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HARMONIC MEAN CORDIAL LABELING IN THE SCENARIO OF DUPLICATING GRAPH ELEMENTS

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ABSTRACT. All the graphs considered in this article are simple and undirected. Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f : V(G) \to \{1, 2\}$ is called Harmonic Mean Cordial if the induced function $f^* : E(G) \to \{1, 2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1, 2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively. A Graph G is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling. In this article, we have discussed Harmonic Mean Cordial labeling In The Scenario of Duplicating Graph Elements.

Keywords: Harmonic Mean Cordial Labeling, Vertex Duplication, Edge Duplication, Cycle.

AMS Subject Classification: 05C78, 05C76

1. INTRODUCTION

The notion of graph labeling in graph theory has garnered significant attention from scholars because of its wide-ranging and rigorous applications in domains such as communication network design and analysis, military surveillance, social sciences, optimization, and linear algebra. Various graph labelings are documented in the current body of literature. A dynamic survey of graph labeling by Gallian [2] is a condensed compilation of a lengthy bibliography of articles on the subject.

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For terminology and notation not defined here we follow Balakrishnan and Rangnathan [1]. In [5] J. Gowri and J. Jayapriya defined Harmonic Mean Cordial labeling of graph G. Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f : V(G) \to \{1, 2\}$ is called *Harmonic Mean Cordial* if the induced function $f^* : E(G) \to \{1, 2\}$ defined by

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$f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1, 2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ is the floor function. A Graph G is called *Harmonic Mean Cordial graph* if it admits Harmonic Mean Cordial labeling. For the sake of convenience of the reader we use 'HMC' for harmonic mean cordial labeling, '$C_{1,n}$' for One Chord Cycle Graph and $C_{2,n}$ for Twin Chord Cycle Graph. It is useful to recall some useful definitions of graph theory to make this article self-contained.
Motivated by the interesting results proved in [3, 4, 6, 8, 9] and on Root Cube Mean Cordial Labeling in [7, 10].

Definition 1.1. [2] *The neighborhood of a vertex v of a graph is the set of all vertices adjacent to v. It is denoted by $N(v)$.*

Definition 1.2. [2] *The duplication of a vertex v of graph G_1 produces a new graph G_2 by adding a new vertex v^* such that $N(v^*) = N(v)$. In other words a vertex v^* is said to be duplication of v if all the vertices which are adjacent to v in G_1 are also adjacent to v^* in G_2.*

Definition 1.3. [2] *The duplication of vertex v_n by a new edge $e = v_n^* v_n^{**}$ in a graph G_1 produce a new graph G_2 such that $N(v_n{}^*) = \{v_n, v_n^{**}\}$ and $N(v_n^{**}) = \{v_n, v_n^*\}$.*

Definition 1.4. [2] *The duplication of an edge $e = uv$ by a new vertex w in a graph G_1 produce a new graph G_2 such that $N(w) = \{u, v\}$.*

Definition 1.5. [2] *The duplication of an edge $e = uv$ of a graph G_1 produce a new graph G_2 by adding an edge $e^* = u^*v^*$ such that $N(u^*) = \{N(u) \cup v^*\} \setminus \{v\}$and $N(v^*) = \{N(v) \cup u^*\} \setminus \{u\}$.*

Definition 1.6. [2] *Let us consider a pair of a cycle C_n and let $e_k = v_k v_{k+1}$ be an edge in the first copy of C_n with $e_{k-1} = v_{k-1}v_k$ and $e_{k+1} = v_{k+1}v_{k+2}$ be its incident edges. Similarly let $e_k^* = u_k u_{k+1}$ be an edge in the second copy of C_n with $e_{k-1}^* = u_{k-1}u_k$ and $e_{k+1}^* = u_{k+1}u_{k+2}$ be its incident edges. The mutual duplication of a pair of edges e_k, e_k^* between two copies of cycle C_n produces a new graph G in such a way that $N(v_k) \cap N(u_k) = \{v_{k-1}, u_{k-1}\}$ and $N(v_{k+1}) \cap N(u_{k+1}) = \{v_{k+2}, u_{k+2}\}$.*

Definition 1.7. [2] *Let us consider a pair of a cycle C_n. Then the mutual duplication of a pair of vertices v_k and v_k^* respectively from each copy of cycle C_n produces a new graph G such that $N(v_k) = N(v_k^*)$.*

Definition 1.8. [2] *One Chord of a cycle is an edge joining two non−adjacent vertices of cycle C_n. It is denoted by $C_{1,n}$.*

Definition 1.9. [2] *Twin chords of a cycle is said to be twin chords if they form a triangle with an edge of the cycle C_n. It is denoted by $C_{2,n}$.*

In the next section, in Theorems 2.1 and 2.2, we investigate HMC labeling for duplicating vertices by edges of a generalized HMC graph. Also, in Theorems 2.3 and 2.4, we have derived HMC labeling for the duplicating vertice by an edge as well as the duplicating edges by a vertice on C_n , respectively. In Theorem 2.5, we proved that the graph formed by duplicating all the vertices by edges in cycle $C_{n \geq 4}$ is not HMC, and in Theorem 2.6, we proved that the graph formed by duplicating all the edges by vertices in cycle C_n is not HMC. We also derived in Theorems 2.7 and 2.8 that the graph formed by duplicating all the vertices by edges as well as all the edges by vertices in cycle P_n is HMC, respectively. Furthermore, in Theorems 2.9 and 2.10, we prove that the graphs obtained by mutual duplication of a pair of edges, as well as mutual duplication of a pair of vertices from each pair of cycles C_n, admit HMC labeling, respectively.

2. Main Results

Theorem 2.1. *The graph G_2 obtained from a HMC graph G_1 by duplicating each of the vertices with an edge is a HMC graph if* $|V(G_1)| \cong 0 \pmod 2$.

