rho) \vee \mathfrak{A}_y(\rho))$
$= (1 - \mathfrak{A}_y(\sigma+\rho)) \wedge (1 - \mathfrak{A}_y(\rho))$
$= \overline{\mathfrak{A}_y}(\sigma+\rho) \wedge \overline{\mathfrak{A}_y}(\rho).$

Hence $\overline{\mathfrak{A}_y}$ is a FKI.

Conversely, suppose that $\mathfrak{A}_x$ and $\overline{\mathfrak{A}_y}$ are FKI of $\mathfrak{S}$.

Let $\sigma, \rho \in \mathfrak{S}$ and $\gamma \in \Gamma$. Since $\mathfrak{A}_x$ is a FKI of $\mathfrak{S}$.

So,

$$\mathfrak{A}_x(\sigma+\rho) \geq \mathfrak{A}_x(\sigma) \wedge \mathfrak{A}_x(\rho),$$
$$\mathfrak{A}_x(\sigma\gamma\rho) \geq \mathfrak{A}_x(\sigma) \text{ and}$$
$$\mathfrak{A}_x(\sigma\gamma\rho) \geq \mathfrak{A}_x(\rho).$$

Also $\mathfrak{A}_x(\sigma) \geq \mathfrak{A}_x(\sigma+\rho) \wedge \mathfrak{A}_x(\rho)$ for all $\sigma, \rho \in \mathfrak{S}$ and $\gamma \in \Gamma$.

Again $\overline{\mathfrak{A}_y}$ is a FI of $\mathfrak{S}$, So

$\mathfrak{A}_y(\sigma+\rho) = 1 - \overline{\mathfrak{A}_y}(\sigma+\rho)$
$\leq 1 - (\overline{\mathfrak{A}_y}(\sigma) \wedge \overline{\mathfrak{A}_y}(\rho))$
$= (1 - \overline{\mathfrak{A}_y}(\sigma)) \vee (1 - \overline{\mathfrak{A}_y}(\rho))$
$= \mathfrak{A}_y(\sigma) \vee \mathfrak{A}_y(\rho).$

Also

$\mathfrak{A}_y(\sigma\gamma\rho) = 1 - \overline{\mathfrak{A}_y}(\sigma\gamma\rho)$
$\leq 1 - \overline{\mathfrak{A}_y}(\rho)$
$= \mathfrak{A}_y(\rho).$

Similarly

$\mathfrak{A}_y(\sigma\gamma\rho) = 1 - \overline{\mathfrak{A}_y}(\sigma\gamma\rho)$
$\leq 1 - \overline{\mathfrak{A}_y}(\sigma)$
$= \mathfrak{A}_y(\sigma).$

Again $\overline{\mathfrak{A}_y}$ is a FKI of $\mathfrak{S}$.

$\mathfrak{A}_y(\sigma) = 1 - \overline{\mathfrak{A}_y}(\sigma)$
$\leq 1 - (\overline{\mathfrak{A}_y}(\sigma+\rho) \wedge \overline{\mathfrak{A}_y}(\rho))$
$= (1 - \overline{\mathfrak{A}_y}(\sigma+\rho)) \vee (1 - \overline{\mathfrak{A}_y}(\rho))$
$= \mathfrak{A}_y(\sigma+\rho) \vee \mathfrak{A}_y(\rho).$

Hence $\mathfrak{A}$ is a PFKI of $\mathfrak{S}$. □

**Corollary 3.4.** *A PFS $\mathfrak{A}$ be a PFKI of $\mathfrak{S}$ if and only if $\mathfrak{A} = (\mathfrak{A}_x, \overline{\mathfrak{A}_x})$ and $\mathfrak{E} = (\overline{\mathfrak{A}_x}, \mathfrak{A}_y)$ are PFKI of $\mathfrak{S}$.*

**Theorem 3.6.** *$\mathfrak{A}$ be a FKI of $\mathfrak{S}$ if and only if for any $t \in [0,1]$ $U_{\mathfrak{A}}(\mathfrak{A}_x, t) \neq \emptyset$ $U_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t) \neq \emptyset$, $U_{\mathfrak{A}}(\mathfrak{A}_x, t)$ $U_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t)$ are $\mathfrak{K}$ - ideals of $\mathfrak{S}$.*

