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φ–NEARLY LIPSCHITZIAN AND STRONGLY PSEUDO-CONTRACTIVE MAPPINGS IN $CAT_p(0)$ SPACES VIA FIXED POINT THEORY

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Abstract. This paper introduces novel conditions for mappings in $CAT_p(0)$ spaces, $p \geq 2$, that extend existing concepts in the literature. By leveraging the geometric properties of $CAT_p(0)$ spaces, we establish two fixed point theorems for φ–nearly Lipschitzian and generalized strongly pseudo-contractive mappings.

Keywords: φ–nearly Lipschitzian mapping, $CAT(0)$ spaces, generalized strongly pseudo-contractive mapping.

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1. Introduction

The concepts of nearly Lipschitzian mappings, nearly contraction mappings, nearly nonexpansive mappings, nearly asymptotically nonexpansive mappings, nearly uniformly k-Lipschitzian mappings, and nearly uniform k-contraction mappings were introduced by Sahu [12] in 2005.

In 2019, Mostafa Bachar and Mohamed Amine Khamsi applied $\mathrm{CAT}_p(0)$ spaces, where the comparison triangles belong to ℓ_p, for $p \geq 2$. They studied some theorems in $\mathrm{CAT}_p(0)$ spaces and utilized them to investigate the existence of fixed points in $\mathrm{CAT}_p(0)$ spaces.

A $\mathrm{CAT}_p(0)$ space is a specific type of geodesic metric space such that every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. For example, every complete, connected Riemannian manifold with non-positive sectional curvature is a $\mathrm{CAT}_p(0)$ space

This manuscript is a study on all classes of W-hyperbolic spaces. The $\mathrm{CAT}_p(0)$ spaces are the specific types of W-hyper-convex spaces in the sense that a unique metric interval joins every two points of such a space. In continuation and line with the works of $\mathrm{CAT}_p(0)$ space, many researchers introduce and research the contents.

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The non-local property for the $CAT(0)$ space is significant in application because it is useful for modeling the dynamics of many problems in physics, engineering, medicine, economics, control theory, etc. To see more details, readers can refer to the references ([1]- [5], [6], [7]- [10]).

We establish two fixed point theorems for φ–nearly Lipschitzian and generalized strongly pseudo-contractive mappings in $\mathrm{CAT}_p(0)$ spaces, which help to extend some concepts in the literature and are very helpful for some geometrical properties of $\mathrm{CAT}_p(0)$ spaces, for $p \geq 2$.

2. Preliminaries

Definition 1 : Let C be a nonempty subset of a Banach space X, $\varphi : R^+ = [0, \infty) \to R^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ and $J : C \to C$. Fix a sequence $\{A_n\}$ in R^+ with $A_n \to 0$. Let

$$\eta(J^n) = sup\{\frac{d(J^n x, J^n y)}{\varphi(d(x, y) + A_n)} : x, y \in C, x \neq y\},$$

is $\varphi-$nearly Lipschitzian constant of J^n. J is said to be $\varphi-$nearly nonexplosive if $\eta(J^n) = 1$, also J is $\varphi-$nearly Lipschitzian if

$$d(J^n x, J^n y) \leq \eta(J^n).\varphi(d(x, y) + A_n),$$

for $x, y \in C$ and $n \in N$. J is $\varphi-$nearly asymptotically non-expansive with sequence $\{(\eta(J^n), A_n)\}$ if $\eta(J^n) \geq 1$ with $\lim_{n\to\infty} \eta(J^n) = 1$ and

$$d(J^n x, J^n y) \leq \eta(J^n).\varphi(d(x, y) + A_n).$$

Example 1 : Let $X = R$, $C = [0, 1]$ and $J : C \to C$ defined by

$$Jx = \begin{cases} 1 & x \in [0, \frac{1}{2}], \\ \frac{1}{2} & x \in (\frac{1}{2}, 1]. \end{cases} \tag{1}$$

Obviously, J is discontinuous and non-Lipschitzian. Although, it is $\varphi-$nearly non-expansive such that, for a sequence a_n with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we get

$$d(Jx, Jy) \leq d(x, y) + a_1$$

for $x, y \in C$ and

$$d(J^n x, J^n y) \leq d(x, y) + a_1$$

for $x, y \in C$ and $n \geq 2$.