Proof. Let G_1 be a HMC graph with $|V(G_1)| = p$ and $|E(G_1)| = q$. Let $f : V(G_1) \to \{1, 2\}$ be corresponding HMC labeling such that $|v_f(1) - v_f(2)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$. Let $G_2 = (V, E)$ be the graph obtained from G_1 by duplicating each of the vertices by an edge. Note that $|V(G_2)| = p + 2p = 3p$ and $|E(G_2)| = q + 3p$. We have $|v_g(1)| = \frac{3p}{2} = |v_g(2)|$.
Case 1: $q \cong 0 \pmod 2$
Define a labeling function $g : V(G_2) \to \{1, 2\}$ as follows

$$g(u) = \begin{cases} f(u) & \text{if } u \in V(G_1) \\ 1 & \text{if } u \textit{ is adjacent to } v \textit{ where } u \notin V(G_1), v \in V(G_1) \textit{ and } f(v) = 1 \\ 2 & \text{if } u \textit{ is adjacent to } v \textit{ where } u \notin V(G_1), v \in V(G_1) \textit{ and } f(v) = 2 \end{cases}$$

Then we have $e_g(1) = \frac{q+3p}{2} = e_g(2)$. So, we have $|v_g(1) - v_g(2)| \leq 1$ and $|e_g(1) - e_g(2)| \leq 1$.
Case 2: $q \cong 1 \pmod 2$
Define a labeling function $g : V(G_2) \to \{1, 2\}$ as follows

$$g(u) = \begin{cases} f(u) & \text{if } u \in V(G_1) \\ 1 & \text{if } u \textit{ is adjacent to } v \textit{ where } u \notin V(G_1), v \in V(G_1) \textit{ and } f(v) = 1 \\ 2 & \text{if } u \textit{ is adjacent to } v \textit{ where } u \notin V(G_1), v \in V(G_1) \textit{ and } f(v) = 2 \end{cases}$$

Then we have $e_g(1) = \frac{q+3p+1}{2}$ and $e_g(2) = \frac{q+3p-1}{2}$. So, we have $|v_g(1) - v_g(2)| \leq 1$ and $|e_g(1) - e_g(2)| \leq 1$. Hence, G_2 is HMC. □

Example 2.1. *HMC labeling of the graph G_2 obtained from a HMC graph $G_1 = C_{2,8}$ by duplicating each of the vertices with an edge is a HMC graph if* $|V(G_1)| \cong 0 \pmod 2$ *is shown in figure - 1.*

Example 2.2. *HMC labeling of the graph G_2 obtained from a HMC graph $G_1 = C_{1,8}$ by duplicating each of the vertices with an edge is a HMC graph if* $|V(G_1)| \cong 0 \pmod 2$ *is shown in figure - 2.*

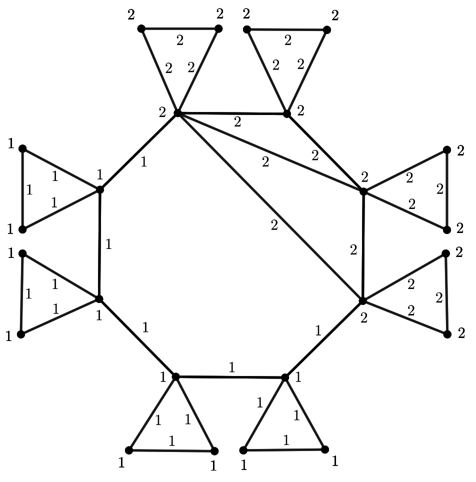

Figure − 1 : $C_{2,8}$

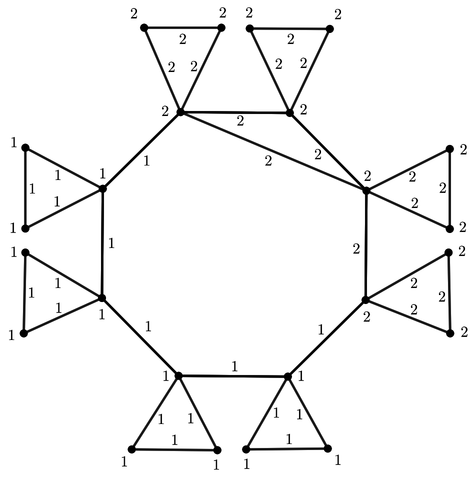

Figure − 2 : $C_{1,8}$

Theorem 2.2. *The graph G_2 obtained from a HMC graph G_1 by duplicating each of the vertices with an edge is a HMC graph if* $|V(G_1)| \cong 1 \pmod 2$ *and* $|E(G_1)| \cong 0 \pmod 2$.

Proof. Let G_1 be a HMC graph with $|V(G_1)| = p$ and $|E(G_1)| = q$. Let $f : V(G_1) \to \{1, 2\}$ be corresponding HMC labeling such that $|v_f(1) - v_f(2)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$. Let

$G_2 = (V, E)$ be the graph obtained from G_1 by duplicating each of the vertices by an edge. Note that $|V(G_2)| = p + 2p = 3p$ and $|E(G_2)| = q + 3p$. As $|V(G_1)| = p = 2k + 1$ have two possibilities:
(1) $|v_f(1)| = 2k$, $|v_f(2)| = 2k + 1$
(2) $|v_f(1)| = 2k + 1$, $|v_f(2)| = 2k$
Case 1: $|v_f(1)| = 2k$, $|v_f(2)| = 2k + 1$
Suppose $v_f(1) = \{v_1, v_2, ..., v_k\}$ and $v_f(2) = \{u_1, u_2, ..., u_{k+1}\}$ in G_1 where $p = 2k + 1$. So we have $|v_f(1)| = \frac{p-1}{2}$ and $|v_f(2)| = \frac{p+1}{2}$. Also $|e_f(1)| = \frac{q}{2} = |e_f(2)|$.
Now, $V(G_2) = V(G_1) \cup \{v_i^*, v_i^{**} \mid i \in \{1, 2, ..., k\} \cup \{u_i^*, u_i^{**} \mid i \in \{1, 2, ..., k+1\}$ and $E(G_2) = E(G_1) \cup \{v_i^* v_i, v_i^{**} v_i, v_i^* v_i^{**} \mid i \in \{1, 2, ..., k\} \cup \{u_i^* u_i, u_i^{**} u_i, u_i^* u_i^{**} \mid i \in \{1, 2, ..., k + 1\}$.
Define a labeling function $g : V(G_2) \to \{1, 2\}$ as follows