*Proof.* Let $\mathfrak{A} = (\mathfrak{A}_x, \mathfrak{A}_y)$ be a PFKI of $\mathfrak{S}$. So by thm2.2 $\mathfrak{A}_x$ and $\overline{\mathfrak{A}_y}$ are FKI of $\mathfrak{S}$.
Then $\mathfrak{A}_x$ and $\overline{\mathfrak{A}_y}$ are FI. Also for any $t \in [0,1]$, $U_{\mathfrak{A}}(\mathfrak{A}_x, t) \neq \emptyset$ $U_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t) \neq \emptyset$ $U_{\mathfrak{A}}(\mathfrak{A}_x, t)$ $U_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t)$ are ideals of $\mathfrak{S}$.
We have to prove that $U_{\mathfrak{A}}(\mathfrak{A}_x, t)$ $U_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t)$ are $\mathfrak{K}$ - ideals of $\mathfrak{S}$.
For let $\rho \in S$ and $\sigma, \sigma+\rho \in U_{\mathfrak{A}}(\mathfrak{A}_x, t)$ implies $\mathfrak{A}_x(\sigma) \geq t$ and $\mathfrak{A}_x(\sigma+\rho) \geq t$

Since $\mathfrak{A}_x$ is FKI of $\mathfrak{S}$. We have $\mathfrak{A}_x(\rho) \geq \mathfrak{A}_x(\sigma+\rho) \wedge \mathfrak{A}_x(\rho) \geq t \wedge t \geq t$ implies $\mathfrak{A}_x(\rho) \geq t$ then $\rho \in U_{\mathfrak{A}}(\mathfrak{A}_x, t)$. Hence $U_{\mathfrak{A}}(\mathfrak{A}_x, t)$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$. Similarly $U_{\mathfrak{A}}(\overline{\mathfrak{A}_x}, t)$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$.
Conversely let us assume that for any $t \in [0,1]$ $U_{\mathfrak{A}}(\mathfrak{A}_x, t) \neq \emptyset$ $U_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t) \neq \emptyset$, $U_{\mathfrak{A}}(\mathfrak{A}_x, t)$ $U_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t)$ are $\mathfrak{K}$ - ideals of $\mathfrak{S}$. Then $\mathfrak{A}_x, \overline{\mathfrak{A}_y}$ are FI of $\mathfrak{S}$.
Let $\sigma, \rho \in \mathfrak{S}$ and $\mathfrak{A}_x(\rho) = r_1$, $\mathfrak{A}_x(\sigma+\rho) = r_2$, $r_i \in [0,1]$, $t = r_1 \wedge r_2 = min(r_1, r_2)$. So $\mathfrak{A}_x(\sigma+\rho) = r_1 \geq t$ implies $\rho \in U_{\mathfrak{A}}(\mathfrak{A}_x, t)$. Similarly $\sigma+\rho \in U_{\mathfrak{A}}(\mathfrak{A}_x, t)$. But $U_{\mathfrak{A}}(\mathfrak{A}_x, t)$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$, so $\sigma \in U_{\mathfrak{A}}(\mathfrak{A}_x, t)$
implies $\mathfrak{A}_x(\sigma) \geq t = r_1 \wedge r_2 = \mathfrak{A}_x(\sigma+\rho) \wedge \mathfrak{A}_x(\rho)$.
Hence $\mathfrak{A}_x$ is a fuzzy $\mathfrak{K}$ - ideal of $\mathfrak{S}$. Similarly, $\overline{\mathfrak{A}_y}$ is a FKI of $\mathfrak{S}$. By thm 2.5 $\mathfrak{A}$ be a FKI of $\mathfrak{S}$ □

**Theorem 3.7.** *$\mathfrak{A}$ be a FKI of $\mathfrak{S}$ if and only if for any $t \in [0,1]$
$U_{\mathfrak{A}}(\mathfrak{A}_x, t) \neq \emptyset$ $U_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t) \neq \emptyset$, $L_{\mathfrak{A}}(\mathfrak{A}_x, t)$ $L_{\mathfrak{A}}(\overline{\mathfrak{A}_y}, t)$ are $\mathfrak{K}$ - ideals of $\mathfrak{S}$.*

*Proof.* By follows theorem 2.5 □

**Theorem 3.8.** *Let $\mathfrak{A} = (\mathfrak{A}_x, \mathfrak{A}_y)$ be a PFKI in $\mathfrak{S}$. If $x+y=0$ then $\mathfrak{A}(\sigma) = \mathfrak{A}(\rho)$ for any $\sigma, \rho \in \mathfrak{S}$.*

