TWMS J. App. and Eng. Math. V.15, N.7, 2025, pp. 1616-1630

GEOMETRIC PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. This paper deals with some geometrical properties of certain subclasses of analytic functions having predetermined second coefficient. Firstly, we determine the growth and distortion estimates for such classes. Moreover, we investigate starlikeness of order α, strongly starlikeness, parabolic starlikeness, lemniscate starlikeness, exponential starlikeness, sine starlikeness and rational starlikeness for such classes.

Keywords: Analytic function; starlike function; subordination, fixed second coefficient; starlikeness.

AMS Subject Classification: 30C45, 30C80

1. Introduction

We represent by $\mathcal{A}$ the collection of analytic functions defined on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0)-1 = 0$ and denote by $\mathcal{A}_m$ the collection of functions $f \in \mathcal{A}$ of the type $f(z) = z + mz^2 + a_3z^3 \cdots, (z \in \mathbb{D})$ where m is a fixed number fulfilling the condition $0 \leq m \leq 1$. Let $\mathcal{S}$ be the subfamily of $\mathcal{A}$ consisting of univalent functions. The collection of analytic functions with real part positive is denoted by $\mathcal{P}$. The collection of starlike and convex functions are represented by $\mathcal{S}^*$ and $\mathcal{K}$, respectively. Analytically, the class $\mathcal{S}^*$ is defined as the class of functions $f \in \mathcal{S}$ such that $\tau(z) = zf'(z)/f(z) \in \mathcal{P}$ for every $z \in \mathbb{D}$. The class $\mathcal{K}$ consists of functions $f \in \mathcal{S}$ for which $1 + zf''/f' \in \mathcal{P}$ for all $z \in \mathbb{D}$. $\mathcal{S}^*(\varphi)$ represents the unified class of starlike functions for an analytic function φ such that $\mathrm{Re}(\varphi(z)) > 0$. Analytically, $f \in \mathcal{S}^*(\varphi)$ if $\tau(z) \prec \varphi(z)$, $z \in \mathbb{D}$ where $\prec$ shows subordination. For this class, Ma and Minda [12] investigated the covering, growth, and distortion theorem. The class $\mathcal{S}^*(\varphi)$ contains various well-known subclasses of starlike functions. The collection of starlike functions of order α is denoted by $S^*(\alpha)$ where $0 \leq \alpha < 1$ [18]. The strongly starlike functions of order γ are given by the class

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§ Manuscript received: June 3, 2024; accepted: March 18, 2025.

$S^*_\gamma := S^*((1+z)/(1-z)^\gamma)$ where $0 < \gamma \leq 1$. These functions satisfy $|\arg\tau(z)| < (\alpha\pi/2)$ [4]. The class $S_p := S^*(1+(2/\pi^2)(\log((1-\sqrt{z})/(1+\sqrt{z})))^2$ has parabolic starlike functions linked with the region $\{w \in \mathbb{C} : |w-1| < \mathrm{Re}(w)\}$ [19]. The class of lemniscate starlike functions is given by $S^*((1+z)^{1/2}) := S^*_L$ [22]. The class $S^*(e^z) := S^*_e$ is connected with the region $\{w \in \mathbb{C} : |\log w| < 1\}$ [16]. The class $S^*(1+\sin z) := S^*_{sin}$ is related to the sine function [5]. $S^*(\varphi_k = 1 + (z(k+z))/(k(k-z))) := S^*_R$, where $k = 1+2^{1/2}$ is the class related to the rational function φ_k [10].

The bound on the initial coefficient a_2 is important for studying the geometrical and analytical properties of normalized analytic functions. In 1916, Bieberbach [3] gave the bound on the initial coefficient of univalent functions. It was proven that $|a_2| \leq 2$ for each $f \in \mathcal{S}$ and it was conjectured that $|a_n| \leq n \in \mathbb{N} \setminus \{1\}$. This bound has made a major contribution to proving the distortion, growth, and covering theorems and obtaining the radius of convexity for the class $\mathcal{S}$. In [9], the author determined the sharp bounds for growth and distortion for the classes $\mathcal{S}$ and $\mathcal{K}$. In 1967, Finkelstein [6] obtained the growth estimates for the class $\mathcal{S}^*$. Later, Tepper [23] generalized this result for the class $\mathcal{S}^*(\alpha)$. Recently, authors [11] discussed the coefficient problems for starlike functions having fixed second coefficient. For $0 \leq \alpha < 1$, $\mathcal{P}(\alpha)$ represents the class of analytic functions $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ satisfying $\mathrm{Re}(p(z)) > \alpha$. It can be seen that $\mathcal{P}(0) = \mathcal{P}$. Since $p \in \mathcal{P}(\alpha)$ [17, p.170], thus $|b_1| \leq 2(1-\alpha)$. The subclass of $\mathcal{P}(\alpha)$ consisting of functions of the form $p(z) = 1 + 2m(1-\alpha)z + \cdots, |m| \leq 1$ is denoted by $P_m(\alpha)$ and let $\mathcal{P}_m := \mathcal{P}_m(0)$. Let $\chi(z) = f(z)/g(z)$. Consider the functions f and g with Taylor series expansions as