$$g(u) = \begin{cases} f(u) & \text{if } u \in V(G_1) \\ 1 & \text{if } x \in \{v_i^*, v_i^{**} \mid i \in \{1, 2, ..., k\} \\ 1 & \text{if } x = u_{k+1}^{**} \\ 2 & \text{if } x \in \{u_i^*, u_i^{**} \mid i \in \{1, 2, ..., k\} \\ 2 & \text{if } x = u_{k+1}^* \end{cases}$$

Then $v_g(1) = \frac{3p-1}{2}$, $v_g(2) = \frac{3p+1}{2}$ and $e_g(1) = \frac{q+3p+1}{2}$, $e_g(2) = \frac{q+3p-1}{2}$. So, we have $|v_g(1) - v_g(2)| \leq 1$ and $|e_g(1) - e_g(2)| \leq 1$. In this case G_2 admits HMC Labeling.
Case 2: $|v_f(1)| = 2k + 1$, $|v_f(2)| = 2k$
Suppose $v_f(1) = \{v_1, v_2, ..., v_k, v_{k+1}\}$ and $v_f(2) = \{u_1, u_2, ..., u_k\}$ in G_1 where $p = 2k + 1$. So we have $|v_f(1)| = \frac{p+1}{2}$ and $|v_f(2)| = \frac{p-1}{2}$. Also $|e_f(1)| = \frac{q}{2} = |e_f(2)|$.
Now, $V(G_2) = V(G_1) \cup \{v_i^*, v_i^{**} \mid i \in \{1, 2, ..., k+1\} \cup \{u_i^*, u_i^{**} \mid i \in \{1, 2, ..., k\}$ and $E(G_2) = E(G_1) \cup \{v_i^* v_i, v_i^{**} v_i, v_i^* v_i^{**} \mid i \in \{1, 2, ..., k + 1\} \cup \{u_i^* u_i, u_i^{**} u_i, u_i^* u_i^{**} \mid i \in \{1, 2, ..., k\}$.
Define a labeling function $g : V(G_2) \to \{1, 2\}$ as follows

$$g(u) = \begin{cases} f(u) & \text{if } u \in V(G_1) \\ 1 & \text{if } x \in \{v_i^*, v_i^{**} \mid i \in \{1, 2, ..., k\} \\ 2 & \text{if } x \in \{u_i^*, u_i^{**} \mid i \in \{1, 2, ..., k\} \\ 2 & \text{if } x \in \{v_{k+1}^*, v_{k+1}^{**}\} \end{cases}$$

Then $v_g(1) = \frac{3p+1}{2}$, $v_g(2) = \frac{3p-1}{2}$ and $e_g(1) = \frac{q+3p+1}{2}$, $e_g(2) = \frac{q+3p-1}{2}$. So, we have $|v_g(1) - v_g(2)| \leq 1$ and $|e_g(1) - e_g(2)| \leq 1$. In this case G_2 admits HMC Labeling. Hence, G_2 is HMC. □

Example 2.3. *HMC labeling of the graph G_2 obtained from a HMC graph $G_1 = C_{1,7}$ and $G_1 = C_{2,7}$ by duplicating each of the vertices with an edge is a HMC graph if $|V(G_1)| \cong 1 \pmod 2$ and $|E(G_1)| \cong 0 \pmod 2$ is shown in figure - 3.*

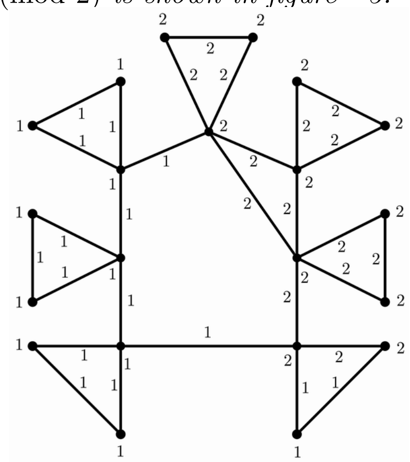

Figure − 3 : $C_{1,7}$

Theorem 2.3. *The graph formed by duplicating an arbitrary vertex with a new edge in cycle C_n is HMC.*

Proof. Let $G = (V, E)$ be the graph formed by duplicating an arbitrary vertex with a new edge in cycle C_n. Note that $|V(G)| = n+2$ and $|E(G)| = n+3$. Let $V = \{v_1, v_2, ..., v_{n+2}\}$ be the vertex set of G. Without loss of generality we fix duplicating vertex v_1 of C_n by an edge e_{n+1} with end vertices v_{n+1} and v_{n+2} as shown in following figure - 4.