*Proof.* Since $\mathfrak{A} = (\mathfrak{A}_x, \mathfrak{A}_y)$ be a PFKI in $\mathfrak{S}$. Then

$$\begin{aligned}\mathfrak{A}_x(\sigma) &\geq \mathfrak{A}_x(\sigma+\rho) \wedge \mathfrak{A}_x(\rho)\\ &= \mathfrak{A}_x(0) \wedge \mathfrak{A}_x(\rho)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{A}_y(\sigma) &\leq \mathfrak{A}_y(\sigma+\rho) \vee \mathfrak{A}_y(\rho)\\ &\mathfrak{A}_y(0) \wedge \mathfrak{A}_y(\rho).\end{aligned}$$

Since $\mathfrak{A}$ is fuzzy ideal in $\mathfrak{S}$, we have $\mathfrak{A}_x(0) \geq \mathfrak{A}_x(\sigma)$ and $\mathfrak{A}_y(0) \leq \mathfrak{A}_y(\sigma)$ for all $\sigma \in \mathfrak{S}$. We have $\mathfrak{A}_x(\sigma) \geq \mathfrak{A}_x(\rho)$ and $\mathfrak{A}_y(\sigma) \leq \mathfrak{A}_y(\rho)$.
Similarly , $\mathfrak{A}_x(\sigma) \leq \mathfrak{A}_x(\rho)$ and $\mathfrak{A}_y(\sigma) \geq \mathfrak{A}_y(\rho)$.
Then $\mathfrak{A}_x(\sigma) = \mathfrak{A}_x(\rho)$ and $\mathfrak{A}_y(\sigma) = \mathfrak{A}_y(\rho)$.
Hence $\mathfrak{A}(\sigma) = \mathfrak{A}(\rho)$. □

**Theorem 3.9.** *Let $\mathfrak{S}$ be a $\Gamma$ - semiring with zero element and $\mathfrak{A}$be a PFKI in $\mathfrak{S}$. Let $\mathfrak{R}_{\mathfrak{A}} = \{\sigma \in \mathfrak{S} : \mathfrak{A}(\sigma) = \mathfrak{A}(0)\}$. Then $\mathfrak{R}_{\mathfrak{A}}$ is a $\mathfrak{K}$ - ideal of $\mathfrak{S}$.*

*Proof.* Let $\sigma, \rho \in \mathfrak{R}_{\mathfrak{A}}$. Then $\mathfrak{A}_x(\sigma+\rho) \geq \mathfrak{A}_x(\sigma) \wedge \mathfrak{A}_x(\rho) = \mathfrak{A}_x(0)$
$\mathfrak{A}_y(\sigma+\rho) \leq \mathfrak{A}_y(\sigma) \vee \mathfrak{A}_y(\rho) = \mathfrak{A}_y(0)$.
Since $\mathfrak{A}$ is a PFKI in $\mathfrak{S}$ so we get $\mathfrak{A}_x(0) \geq \mathfrak{A}_x(\sigma+\rho)$ and $\mathfrak{A}_y(0) \leq \mathfrak{A}_y(\sigma+\rho)$. Then we get $\mathfrak{A}(0) = \mathfrak{A}(\sigma+\rho)$. Hence $\sigma+\rho \in \mathfrak{R}_{\mathfrak{A}}$.
Next we have to prove $\mathfrak{R}_{\mathfrak{A}}$ is a ideal and $\mathfrak{R}_{\mathfrak{A}}$ is a $\mathfrak{K}$ - ideal.
Let $l \in \mathfrak{R}$, $\sigma \in \mathfrak{R}_{\mathfrak{A}}$ and $\gamma \in \Gamma$, then we get $\mathfrak{A}_x(l\gamma\sigma) \geq \mathfrak{A}_x(\sigma) = \mathfrak{A}_x(0)$
and $\mathfrak{A}_y(l\gamma\sigma) \leq \mathfrak{A}_y(\sigma) = \mathfrak{A}_y(0)$. Similarly $\mathfrak{A}_x(0) \geq \mathfrak{A}_x(l\gamma\sigma)$ and $\mathfrak{A}_y(0) \leq \mathfrak{A}_x(l\gamma\sigma)$.
Then $\mathfrak{A}(0) = \mathfrak{A}(l\gamma\sigma)$ implies $l\gamma\sigma \in \mathfrak{R}_{\mathfrak{A}}$
Similarly we prove $\sigma\gamma l \in \mathfrak{R}_{\mathfrak{A}}$. Hence $\mathfrak{R}_{\mathfrak{A}}$ is an ideal.
Next we have to prove $\mathfrak{R}_{\mathfrak{A}}$ is a $\mathfrak{K}$ - ideal. For this let $\sigma \in \mathfrak{R}$, $f \in \mathfrak{R}_{\mathfrak{A}}$ and $f+\sigma \in \mathfrak{R}_{\mathfrak{A}}$, then $\mathfrak{A}_x(\sigma) \geq \mathfrak{A}_x(f+\sigma) \wedge \mathfrak{A}_x(f) = \mathfrak{A}_x(0)$ and $\mathfrak{A}_y(\sigma) \leq \mathfrak{A}_y(f+\sigma) \vee \mathfrak{A}_y(f) = \mathfrak{A}_y(0)$.
Similarly we see that $\mathfrak{A}_x(0) \geq \mathfrak{A}_x(\sigma)$ and $\mathfrak{A}_y(0) \leq \mathfrak{A}_y(\sigma)$.
Then $\mathfrak{A}(\sigma) = \mathfrak{A}(0)$, hence $\sigma \in \mathfrak{R}_{\mathfrak{A}}$. Hence $\mathfrak{R}_{\mathfrak{A}}$ is a $\mathfrak{K}$ - ideal of $\mathfrak{R}$. □