Example 2 : Let $X = R$, $C = [0, 1]$ and $J : C \to C$ defined by

$$Jx = \begin{cases} \frac{\frac{1}{2^{2n}}}{1+\frac{1}{2^n}} & x \in [0, \frac{1}{2}], \\ 0 & x \in (\frac{1}{2}, 1]. \end{cases} \tag{2}$$

Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{t}{1+t}$. Clearly, φ is strictly increasing and $\varphi(0) = 0$. Observe that J is not continuous and non-Lipschitzian. We now show that J is $\varphi-$nearly non-expansive, with sequence $a_n = \frac{1}{2^n} \to 0$ as $n \to \infty$, we have for each $x, y \in C$,

$$\|Jx - Jy\| \leq \|x - y\| + a_n$$

and for each $x, y \in C$ and $n \in [2, \infty)$, we obtain

$$\|J^n x - J^n y\| \leq \frac{\frac{1}{2^{2n}}}{1 + \frac{1}{2^n}} + a_n \leq \frac{\frac{1}{2^n} \cdot \frac{1}{2^n}}{1 + \frac{1}{2^n}} + a_n \leq \varphi(\|x - y\| + a_n).$$

Hence, J is $\varphi-$nearly non-expansive.

Definition 2 : For a metric space (X, d), a geodesic joining $x \in X$ to $y \in X$ is a mapping $\gamma : [0, d(x, y)] \to X$ such that
i_1. $\gamma(0) = x$,
i_2. $\gamma(d(x, y)) = y$,
i_3. $d(\gamma(t_1), \gamma(t_2)) =| t_1 - t_2 |$ for $t_1, t_2 \in [0, d(x, y)]$.

Definition 3 : A metric space (X, d) is geodesic if every two points in X are joined by a geodesic. (X, d) is said to be uniquely geodesic, if, for every $x, y \in X$, there is exactly one geodesic joining x and y for each $x, y \in X$, which we denote by $[x, y]$. The point $\gamma(t)$ in $[x, y]$ is also denoted by $(1 - t)x \oplus ty$.

Definition 4 : Let (X, d) be a geodesic metric space. A geodesic triangle consists of three point $x_1, x_2, x_3 \in X$ and three geodesics $[x_1, x_2], [x_2, x_3], [x_3, x_1]$.
Denote $\Delta([x_1, x_2], [x_2, x_3], [x_3, x_1])$. For such a triangle, there is a comparison triangle $\overline{\Delta}(\overline{x_1}, \overline{x_2}, \overline{x_3}) \subset R^2$:
j_1. $d(x_1, x_2) = d(\overline{x_1}, \overline{x_2})$,
j_2. $d(x_2, x_3) = d(\overline{x_2}, \overline{x_3})$,
j_3. $d(x_3, x_1) = d(\overline{x_3}, \overline{x_1})$.

Definition 5 : A geodesic space (X, d), is a $CAT(0)$ space if for any geodesic triangle $\Delta \subset X$ and $x, y \in \Delta$ the equality $d(x, y) = d(\overline{x}, \overline{y})$,$\overline{x}, \overline{y} \in \overline{\Delta}$ is true and X is a $CAT_p(0)$, for $p > 2$, if for any Δ in X, there exists a comparison triangle Δ in ℓ_p such that the comparison axiom holds, i.e., for $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \Delta$, the following inequality is true.

$$d(x, y) \leq \|\bar{x} - \bar{y}\|.$$

Definition 6 : Let X be a metric space and J be a mapping on X. Then J is generalized strongly pseudo-contractive if for $x, y \in Dom(J)$ the following inequality holds:

$$d(J^n x, J^n y) \leqslant \eta(J^n)\Big[d(x, y) + d(J^n x, x) + d(J^n y, y) + \frac{1}{2}(d(J^n x, y) + d(J^n y, x))\Big],$$

such that $\eta(J^m) \geqslant \frac{1}{4}, \lim_{m\to\infty} \eta(J^m) = \frac{1}{4}$.

Definition 7 : $\xi : X \to [0, +\infty)$ is a type function if there exists a sequence $\{y_n\}$ in X with the following property.

$$\xi(A) = \lim_{n\to\infty} \sup d(y_n, A),$$

A sequence $\{t_n\}$ in X is said to be a minimizing of ξ if

$$\lim_{n\to\infty} \xi(t_n) = \inf\{\xi(A); A \in X\}.$$

Example 3 : The unit sphere S_{ℓ_2} of the Hilbert space ℓ_2 provided with the intrinsic metric ℓ_d is a $CAT(1)$ space.