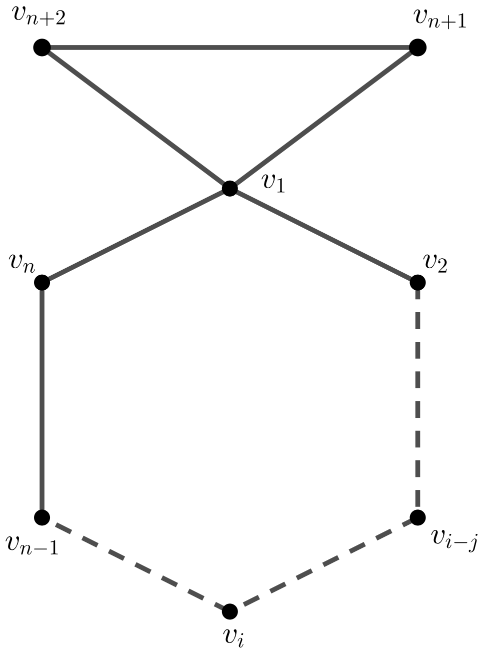

Figure – 4 : Duplicating an arbitrary vertex with a new edge in cycle C_n

Case 1: n is even
Define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & \text{if } 1 \leq i \leq \frac{n-2}{2} \\ 2 & \text{if } n+1 \leq i \leq n+2 \\ 1 & \text{if } \frac{n+2}{2} \leq i \leq n \end{cases}$$

Then $v_f(1) = v_f(2) = \frac{n+2}{2}$ and $e_f(1) = \frac{n+4}{2}$, $e_f(2) = \frac{n+2}{2}$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 1$.

Case 2 : n is odd
Define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & \text{if } 1 \leq i \leq \frac{n-1}{2} \\ 2 & \text{if } n+1 \leq i \leq n+2 \\ 1 & \text{if } \frac{n+1}{2} \leq i \leq n \end{cases}$$

Then $v_f(1) = \frac{n-1}{2}$, $v_f(2) = \frac{n+1}{2}$ and $e_f(1) = e_f(2) = \frac{n+3}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 0$.
Hence, G is HMC. □

Example 2.4. *HMC labeling of the graph formed by duplication of a vertex by an edge in C_5 and C_6 is shown in figure - 5.*

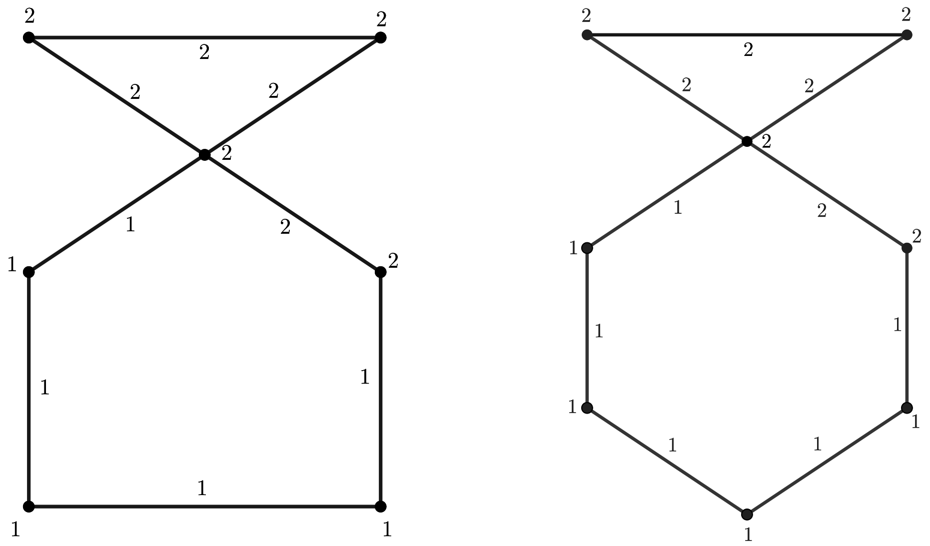

Figure – 5 : Duplicating an arbitrary vertex with a new edge in cycle C_5 and C_6

Theorem 2.4. *The graph formed by duplicating an arbitrary edge with a new vertex in cycle $C_{n\geq 4}$ is HMC.*

Proof. Let $G = (V, E)$ be the graph formed by duplicating an arbitrary edge with a new vertex in cycle C_n. Note that $|V(G)| = n+1$ and $|E(G)| = n+2$. Let $V = \{v_1, v_2, ..., v_{n+1}\}$ be the vertex set of G. Without loss of generality we fix duplicating edge v_1v_2 of C_n by an vertex v_{n+1} as shown in following figure - 6.

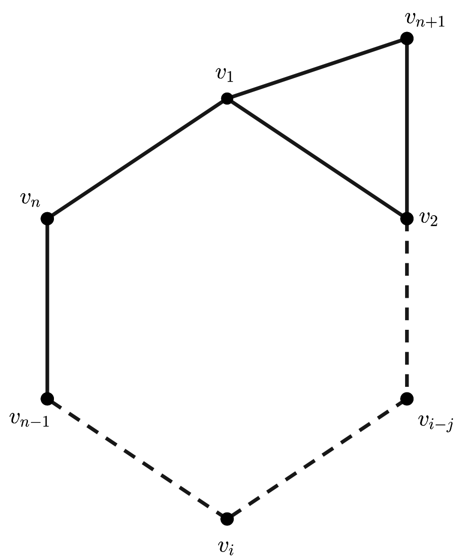

Figure – 6 : Duplicating an arbitrary edge with a new vertex in cycle C_n

Case 1: n is even

Define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & \text{if } 1 \le i \le \frac{n}{2} \\ 2 & \text{if } i = n+1 \\ 1 & \text{if } \frac{n+2}{2} \le i \le n \end{cases}$$

Then $v_f(1) = \frac{n}{2}$, $v_f(2) = \frac{n+2}{2}$ and $e_f(1) = e_f(2) = \frac{n+2}{2}$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 1$.

Case 2 : n is odd

Define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & \text{if } 1 \le i \le \frac{n-1}{2} \\ 2 & \text{if } i = n+1 \\ 1 & \text{if } \frac{n+1}{2} \le i \le n \end{cases}$$

Then $v_f(1) = v_f(2) = \frac{n+1}{2}$ and $e_f(1) = e_f(2) = \frac{n+3}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 0$.