## 4. Properties of Pythagorean fuzzy $\mathfrak{K}$ - ideals in $\Gamma$ - semirings

In this section we discuss some interesting properties of PFKI in $\Gamma$ - semirings.

**Proposition 4.1.** *Non empty intersection of PFKI of $\mathfrak{S}$ is PFKI of $\mathfrak{S}$*

*Proof.* Assume that $\{\mathfrak{A}^i : i \in I\}$ be a family of a PFKI of $\mathfrak{S}$ . Let $\sigma, \rho \in \mathfrak{S}$ and $\gamma \in \Gamma$.
Then

$$\begin{aligned}(\bigcap_{i\in I} \mathfrak{A}^i_x)(\sigma+\rho) &= \inf_{i\in I} \mathfrak{A}^i_x(\sigma+\rho)\\ &\geq \inf_{i\in I} \big(\mathfrak{A}^i_x(\sigma) \wedge \mathfrak{A}^i_x(\rho)\big)\\ &= \inf_{i\in I} \mathfrak{A}^i_x(\sigma) \wedge \inf_{i\in I} \mathfrak{A}^i_x(\rho)\\ &= \bigcap_{i\in I} \mathfrak{A}^i_x(\sigma) \wedge \bigcap_{i\in I} \mathfrak{A}^i_x(\rho)\end{aligned}$$

and

$$\begin{aligned}(\bigcap_{i\in I} \mathfrak{A}^i_y)(\sigma+\rho) &= \sup_{i\in I} \mathfrak{A}^i_y(\sigma+\rho)\\ &\leq \sup_{i\in I} \big(\mathfrak{A}^i_y(\sigma) \vee \mathfrak{A}^i_y(\rho)\big)\\ &= \sup_{i\in I} \mathfrak{A}^i_y(\sigma) \vee \sup_{i\in I} \mathfrak{A}^i_y(\rho)\\ &= \bigcap_{i\in I} \mathfrak{A}^i_y(\sigma) \vee \bigcap_{i\in I} \mathfrak{A}^i_y(\rho).\end{aligned}$$

Moreover

$$\begin{aligned}(\bigcap_{i\in I} \mathfrak{A}^i_x)(\sigma\gamma\rho) &= \inf_{i\in I} \mathfrak{A}^i_x(\sigma\gamma\rho)\\ &\geq \mathfrak{A}^i_x(\rho)\\ &= \bigcap_{i\in I} \mathfrak{A}^i_x(\rho)\end{aligned}$$

and

$$\begin{aligned}(\bigcap_{i\in I} \mathfrak{A}^i_x)(\sigma\gamma\rho) &= \inf_{i\in I} \mathfrak{A}^i_x(\sigma\gamma\rho)\\ &\geq \mathfrak{A}^i_x(\sigma)\\ &= \bigcap_{i\in I} \mathfrak{A}^i_x(\sigma).\end{aligned}$$

Finally

$$\begin{aligned}(\bigcap_{i\in I} \mathfrak{A}^i_y)(\sigma\gamma\rho) &= \sup_{i\in I} \mathfrak{A}^i_y(\sigma\gamma\rho)\\ &\leq \mathfrak{A}^i_y(\rho)\\ &= \bigcap_{i\in I} \mathfrak{A}^i_y(\rho)\end{aligned}$$

and

$$\begin{aligned}(\bigcap_{i\in I} \mathfrak{A}^i_y)(\sigma\gamma\rho) &= \sup_{i\in I} \mathfrak{A}^i_y(\sigma\gamma\rho)\\ &\leq \mathfrak{A}^i_y(\sigma)\\ &= \bigcap_{i\in I} \mathfrak{A}^i_y(\sigma).\end{aligned}$$