The following lemmas demonstrates why type functions are stronger tools.

Lemma 1: [1] Suppose that (X, d) is a metric space. Considering the $CAT_p(0)$ metric space, $2 \leq p$, for $x, y_1, y_2 \in X$ and $\alpha \in [0, 1]$, the following equality is true.

$$d(x, \alpha y_1 \oplus (1 - \alpha)y_2)^p + \frac{1}{2^{p-1}}\alpha(1 - \alpha)d(y_1, y_2)^p = \alpha d(x, y_1)^p + (1 - \alpha)d(x, y_2)^p.$$

Lemma 2: [1] Suppose that (X, d) is a complete $CAT_p(0)$ metric space, with $2 \leq p$. Let C is a nonempty closed convex subset of X and $\xi : C \to [0, +\infty)$ is a type function generated by a limited sequence $x_n \subset X$. Then, we have these items.
k_1. Every minimization sequence of ξ is convergent.
k_2. All minimization sequences ξ converge to the same extent $z \in C$.
k_3. z is a minimum point of $\xi, i.e., \xi(z) = \inf\{\xi(x) : x \in C\}$.

3. Main results

Now we obtain two fixed point results for $\varphi-$nearly Lipschitzian mapping and generalized strongly pseudo-contractive mapping.

Theorem 1 : Let (X, d) be a complete $CAT_p(0)$ metric space, with $p \geqslant 2$. Let C be nonempty closed bounded convex subset of X. Let $J : C \to C$ be a $\varphi-$nearly Lipschitzian mapping. Then, J has a fixed point. In addition, $Fix(J)$ is closed and convex.
Proof: Let $\eta(J^m)$ be $\varphi-$nearly Lipschitzian constant with sequence $(\eta(J^m), A_n)$. Fix $x \in C$, consider the type function ξ generated by $J^m(x)$. Let z be the minimum point of ξ which exists by using Lemma 2. Therefore,

$$d(J^{n+m}x, J^m z) \leqslant \eta(J^n).\varphi(d(J^n(x), z) + A_n),$$

for any $n, m \in N$. Letting $n \to \infty$, we get $\xi(J^m(z)) \leqslant \eta(J^n)\varphi(\xi(z)) = \eta(J^m)\varphi(\xi_0)$, for any $m \in N$. If $m \to \infty$, by using $\lim_{n\to\infty} \eta(J^n) = 1$, we get
$\lim_{m\to\infty} \xi(J^m(z)) = \varphi(\xi_0)$, i.e., $J^m(z)$ is a minimizing sequence of ξ. Using Lemma 2, we conclude that $J^m(z)$ converges to z. As J is continuous, we conclude that $J(z) = z$. The fact that $Fix(J)$ is closed by continuity of J. Let us prove that $Fix(J)$ is convex. Suppose $h_1, h_2 \in Fix(J)$ are different. Since $Fix(J)$ is closed, We just prove that $\omega = \frac{1}{2}h_1 + \frac{1}{2}h_2 \in Fix(J)$. Note that

$$d(h_i, J^n(\omega)) = d(J^n(h_i), J^n(\omega)) \leqslant \eta(J^n).\varphi(d(h_i, \omega)) = \frac{\eta(J^n)}{2}.\varphi(d(h_1, h_2) + A_n),$$

for $n \in N$ and $i = 1, 2$. Therefore,

$$d(h_1, h_2) \leqslant d(h_1, J^n(\omega)) + d(J^n(\omega), h_2) \leqslant \eta(J^n).\varphi(d(h_1, h_2) + A_n),$$

for any $n \in N$. As $\lim_{n\to\infty} \eta(J^n) = 1$ and, $\lim_{n\to\infty} A_n = 0$, we conclude that
$\lim_{n\to\infty} d(h_i, J^n(\omega)) = \eta(J^n).\varphi(d(h_1, h_2))$, for $i = 1, 2$. We have the same

$$\lim_{n\to\infty} d(h_i, \frac{1}{2}\omega \oplus \frac{1}{2}J^n(\omega)) = \frac{\varphi(d(h_1, h_2))}{2},$$