Hence, G is HMC. □

Example 2.5. *HMC labeling of the graph formed by duplication of a edge by a vertex in C_7 and C_8 is shown in figure - 7.*

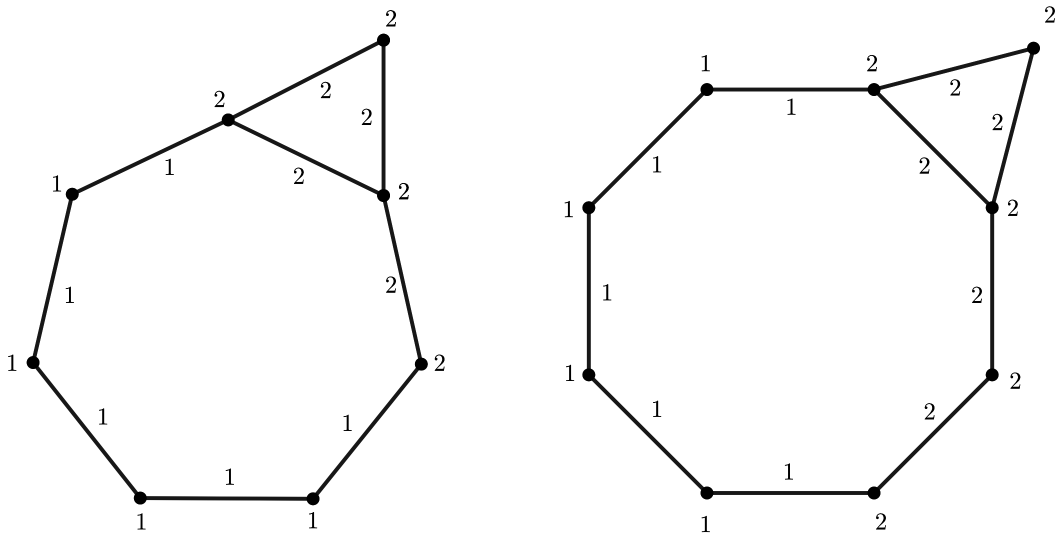

Figure − 7 : Duplicating an arbitrary edge with a new vertex in cycle C_7 and C_8

Theorem 2.5. *The graph formed by duplicating all the vertices by edges in cycle $C_{n\ge4}$ is not HMC.*

Proof. Let $G = (V, E)$ be the graph formed by duplicating all vertices by edge in cycle $C_{n\ge4}$. Note that $|V(G)| = 3n$ and $|E(G)| = 4n$. Let $V = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n, v''_1, v''_2, \ldots, v''_n\}$ be the vertex set of G.

If possible, let there be a HMC labeling $f : V(G) \to \{1, 2\}$ for graph G which is formed by duplicating all vertices by edge in cycle $C_{n\ge4}$.

Case 1: $n \cong 0 \bmod 2$

So, $v_f(1) = v_f(2) = \frac{3n}{2}$. If we assign consecutive labeling 2 on $\frac{3n}{2}$ vertices of G, then we have $e_f(2) = 2n - 1$ and $e_f(1) = 2n + 1$. Therefore $|e_f(1) - e_f(2)| = 2 > 1$.

Without assuming consecutive labeling 2 on $\frac{3n}{2}$ vertices of G, we get $|e_f(1) - e_f(2)| > 2$. Thus G does not satisfies HMC labeling.

Case 2: $n \cong 1 \bmod 2$

As we have $|v_f(1) - v_f(2)| \le 1$, we have two possibilities,

(1) $v_f(1)\frac{3n-1}{2}, v_f(2)\frac{3n+1}{2}$ and
(2) $v_f(1)\frac{3n+1}{2}, v_f(2)\frac{3n-1}{2}$
Subcase 2.1: $v_f(1)\frac{3n-1}{2}, v_f(2)\frac{3n+1}{2}$
If we assign consecutive labeling 2 on $\frac{3n+1}{2}$ vertices of G, then we have $e_f(2) = 2n-1$ and $e_f(1) = 2n+1$. Therefore $|e_f(1) - e_f(2)| = 2 > 1$.
Without assuming consecutive labeling 2 on $\frac{3n+1}{2}$ vertices of G, we get $|e_f(1) - e_f(2)| > 2$. Thus G does not satisfies HMC labeling.
Subcase 2.2: $v_f(1)\frac{3n+1}{2}, v_f(2)\frac{3n-1}{2}$
If we assign consecutive labeling 2 on $\frac{3n-1}{2}$ vertices of G, then we have $e_f(2) = 2n-2$ and $e_f(1) = 2n+2$. Therefore $|e_f(1) - e_f(2)| = 4 > 1$.
Without assuming consecutive labeling 2 on $\frac{3n-1}{2}$ vertices of G, we get $|e_f(1) - e_f(2)| > 4$. Thus G does not satisfies HMC labeling.
Hence, the graph formed by duplicating all the vertices by edges in cycle $C_{n\geq 4}$ is not HMC. □

Corollary 2.1. *The graph formed by duplicating all the vertices by edges in cycle C_3 and its HMC labeling is shown in figure - 8.*

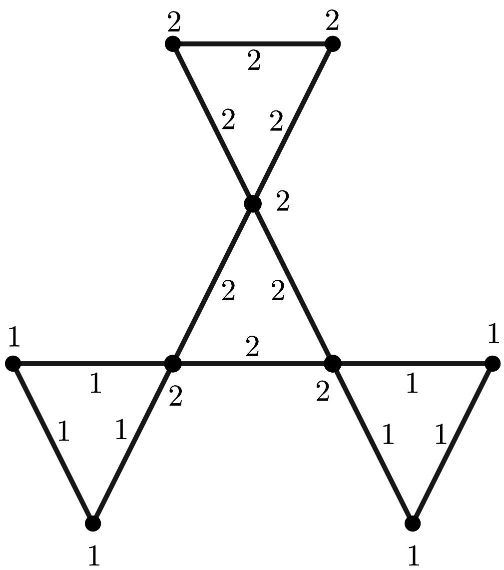

Figure – 8 : Duplicating all the vertices by edge in cycle C_3

Theorem 2.6. *The graph formed by duplicating all the edges by vertices in cycle C_n is not HMC.*