Hence $\{\mathfrak{A}^i : i \in I\}$ is a PFI of $\mathfrak{S}$. Next we prove $\{\mathfrak{A}^i : i \in I\}$ is a PFKI of $\mathfrak{S}$.
For that let $\sigma, \rho \in \mathfrak{S}$ and $\gamma \in \Gamma$.
Then

$$\begin{aligned}(\bigcap_{i\in I} \mathfrak{A}^i_x)(\sigma) &= \inf_{i\in I} \mathfrak{A}^i_x(\sigma)\\ &\geq \inf_{i\in I} \big(\mathfrak{A}^i_x(\sigma+\rho) \wedge \mathfrak{A}^i_x(\rho)\big)\\ &= \inf_{i\in I} \mathfrak{A}^i_x(\sigma+\rho) \wedge \inf_{i\in I} \mathfrak{A}^i_x(\rho)\\ &= \bigcap_{i\in I} \mathfrak{A}^i_x(\sigma+\rho) \wedge \bigcap_{i\in I} \mathfrak{A}^i_x(\rho)\end{aligned}$$

and

$$\begin{aligned}(\bigcap_{i\in I}\mathfrak{A}^i_y)(\sigma+\rho) &= \sup_{i\in I}\mathfrak{A}^i_y(\sigma)\\ &\leq \sup_{i\in I}\big(\mathfrak{A}^i_y(\sigma+\rho)\vee\mathfrak{A}^i_y(\rho)\big)\\ &= \sup_{i\in I}\mathfrak{A}^i_y(\sigma+\rho)\vee\sup_{i\in I}\mathfrak{A}^i_y(\rho)\\ &= \bigcap_{i\in I}\mathfrak{A}^i_y(\sigma+\rho)\vee\bigcap_{i\in I}\mathfrak{A}^i_y(\rho).\end{aligned}$$

Hence Proved. □

**Proposition 4.2.** *Union of PFKI of $\mathfrak{S}$ is PFKI of $\mathfrak{S}$.*

*Proof.* Assume that $\{\mathfrak{A}^i : i\in I\}$ be a family of a PFKI of $\mathfrak{S}$. Let $\sigma,\rho\in\mathfrak{S}$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}(\bigcup_{i\in I}\mathfrak{A}^i_x)(\sigma+\rho) &= \sup_{i\in I}\mathfrak{A}^i_x(\sigma+\rho)\\ &\geq \sup_{i\in I}\big(\mathfrak{A}^i_x(\sigma)\wedge\mathfrak{A}^i_x(\rho)\big)\\ &= \sup_{i\in I}\mathfrak{A}^i_x(\sigma)\wedge\sup_{i\in I}\mathfrak{A}^i_x(\rho)\\ &= \bigcup_{i\in I}\mathfrak{A}^i_x(\sigma)\wedge\bigcup_{i\in I}\mathfrak{A}^i_x(\rho)\end{aligned}$$

and

$$\begin{aligned}(\bigcup_{i\in I}\mathfrak{A}^i_y)(\sigma+\rho) &= \inf_{i\in I}\mathfrak{A}^i_y(\sigma+\rho)\\ &\leq \inf_{i\in I}\big(\mathfrak{A}^i_y(\sigma)\vee\mathfrak{A}^i_y(\rho)\big)\\ &= \inf_{i\in I}\mathfrak{A}^i_y(\sigma)\vee\inf_{i\in I}\mathfrak{A}^i_y(\rho)\\ &= \bigcup_{i\in I}\mathfrak{A}^i_y(\sigma)\vee\bigcup_{i\in I}\mathfrak{A}^i_y(\rho).\end{aligned}$$

Moreover

$$\begin{aligned}(\bigcup_{i\in I}\mathfrak{A}^i_x)(\sigma\gamma\rho) &= \sup_{i\in I}\mathfrak{A}^i_x(\sigma\gamma\rho)\\ &\geq \mathfrak{A}^i_x(\rho)\\ &= \bigcup_{i\in I}\mathfrak{A}^i_x(\rho)\end{aligned}$$

and

$$\begin{aligned}(\bigcup_{i\in I}\mathfrak{A}^i_x)(\sigma\gamma\rho) &= \sup_{i\in I}\mathfrak{A}^i_x(\sigma\gamma\rho)\\ &\geq \mathfrak{A}^i_x(\sigma)\\ &= \bigcup_{i\in I}\mathfrak{A}^i_x(\sigma).\end{aligned}$$