for $i = 1, 2$. Using Lemma 1, we get

$$d(h_1, \frac{1}{2}\omega \oplus \frac{1}{2}J^n(\omega))^p + \frac{1}{2^{p+1}}d(\omega, J^n(\omega))^p \leqslant \frac{1}{2}d(h_1, \omega)^p + \frac{1}{2}d(h_1, J^n(\omega))^p,$$

for any $n \in N$. If $n \to \infty$, we get

$$\lim_{n\to\infty} d(\omega, J^n(\omega)) = 0.$$

As J is continuous, then $\omega \in Fix(J)$ as claimed, which completes the proof of Theorem 1.

Theorem 2 : Suppose that (X, d) is a complete $CAT_p(0)$ metric space, with $p \geqslant 2$. Let C be closed bounded convex subset (CBCS) of X and let $J : C \to C$ be generalized strongly pseudo-contractive mapping.
Then J has a fixed point. Moreover $Fix(J)$ is closed and convex.

Proof: Let $J : C \to C$ is generalized strongly pseudo-contractive mapping with sequence $(\eta(J^m), A_m)$. Fix $x \in C$, consider the type function ξ generated by $J^m(x)$. Let x be the minimum point of ξ which exists by using Lemma 2. Hence,

$$d(J^m x, J^m y) \leqslant \eta(J^m)\Big[\frac{1}{6}d(x,y) + \frac{1}{4}(d(J^m x, x) + d(J^m y, y)) + \frac{1}{6}(d(J^m x, y) + d(J^m y, x))\Big],$$

for any $m \in N$. Letting $m \to \infty$, we get $\xi(J^m(x)) \leqslant \eta(J^m)\xi(x) = \eta(J^m)\xi_0$, for any $m \in N$. If $m \to \infty$, by using $\lim_{m\to\infty} \eta(J^m) = 1$, we get
$\lim_{m\to\infty} \xi(J^m(x)) = \xi_0$, i.e., $J^m(x)$ is a minimizing sequence of ξ. Using Lemma 2, we conclude that $J^m(x)$ converges to x. As J is continuous, we conclude that $J(x) = x$. The fact that $Fix(J)$ is closed by continuity of J. Let us prove that $Fix(J)$ is convex. Let $x_1, x_2 \in Fix(J)$ be different. As $Fix(J)$ is closed, we only need to prove that $y = \frac{1}{2}x_1 + \frac{1}{2}x_2 \in Fix(J)$. By definition 2, we have:

$$\begin{aligned}
d(x_i, J^m y) &= d(J^m x_i, J^m y) \\
&\leqslant d(J^m x, J^m y) \\
&\leqslant \eta(J^m)\Big[\frac{1}{6}d(x,y) + \frac{1}{4}(d(J^m x, x) \\
&+ d(J^m y, y)) + \frac{1}{6}(d(J^m x, y) + d(J^m y, x))\Big] \\
&= \eta(J^m)\Big[\frac{1}{6}d(x,y) + \frac{1}{6}(d(x,y) + d(y,x))\Big],
\end{aligned}$$

$$\lim_{m\to\infty} d(J^m(x_i), J^m(y)) = d(x_i, \lim_{m\to\infty} J^m(y)) \leq \lim_{m\to\infty} \eta(J^m)\Big[\frac{1}{2}d(x,y)\Big],$$

for $m \in N$ and $i = 1, 2$.
Therefore,

$$\begin{aligned}
d(x_1, x_2) &\leq d(x_1, J^m(y)) + d(J^m(y), x_2) \\
&\leq \frac{1}{2}\eta(J^m)\Big[d(x_1, y) + d(x_2, y)\Big],
\end{aligned}$$

for any $m \in N$. As $\lim_{m\to\infty} \eta(J^m) = 1$, we conclude that

$$\lim_{m\to\infty} d(x_i, J^m(y)) = d(x_1, x_2),$$

for $i = 1, 2$.
Similarly, we can show that

$$\lim_{m\to\infty} d(x_i, \frac{1}{2}y \oplus \frac{1}{2}J^m(y)) = d(x_1, x_2),$$

for $i = 1, 2$.
Using Lemma 1, we get

$$d(x_1, \frac{1}{2}y \oplus \frac{1}{2}J^m(y))^p + \frac{1}{2^{p+1}}d(y, J^m(y))^p \leq \frac{1}{2}d(x_1, y)^p + \frac{1}{2}d(x_1, J^m(y))^p,$$

then, we have $d(y, J^m y) = 0$, by continuity of J, $y \in Fix(J)$.