Proof. Let $G = (V, E)$ be the graph formed by duplicating all edges by vertices in cycle C_n. Note that $|V(G)| = 2n$ and $|E(G)| = 3n$. Let $V = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n'\}$ be the vertex set of G.
If possible, let there be a HMC labeling $f : V(G) \to \{1, 2\}$ for graph G which is formed by duplicating all edges by vertices in cycle C_n.
As we have $|v_f(1) - v_f(2)| = 0$. So, $|v_f(1)| = n = |v_f(2)|$, we have two possibilities,
(1) $n \cong 0$ mod 2 and
(2) $n \cong 1$ mod 2.
Case 1: $n \cong 0$ mod 2

If we assign consecutive labeling 2 on n vertices of G, then we have $e_f(2) = \frac{3n-4}{2}$ and $e_f(1) = \frac{3n+4}{2}$. Therefore $|e_f(1) - e_f(2)| = 4 > 1$.
Without assuming consecutive labeling 2 on n vertices of G, we get $|e_f(1) - e_f(2)| > 4$. Thus G does not satisfies HMC labeling.
Case 2: $n \cong 1 \bmod 2$
If we assign consecutive labeling 2 on n vertices of G, then we have $e_f(2) = \frac{3n-3}{2}$ and $e_f(1) = \frac{3n+3}{2}$. Therefore $|e_f(1) - e_f(2)| = 6 > 1$.
Without assuming consecutive labeling 2 on n vertices of G, we get $|e_f(1) - e_f(2)| > 6$. Thus G does not satisfies HMC labeling.
Hence,the graph formed by duplicating all the edges by vertices in cycle C_n is not HMC. □

Theorem 2.7. *The graph formed by duplicating all the vertices by edges in path P_n is HMC.*

Proof. Let $G = (V, E)$ be the graph formed by duplicating all arbitrary vertices by edge in path P_n. Note that $|V(G)| = 3n$ and $|E(G)| = 4n-1$. Let $V = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n, v''_1, v''_2, \ldots, v''_n\}$ as shown in figure - 9.

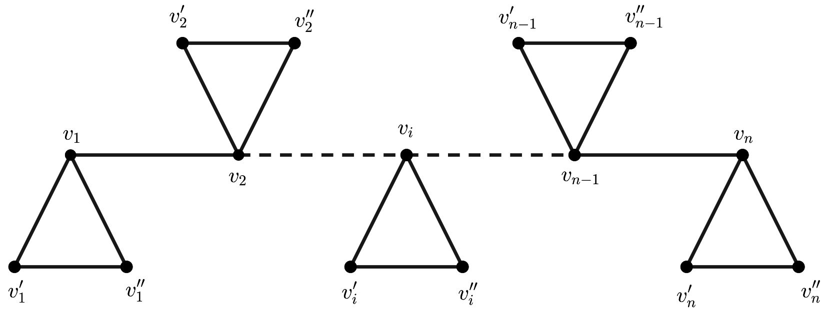

Figure – 9 : Duplicating all the vertices with by edges in Path P_n

Case 1: $n \cong 0 \bmod 2$
Proof Follows from Theorem 2.1.
Case 2: $n \cong 1 \bmod 2$
Proof Follows from Theorem 2.2. □

Example 2.6. *HMC labeling of the graph formed by duplicating all the vertices by edges in path P_6 and P_7 is shown in figure - 10 and figure - 11.*

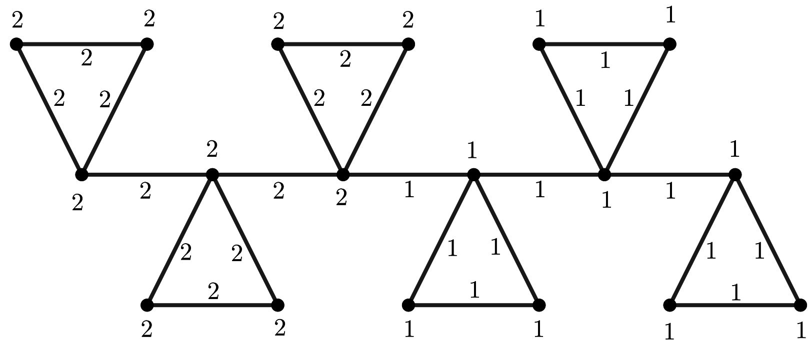

Figure – 10 : Duplicating all the vertices with by edges in Path P_6

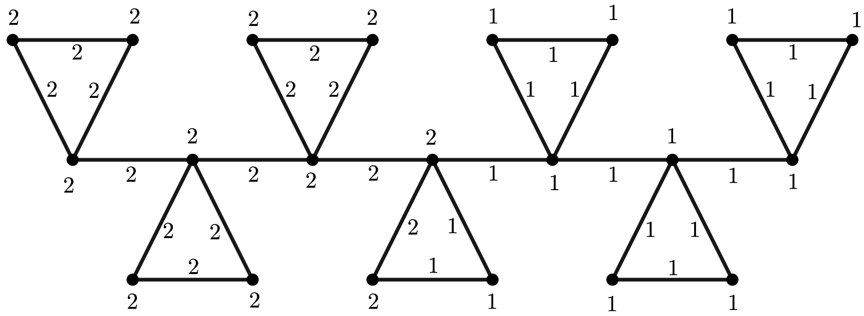

Figure – 11 : Duplicating all the vertices with by edges in Path P_7

Theorem 2.8. *The graph formed by duplicating all the edges by vertices in path P_n is HMC.*

Proof. The graph formed by duplicating all the edges by vertices in path P_n is Triangular Snake TS_n which is HMC proved by J. Gowri and J Jayapriya in [6]. □

Theorem 2.9. *The graph formed by mutual duplication of a pair of edges in cycle $C_{n\geq 8}$ is HMC.*

Proof. Let $G = (V, E)$ be the graph formed by mutual duplication of a pair of edges in cycle $C_{n\geq 8}$. Note that $|V(G)| = 2n$ and $|E(G)| = 2n + 4$. Let $V = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$ be the vertex set of G. Without loss of generality we fix mutual duplicating edge v_1v_n and u_1u_n of $C_{n\geq 8}$ as shown in following figure - 12.