Finally

$$\begin{aligned}(\bigcup_{i\in I}\mathfrak{A}^i_y)(\sigma\gamma\rho) &= \inf_{i\in I}\mathfrak{A}^i_y(\sigma\gamma\rho)\\ &\leq \mathfrak{A}^i_y(\rho)\\ &= \bigcup_{i\in I}\mathfrak{A}^i_y(\rho)\end{aligned}$$

and

$$\begin{aligned}(\bigcup_{i\in I}\mathfrak{A}^i_y)(\sigma\gamma\rho) &= \inf_{i\in I}\mathfrak{A}^i_y(\sigma\gamma\rho)\\ &\leq \mathfrak{A}^i_y(\sigma)\\ &= \bigcup_{i\in I}\mathfrak{A}^i_y(\sigma).\end{aligned}$$

Hence $\{\mathfrak{A}^i : i\in I\}$ is a PFI of $\mathfrak{S}$. Next we prove $\{\mathfrak{A}^i : i\in I\}$ is a PFKI of $\mathfrak{S}$. For that let $\sigma,\rho\in\mathfrak{S}$ and $\gamma\in\Gamma$.

Then

$$
\begin{aligned}
(\bigcup_{i\in I}\mathfrak{A}^i_x)(\sigma) &= \sup_{i\in I}\mathfrak{A}^i_x(\sigma)\\
&\geq \sup_{i\in I}\big(\mathfrak{A}^i_x(\sigma+\rho)\wedge\mathfrak{A}^i_x(\rho)\big)\\
&= \sup_{i\in I}\mathfrak{A}^i_x(\sigma+\rho)\wedge\sup_{i\in I}\mathfrak{A}^i_x(\rho)\\
&= \bigcup_{i\in I}\mathfrak{A}^i_x(\sigma+\rho)\wedge\bigcup_{i\in I}\mathfrak{A}^i_x(\rho)
\end{aligned}
$$

and

$$
\begin{aligned}
(\bigcup_{i\in I}\mathfrak{A}^i_y)(\sigma) &= \inf_{i\in I}\mathfrak{A}^i_y(\sigma)\\
&\leq \inf_{i\in I}\big(\mathfrak{A}^i_y(\sigma+\rho)\vee\mathfrak{A}^i_y(\rho)\big)\\
&= \inf_{i\in I}\mathfrak{A}^i_y(\sigma+\rho)\vee\inf_{i\in I}\mathfrak{A}^i_y(\rho)\\
&= \bigcup_{i\in I}\mathfrak{A}^i_y(\sigma+\rho)\vee\bigcup_{i\in I}\mathfrak{A}^i_y(\rho).
\end{aligned}
$$

Hence Proved. □

**Theorem 4.1.** *Let $G$ and $H$ be two $\Gamma$ - semirings and $\xi$ be a homomorphism of $G$ onto $H$. If $\mathfrak{A}$ is a PFKI of $H$, then $\xi^{-1}(\mathfrak{A})$ is a PFKI of $G$.*

*Proof.* Let $\sigma,\rho\in G$. Then

$$
\begin{aligned}
\xi^{-1}(\mathfrak{A}_x)(\sigma+\rho) &= \mathfrak{A}_x(\xi(\sigma+\rho))\\
&= \mathfrak{A}_x(\xi(\sigma)+\xi(\rho))\\
&\geq \mathfrak{A}_x(\xi(\sigma))\wedge\mathfrak{A}_x(\xi(\rho))\\
&= \xi^{-1}(\mathfrak{A}_x)(\sigma)\wedge\xi^{-1}(\mathfrak{A}_x)(\rho)
\end{aligned}
$$

$$
\begin{aligned}
\xi^{-1}(\mathfrak{A}_y)(\sigma+\rho) &= \mathfrak{A}_y(\xi(\sigma+\rho))\\
&= \mathfrak{A}_y(\xi(\sigma)+\xi(\rho))\\
&\leq \mathfrak{A}_y(\xi(\sigma))\vee\mathfrak{A}_y(\xi(\rho))\\
&= \xi^{-1}(\mathfrak{A}_y)(\sigma)\vee\xi^{-1}(\mathfrak{A}_y)(\rho)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\xi^{-1}(\mathfrak{A}_x)(\sigma\gamma\rho) &= \mathfrak{A}_x(\xi(\sigma\gamma\rho))\\
&= \mathfrak{A}_x(\xi(\sigma)\gamma\xi(\rho))\\
&\geq \mathfrak{A}_x(\xi(\rho))\\
&= \xi^{-1}(\mathfrak{A}_x)(\rho).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\xi^{-1}(\mathfrak{A}_y)(\sigma\gamma\rho) &= \mathfrak{A}_y(\xi(\sigma\gamma\rho))\\
&= \mathfrak{A}_y(\xi(\sigma)\gamma\xi(\rho))\\
&\leq \mathfrak{A}_y(\xi(\rho))\\
&= \xi^{-1}(\mathfrak{A}_y)(\rho).
\end{aligned}
$$