$$f(z) = z + a_2 z^2 + \cdots \quad \text{and} \quad g(z) = z + g_2 z^2 + \cdots \tag{1}$$

such that $|\chi(z) - 1| < 1$ and $\mathrm{Re}((1+z)g(z)/z) > 0$. It is noted that $|\chi(z)-1| < 1$ if and only if $\mathrm{Re}(1/\chi(z)) > 1/2$ thus $|a_2| \leq 1 + |g_2| \leq 4$. We take into consideration the collection of functions f and g with fixed second coefficient such that $f(z) = z + 4mz^2 + \cdots$ and $g(z) = z + 3nz^2 + \cdots$, where $|m| \leq 1$ and $|n| \leq 1$. Next, we are considering following two subclasses of analytic functions:

Definition 1.1. *The class $\mathcal{H}^1_{m,n}$ is defined as follows for $|m| \leq 1$ and $|n| \leq 1$,*

$$\mathcal{H}^1_{m,n} := \left\{ f \in \mathcal{A}_{4m} : |\chi(z) - 1| < 1 \text{ and } \mathrm{Re}\left(\frac{1+z}{z} g(z)\right) > 0, \;\; g \in \mathcal{A}_{3n}, z \in \mathbb{D} \right\}$$

where $\chi(z) = f(z)/g(z)$.

Consider the functions f_1 and g_1 defined as

$$f_1(z) = \frac{z(1+(1+3n)z+z^2)}{(1+z)(1+(3n-4m)z)} \qquad \text{and} \qquad g_1(z) = \frac{z(1+(1+3n)z+z^2)}{(1-z^2)(1+z)}. \tag{2}$$

For the functions f_1 and g_1 given by (2) and $|3n-4m| < 2$, we have

$$\left|\frac{f_1(z)}{g_1(z)} - 1\right| = \left|\frac{z(z+(4m-3n))}{-1+(4m-3n)z}\right| < 1 \quad \text{and} \quad \frac{1+z}{z} g_1(z) = \frac{1-w_1(z)}{1+w_1(z)}$$

where $w_1(z) = \dfrac{z(z+(1+3n)/2)}{(1+(1+3n)z/2)}$ with $|1+3n| \leq 2$ is an analytic function fulfilling the criteria of Schwarz lemma in $\mathbb{D}$ and $\mathrm{Re}((1+z)g_1(z)/z) > 0$ in $\mathbb{D}$. Thus the function $f_1 \in \mathcal{H}^1_{m,n}$ and therefore the class $\mathcal{H}^1_{m,n}$ is non-empty.

Next, we consider function f given by (1) fulfilling the condition $\mathrm{Re}((1+z)f(z)/z) > 0$ for all $z \in \mathbb{D}$. Then $(1+z)f(z)/z = 1 + (1+a_2)z + \cdots \in \mathcal{P}$ and thus we have $-3 \leq a_2 \leq 1$.

We take into consideration the collection of functions f given as $f(z) = z + 3mz^2 + \cdots$, where $|m| \leq 1$.

Definition 1.2. *The class $\mathcal{H}^2_m$ is defined as follows for $|m| \leq 1$,*

$$\mathcal{H}^2_m := \left\{ f \in \mathcal{A}_{3m} : \operatorname{Re}\left(\frac{1+z}{z} f(z)\right) > 0, z \in \mathbb{D} \right\}.$$

The function

$$f_2(z) = \frac{z(1 + (1+3m)z + z^2)}{(1+z)(1-z^2)} \tag{3}$$

so that $\dfrac{1+z}{z} f_2(z) = \dfrac{1+w_2(z)}{1-w_2(z)}$ where $w_2(z) = \dfrac{z(z + (1+3m)/2)}{(1 + (1+3m)z/2)}$ with $|1+3m| \leq 2$. It is noted that $w_2 \in \mathcal{A}$ fulfils the criteria of the Schwarz lemma in $\mathbb{D}$. Thus, we have $\operatorname{Re}((1+z)f_2(z)/z) > 0$ in $\mathbb{D}$, which implies that $f_2 \in \mathcal{H}^2_m$. Hence, the class $\mathcal{H}^2_m$ is non-empty, and f_2 is an extremal function.