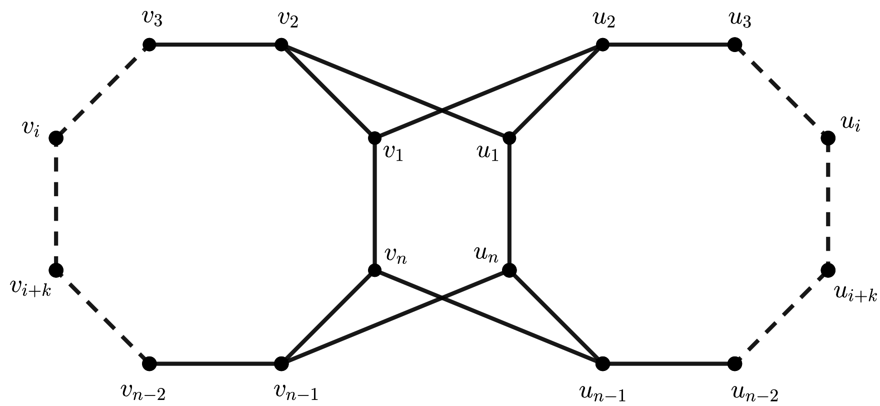

Figure – 12 : Mutual Duplication of a Pair of Edges in Cycle $C_{n\geq 8}$

Define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 1 & \text{if } 3 \leq i \leq 6 \\ 2 & \text{if } i \in \{1, 2, 7, \ . \ . \ . \ , n\} \end{cases}$$

$$f(u_i) = \begin{cases} 1 & \text{if } 3 \leq i \leq n-2 \\ 2 & \text{if } i \in \{1, 2, n-1, n\} \end{cases}$$

Then $v_f(1) = v_f(2) = \frac{n}{2}$ and $e_f(1) = e_f(2) = \frac{2n+4}{2} = n+2$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 0$. Hence, G is HMC. □

Example 2.7. *HMC labeling of the mutual duplication of a pair of edges of cycle C_9 and C_{10} shown in figure - 13 and figure - 14.*

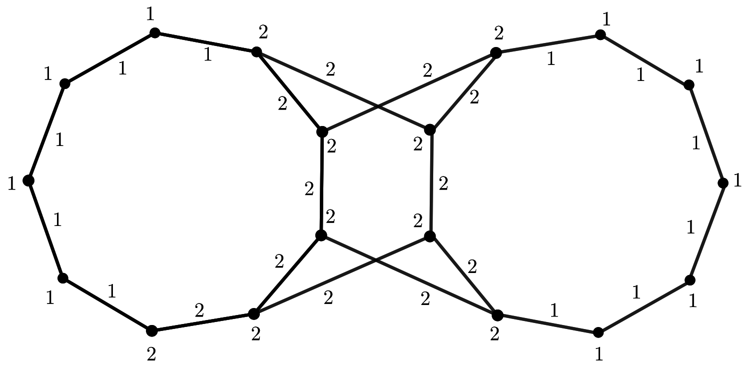

Figure – 13 : Mutual Duplication of a Pair of Edges in Cycle C_9

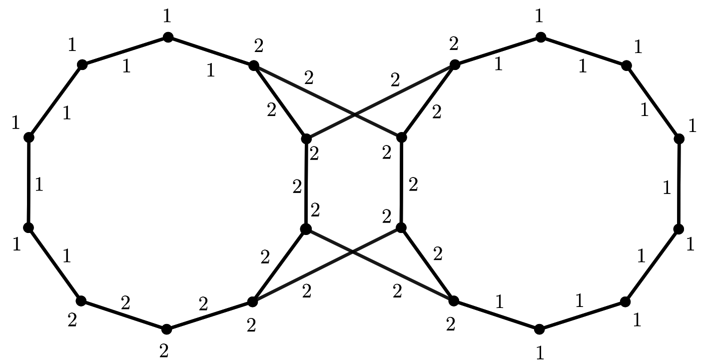

Figure – 14 : Mutual Duplication of a Pair of Edges in Cycle C_{10}

Corollary 2.2. *The graph formed by mutual duplication of a pair of edges in cycle $C_{n\leq 7}$ is not HMC.*

Proof. The graph formed by mutual duplication of a pair of edges in cycle $C_{n\leq 7}$ under consideration violates the condition as shown in Table - 1.

n	$\|V(G)\|$	$\|E(G)\|$	$\|v_f(1)\|$	$\|v_f(2)\|$	$\|e_f(1)\|$	$\|e_f(2)\|$	$\|e_f(1) - e_f(2)\|$
3	6	10	3	3	7	3	4
4	8	12	4	4	8	4	4
5	10	14	5	5	9	5	4
6	12	16	6	6	10	6	4
7	14	18	7	7	10	8	2

Table – 1

Hence, The graph formed by mutual duplication of a pair of edges in cycle $C_{n\leq 7}$ is not HMC. □

Theorem 2.10. *The graph formed by mutual duplication of a pair of vertices in cycle $C_{n\geq 6}$ is HMC.*

Proof. Let $G = (V, E)$ be the graph formed by mutual duplication of a pair of vertices in cycle $C_{n\geq 6}$. Note that $|V(G)| = 2n$ and $|E(G)| = 2n+4$. Let $V = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$ be the vertex set of G. Without loss of generality we fix mutual duplicating vertex v_1 and u_1 of $C_{n\geq 6}$ as shown in following figure - 15.