Hence $\xi^{-1}(\mathfrak{A})$ is a PFI of $G$.
Next we prove $\xi^{-1}(\mathfrak{A})$ is a PFKI of $G$.

$$
\begin{aligned}
\xi^{-1}(\mathfrak{A}_x)(\sigma) &= \mathfrak{A}_x(\xi(\sigma))\\
&\geq \mathfrak{A}_x(\xi(\sigma)+\xi(\rho))\wedge\mathfrak{A}_x(\xi(\rho))\\
&\geq \mathfrak{A}_x(\xi(\sigma+\rho))\wedge\mathfrak{A}_x(\xi(\rho))\\
&= \xi^{-1}(\mathfrak{A}_x)(\sigma+\rho)\wedge\xi^{-1}(\mathfrak{A}_x)(\rho).
\end{aligned}
$$

$$
\begin{aligned}
\xi^{-1}(\mathfrak{A}_y)(\sigma) &= \mathfrak{A}_y(\xi(\sigma))\\
&\leq \mathfrak{A}_y(\xi(\sigma)+\xi(\rho))\wedge\mathfrak{A}_y(\xi(\rho))\\
&\leq \mathfrak{A}_y(\xi(\sigma+\rho))\wedge\mathfrak{A}_y(\xi(\rho))\\
&= \xi^{-1}(\mathfrak{A}_y)(\sigma+\rho)\wedge\xi^{-1}(\mathfrak{A}_y)(\rho)
\end{aligned}
$$

Hence Proved. □

**Theorem 4.2.** *Let $G$ and $H$ be two $\Gamma$ - semirings and $\xi$ be a homomorphism of $G$ onto $H$. If $\mathfrak{A}$ is a PFKI of $G$, then $\xi(\mathfrak{A})$ is a PFKI of $H$.*

*Proof.* Let $\sigma, \rho \in H$. Then,

$$\begin{aligned}
\xi(\mathfrak{A}_x(\sigma^{'}+\rho^{'})) &= \inf_{s+q\in\xi^{-1}(\sigma^{'}+\rho^{'})} \mathfrak{A}_x(s+q)\\
&\geq \inf_{s\in\xi^{-1}(\sigma^{'}),q\in\xi^{-1}(\rho^{'})} \mathfrak{A}_x(s+q)\\
&\geq \inf_{(s\in\xi^{-1}(\sigma^{'}),q\in\xi^{-1}(\rho^{'}))} (\mathfrak{A}_x(s)\wedge\mathfrak{A}_x(q))\\
&= \left(\inf_{s\in\xi^{-1}(\sigma^{'})}(\mathfrak{A}_x(s))\right)\wedge\left(\inf_{q\in\xi^{-1}(\rho^{'})}(\mathfrak{A}_x(q))\right)\\
&= \xi(\mathfrak{A}_x(\sigma^{'}))\wedge\rho(\mathfrak{A}_x(\rho^{'}))
\end{aligned}$$

and

$$\begin{aligned}
\xi(\mathfrak{A}_y(\sigma^{'}+\rho^{'})) &= \sup_{s+q\in\xi^{-1}(\sigma^{'}+\rho^{'})} \mathfrak{A}_y(s+q)\\
&\leq \sup_{s\in\xi^{-1}(\sigma^{'}),q\in\xi^{-1}(\rho^{'})} \mathfrak{A}_y(s+q)\\
&\leq \sup_{(s\in\xi^{-1}(\sigma^{'}),q\in\xi^{-1}(\rho^{'}))} (\mathfrak{A}_y(s)\vee\mathfrak{A}_y(q))\\
&= \left(\sup_{s\in\xi^{-1}(\sigma^{'})}(\mathfrak{A}_y(s))\right)\vee\left(\sup_{q\in\xi^{-1}(\rho^{'})}(\mathfrak{A}_y(q)))\right)\\
&= \xi(\mathfrak{A}_y(\sigma^{'}))\vee\rho(\mathfrak{A}_y(\rho^{'})).
\end{aligned}$$