Consider a set of functions $\mathcal{F}$ and a property $\mathcal{G}$. The radius of the property $\mathcal{G}$ for the set $\mathcal{F}$ is defined as $R_{\mathcal{G}}(\mathcal{F}) = \sup\{r > 0 : f \text{ has the property } \mathcal{G} \text{ in the disk } \mathbb{D}_r, \forall f \in \mathcal{F}\}$. In [13], the author determined the radius of starlikeness of the class of normalized analytic functions with $\operatorname{Re}(f(z)/z) > 0$ for all $z \in \mathbb{D}$. Gangadharan *et al.* [7] obtained the radius of starlikeness of order β, $0 \leq \beta < 1$ for the class of normalized analytic functions f satisfying $\tau(z) \prec \psi(z)$, where ψ is a univalent function with real part positive in $\mathbb{D}$ and $\psi(0) = 1$ and $\psi'(0) > 0$. The function ψ is symmetric with respect to the real axis and it maps $\mathbb{D}$ onto a region starlike with respect to $\psi(0) = 1$. For the functions $f \in \mathcal{A}$ that satisfy one of the following conditions: (i) $|\chi(z) - 1| < 1$ where $(1 + (1/z))g(z)$ has positive real part where $g \in \mathcal{A}$, or (ii) $((1+z)/z)f(z)$ has positive real part Sebastian *et al.* [20] determined the radius constants. In 1920, Gronwall [8] initiated the study of univalent functions having fixed second coefficient. The author obtained the growth and distortion theorems for the subclasses of $\mathcal{S}$ and $\mathcal{K}$ with fixed second coefficient. For $p \in \mathcal{P}_m$, Tepper [23] obtained lower bound for $\operatorname{Re} p(z)$ and upper bound for $|p(z)|$ and $|zp'(z)/p(z)|$. The bounds obtained were sharp. Furthermore, the author obtained the sharp radius of convexity for the class of starlike functions with preassigned second coefficient. McCarty [15] examined the radius of convexity for the class of starlike functions of order α, $0 \leq \alpha < 1$ with predetermined second coefficient. Let $0 \leq m \leq 1$, set $\mathcal{S}^*_m(\psi) = \{f \in \mathcal{S}^*(\psi) : f(z) = z + a_2 z^2 + \cdots, |a_2| = \psi'(0)m\}$. For $0 \leq \alpha < 1$, Ali *et al.* [1] computed the radius of starlikeness of order α, for the class $\mathcal{S}^*_m(\psi)$.

Inspired by the works mentioned above, we investigate the growth and distortion estimates for the subclasses $\mathcal{H}^1_{m,n}$ and $\mathcal{H}^2_m$. Furthermore, we examine the radius constants for the functions in the class $\mathcal{H}^1_{m,n}$ to be in the subclasses $S^*(\alpha)$, S^*_γ, S_p, S^*_L, S^*_e, $S^*_{\sin}$, S^*_R of starlike functions. Moreover, we also obtain the radius constants for the functions in the classes $\mathcal{H}^2_m$ to be in the classes $S^*(\alpha)$, S_p, S^*_L, S^*_e.

2. Growth and Distortion Estimates

This section gives the growth and distortion estimates for the classes $\mathcal{H}^1_{m,n}$ and $\mathcal{H}^2_m$. The following results by McCarty are required to establish our main results:

Lemma 2.1. *[14, Lemma 2, p. 212] Let $|m| \leq 1$. If $p \in \mathcal{P}_m(0)$. Then, for $|z| = \rho < 1$, we have*

$$|p(z)| \leq \frac{1 + m|z| + |z|^2}{1 - |z|^2}.$$

Lemma 2.2. *[14, Corollary 1, p. 213] If $p \in \mathcal{P}_m(\alpha)$, then for $|z| = \rho < 1$, we have*

$$\operatorname{Re}(p(z)) \geq \frac{1 + m\alpha|z| - (1-2\alpha)|z|^2}{1 + m|z| + |z|^2}.$$

Lemma 2.3. *[14, Theorem 2, p. 213] Let $|m| \leq 1$ and $0 \leq \alpha < 1$. If $p \in \mathcal{P}_m(\alpha)$, then for $|z| = \rho < 1$, we have*

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)r}{1-\rho^2}\frac{|m|\rho^2 + 2\rho + |m|}{(1-2\alpha)\rho^2 + 2(1-\alpha)|m|\rho + 1}.$$

Theorem 2.1. *Let $r = |1+3n| \leq 2$ and $s = |3n - 4m| \leq 2$ and let $|z| = \rho < 1$ and $\tau(z) = zf'(z)/f(z)$. For the functions $f \in \mathcal{H}^1_{m,n}$, the following inequalities hold:*

(i)

$$\operatorname{Re}(\tau(z)) \geq \frac{-s\rho^5 - (2+s+2sr)\rho^4 - (1+3r+5s+sr)\rho^3 - (5+r+s+sr)\rho^2 - \rho + 1}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}.$$