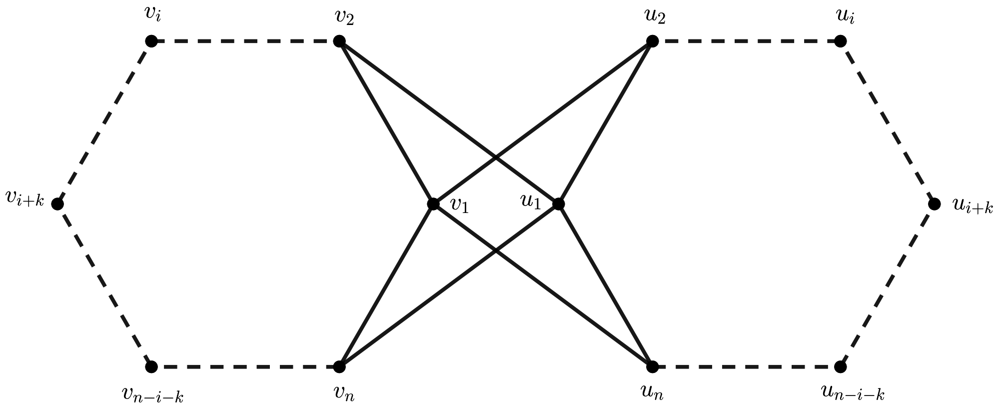

Figure – 15 : Mutual Duplication of a Pair of Vertices in Cycle $C_{n\geq 6}$

Define a labeling function $f : V(G) \rightarrow \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 1 & \text{if } 3 \leq i \leq 5 \\ 2 & \text{if } i \in \{1, 2, 6, \ . \ . \ . \ , n\} \end{cases}$$

$$f(u_i) = \begin{cases} 1 & \text{if } 3 \leq i \leq n-1 \\ 2 & \text{if } i \in \{1, 2, n\} \end{cases}$$

Then $v_f(1) = v_f(2) = \frac{n}{2}$ and $e_f(1) = e_f(2) = \frac{2n+4}{2} = n+2$. So, we have $|v_f(1)-v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 0$. Hence, G is HMC. □

Example 2.8. *HMC labeling of the mutual duplication of a pair of vertices of cycle C_7 and C_8 shown in figure - 16 and figure - 17.*

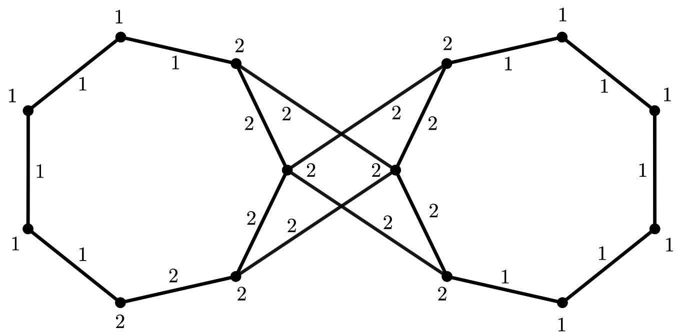

Figure – 16 : Mutual Duplication of a Pair of Vertices in Cycle C_7

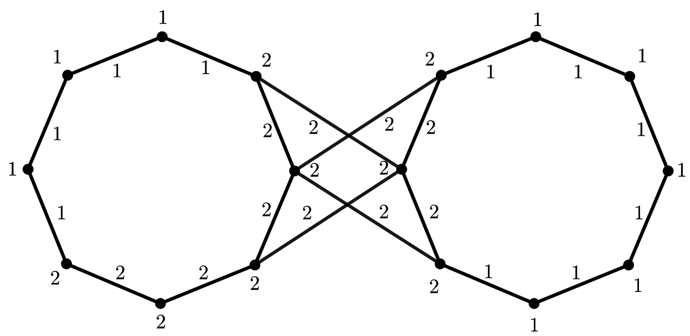

Figure – 17 : Mutual Duplication of a Pair of Vertices in Cycle C_8

Corollary 2.3. *The graph formed by mutual duplication of a pair of vertices in cycle* $C_{n\leq 5}$ *is not HMC.*

Proof. The graph formed by mutual duplication of a pair of vertices in cycle $C_{n\leq 5}$ under consideration violates the condition as shown in Table - 2.

n	$\|V(G)\|$	$\|E(G)\|$	$\|v_f(1)\|$	$\|v_f(2)\|$	$\|e_f(1)\|$	$\|e_f(2)\|$	$\|e_f(1) - e_f(2)\|$
3	6	10	3	3	7	3	4
4	8	12	4	4	8	4	4
5	10	14	5	5	8	6	2

$Table - 2$

Hence, The graph formed by mutual duplication of a pair of vertices in cycle $C_{n\leq 5}$ is not HMC. □

3. Conclusions

In this article, we have discussed HMC labeling for duplicating vertices by edges in a generalized HMC graph. We also explain that the duplicating vertices by an edge and the duplicating edges by a vertice on cycles C_n are HMC. We proved that the graph formed by duplicating all vertices by edges in cycle $C_{n\geq 4}$ is not HMC, and the graph formed by duplicating all edges by vertices in cycles in C_n is not HMC. We also prove that the graph formed by duplicating all vertices by edges and all the edges by vertices in path P_n is HMC. In addition, we have derived the condition from the natural number n The graph formed by mutual duplication of a pair of edges as well as mutual duplication of a pair of vertices in cycle C_n is HMC.

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Mahesh M Jariya for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.14, N.2.