Moreover

$$\begin{aligned}
\xi(\mathfrak{A}_x(\sigma^{'}\gamma\rho^{'})) &= \inf_{s\in\xi^{-1}(\sigma^{'}\gamma\rho^{'})} \mathfrak{A}_x(s)\\
&\geq \inf_{s\in\xi^{-1}(\sigma^{'}),q\in\xi^{-1}(\rho^{'})} \mathfrak{A}_x(s\gamma q)\\
&\geq \inf_{s\in\xi^{-1}(\sigma^{'})} \mathfrak{A}_x(s)\\
&= \xi(\mathfrak{A}_x(\sigma^{'})).
\end{aligned}$$

Also we can prove

$$\begin{aligned}
\xi(\mathfrak{A}_x(\sigma^{'}\gamma\rho^{'})) &= \inf_{s\in\xi^{-1}(\sigma^{'}\gamma\rho^{'})} \mathfrak{A}_x(s)\\
&\geq \inf_{s\in\xi^{-1}(\sigma^{'}),q\in\xi^{-1}(\rho^{'})} \mathfrak{A}_x(s\gamma q)\\
&\geq \inf_{s\in\xi^{-1}(\sigma^{'})} \mathfrak{A}_x(q)\\
&= \xi(\mathfrak{A}_x(\rho^{'})).
\end{aligned}$$

Similarly

$$\begin{aligned}
\xi(\mathfrak{A}_y(\sigma^{'}\gamma\rho^{'})) &= \sup_{s\in\xi^{-1}(\sigma^{'}\gamma\rho^{'})} \mathfrak{A}_y(s)\\
&\leq \sup_{s\in\xi^{-1}(\sigma^{'}),q\in\xi^{-1}(\rho^{'})} \mathfrak{A}_y(s\gamma q)\\
&\leq \sup_{s\in\xi^{-1}(\sigma^{'})} \mathfrak{A}_y(s)\\
&= \xi(\mathfrak{A}_y(\sigma^{'}))
\end{aligned}$$

and

$$\begin{aligned}
\xi(\mathfrak{A}_y(\sigma^{'}\gamma\rho^{'})) &= \sup_{s\in\xi^{-1}(\sigma^{'}\gamma\rho^{'})} \mathfrak{A}_y(s)\\
&\leq \sup_{s\in\xi^{-1}(\sigma^{'}),q\in\xi^{-1}(\rho^{'})} \mathfrak{A}_y(s\gamma q)
\end{aligned}$$

$$\leq \sup_{s\in\xi^{-1}(\sigma')} \mathfrak{A}_y(q)$$
$$= \xi(\mathfrak{A}_y(\rho')).$$

Next we have to prove $\xi(\mathfrak{A})$ is a PFKI of $H$.

$$\begin{aligned}
\xi(\mathfrak{A}_x(\sigma')) &= \inf_{s\in\xi^{-1}(\sigma'+\rho')} \mathfrak{A}_x(s) \\
&\geq \inf_{s\in\xi^{-1}(\sigma'),q\in\xi^{-1}(\rho')} \mathfrak{A}_x(s) \\
&\geq \inf_{(s\in\xi^{-1}(\sigma'),q\in\xi^{-1}(\rho'))} (\mathfrak{A}_x(s+q) \wedge \mathfrak{A}_x(q)) \\
&= \left(\inf_{s\in\xi^{-1}(\sigma')} (\mathfrak{A}_x(s+q))\right) \wedge \left(\inf_{q\in\xi^{-1}(\rho')} (\mathfrak{A}_x(q))\right) \\
&= \xi(\mathfrak{A}_x(\sigma' + \rho')) \wedge \rho(\mathfrak{A}_x(\rho'))
\end{aligned}$$

and

$$\begin{aligned}
\xi(\mathfrak{A}_y(\sigma')) &= \sup_{s\in\xi^{-1}(\sigma'+\rho')} \mathfrak{A}_y(s) \\
&\leq \sup_{s\in\xi^{-1}(\sigma'),q\in\xi^{-1}(\rho')} \mathfrak{A}_y(s) \\
&\leq \sup_{(s\in\xi^{-1}(\sigma'),q\in\xi^{-1}(\rho'))} (\mathfrak{A}_y(s+q) \vee \mathfrak{A}_y(q)) \\
&= \left(\sup_{s\in\xi^{-1}(\sigma')} (\mathfrak{A}_y(s+q))\right) \vee \left(\sup_{q\in\xi^{-1}(\rho')} (\mathfrak{A}_y(q))\right) \\
&= \xi(\mathfrak{A}_y(\sigma' + \rho')) \vee \rho(\mathfrak{A}_y(\rho')).
\end{aligned}$$

Hence proved. □

## 5. Conclusion

In this paper, we apply the concept of PFS to Γ - semirings. We introduced the notion of algebraic structures of PFS in Γ - semirings. We investigate the properties of PFKI of Γ - semirings.

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