Corollary 1 : Suppose that (X, d) is a complete $CAT_p(0)$ metric space, with $p \geqslant 2$. Let C be CBCS of X and $J : C \to C$ such that $x, y \in Dom(J)$ the following inequality holds:

$$d(J^m x, J^m y) \leqslant \eta(J^m)\Big[\frac{1}{3}d(x,y) + \frac{1}{3}d(J^m x, x) + \frac{1}{3}d(J^m y, y)\Big],$$

such that $\eta(J^m) \geqslant 1, \lim_{m\to\infty} \eta(J^m) = 1$.
Then, J has a fixed point. Furthermore, $Fix(J)$ is closed and convex.

Corollary 2 : Suppose that (X, d) is a complete $CAT_p(0)$ metric space, with $p \geqslant 2$. Let C be a CBCS of X and $J : C \to C$ such that for $x, y \in Dom(J)$ the following inequality holds:

$$d(J^m x, J^m y) \leqslant \eta(J^m)\Big[\frac{1}{3}d(x, y) + \frac{1}{3}d(J^m x, y) + \frac{1}{3}d(J^m y, x)\Big],$$

such that $\eta(J^m) \geqslant 1, \lim_{m\to\infty} \eta(J^m) = 1$.
Then, J has a fixed point. Furthermore, $Fix(J)$ is closed and convex.

Example 4 : Define a $CAT(0) = R^2$ and the mapping $J : [a, b] \times [c, d] \to [a, b] \times [c, d]$ as follows:

$$J(x, y) = (\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{1}{4}),$$

where $a, b, c, d \in R$.
Let $(a_n = \frac{1}{2})$ and choose $(\varphi(d(x, y)) = \frac{1}{2}d(x, y))$ for all $(x_1, y_1), (x_2, y_2) \in R^2$:

$$\| J(x_1, y_1) - J(x_2, y_2) \|= \sqrt{\frac{1}{2}(x_1 - x_2) + \frac{1}{2}(y_1 - y_2)} \leq \frac{1}{2} \| (x_1, y_1) - (x_2, y_2) \| +\frac{1}{4}.$$

This confirms that J is φ-nearly Lipschitzian.
Finding a fixed point involves solving:

$$(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{1}{4}) = (x, y)$$

$$\frac{1}{2}x + \frac{1}{4} = x \Rightarrow x = \frac{1}{2},$$
$$\frac{1}{2}y + \frac{1}{4} = y \Rightarrow y = \frac{1}{2}.$$

Thus $(\frac{1}{2}, \frac{1}{2})$ is fixed point of J.

Example 5 : We consider $CAT(0) = R^2$ and the mapping $J : [a, b] \times [c, d] \to [a, b] \times [c, d]$ as follows:

$$J(x, y) = (\frac{2}{3}x + \frac{1}{3}, \frac{2}{3}y + \frac{1}{3}),$$

where $a, b, c, d \in R$.
Then J is φ-nearly Lipschitzian mapping.
Let $(a_n = 1)$ and choose $(\varphi(d(x, y)) = \frac{1}{3}d(x, y))$ for all $(x_1, y_1), (x_2, y_2) \in R^2$:

$$\| J(x_1, y_1) - J(x_2, y_2) \|= \sqrt{\frac{2}{3}(x_1 - x_2) + \frac{2}{3}(y_1 - y_2)} \leq \frac{2}{3} \| (x_1, y_1) - (x_2, y_2) \| +\frac{1}{3}.$$

J is a φ-nearly Lipschitzian mapping.
Finding a fixed point involves solving:

$$(\frac{2}{3}x + \frac{1}{3}, \frac{2}{3}y + \frac{1}{3}) = (x, y)$$

$$\frac{2}{3}x + \frac{1}{3} = x \Rightarrow x = \frac{1}{1} = 1,$$
$$\frac{2}{3}y + \frac{1}{3} = y \Rightarrow y = \frac{1}{1} = 1.$$

Thus $(1,1)$ is fixed point.

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