(ii)

$$|\tau(z)| \leq \frac{s\rho^5 + (2+s+2sr)\rho^4 + (1+3r+7s+sr)\rho^3 + (7+r+s+3sr)\rho^2 + 2(1+r+s)\rho + 1}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}$$

and

$$|\tau(z)| \geq \frac{-s\rho^5 - (2+s+2sr)\rho^4 - (1+3r+5s+sr)\rho^3 - (5+r+s+sr)\rho^2 - \rho + 1}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}.$$

(iii)

$$|f(z)| \leq \frac{2\rho(1+r\rho+\rho^2)(1+s\rho+\rho^2)}{(1-\rho)^2(1+\rho)(2+s\rho)} \text{ and } |f(z)| \geq \rho\left(\frac{e^\rho}{1+\rho}\exp(M(\rho))\right)$$

where $M(\rho) = -\int\limits_0^\rho \frac{u}{1-u^2}\left(\frac{ru^2+4u+r}{u^2+ru+1} + \frac{su^2+2u+s}{su+1}\right)du.$

(iv)

$$|f'(z)| \leq \frac{2\begin{pmatrix} s\rho^7 + (2+s+s^2+2sr)\rho^6 + (1+3r+10s+s^2+sr+2s^2r)\rho^5 \\ +(9+r+3s+s^2+8sr+s^2r)\rho^4 + (2+5r+16s+s^2 \\ +2sr+3s^2r)\rho^3 + (8+r+2s+2s^2+5sr)\rho^2 + (2r+3s+1)\rho + 1 \end{pmatrix}}{(1-\rho)^3(1+\rho)^2(s^2\rho^2+3s\rho+2)} \tag{4}$$

and

$$|f'(z)| \geq \rho\left(\frac{e^\rho}{1+\rho}\exp(M(\rho))N(\rho)\right)$$

where

$$M(\rho) = -\int\limits_0^\rho \frac{u}{1-u^2}\left(\frac{ru^2+4u+r}{u^2+ru+1} + \frac{su^2+2u+s}{su+1}\right)du$$

and

$$N(\rho) = \frac{-s\rho^5 - (2+s+2sr)\rho^4 - (1+3r+5s+sr)\rho^3 - (5+r+s+sr)\rho^2 - \rho + 1}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}.$$

Proof. Consider the functions $f \in \mathcal{H}^1_{m,n}$ and $g \in \mathcal{A}_{3n}$ such that

$$|\chi(z) - 1| < 1 \quad \text{and} \quad \operatorname{Re}\left(\frac{1+z}{z} g(z)\right) > 0 \ (z \in \mathbb{D}) \tag{5}$$

where $\chi(z) = f(z)/g(z)$. Let $p, h : \mathbb{D} \to \mathbb{C}$ be functions defined as

$$p(z) = \left(\frac{1+z}{z}\right) g(z) = 1 + (1+3n)z + \cdots \tag{6}$$

and

$$h(z) = \frac{1}{\chi(z)} = 1 + (3n - 4m)z + \cdots . \tag{7}$$

In view of (5), (6) and (7), it can be seen that $p \in \mathcal{P}_{(1+3n)/2}$ and $h \in \mathcal{P}_{(3n-4m)}(1/2)$. Also, we have

$$f(z) = \frac{g(z)}{h(z)} = \frac{zp(z)}{(1+z)h(z)}. \tag{8}$$

Upon logarithmic differentiation, we have

$$\tau(z) = \frac{zp'(z)}{p(z)} - \frac{zh'(z)}{h(z)} + \frac{1}{1+z}. \tag{9}$$

Considering $\alpha = 0$ and $\alpha = 1/2$ in Lemma 2.3, we have

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{\rho}{1-\rho^2}\frac{r\rho^2+4\rho+r}{\rho^2+r\rho+1} \text{ and } \left|\frac{zh'(z)}{h(z)}\right| \leq \frac{\rho}{1-\rho^2}\frac{s\rho^2+2\rho+s}{s\rho+1}. \tag{10}$$

Since the bilinear mapping $\dfrac{1}{1+z}$ maps $|z| \leq \rho$ onto

$$\left|\frac{1}{1+z} - \frac{1}{1-\rho^2}\right| \leq \frac{\rho}{1-\rho^2} \tag{11}$$

then using (10) and (11) in (9), we obtain

$$\left|\tau(z) - \frac{1}{1-\rho^2}\right| \leq \frac{\rho}{1-\rho^2}\left(\frac{r\rho^2+4\rho+r}{\rho^2+r\rho+1} + \frac{s\rho^2+2\rho+s}{s\rho+1} + 1\right). \tag{12}$$

(i) In view of (12), the minimum of $\operatorname{Re}(\psi(z))$ is given as

$$\operatorname{Re}(\tau(z)) \geq \frac{1}{1-\rho^2} - \frac{\rho}{1-\rho^2}\left(\frac{r\rho^2+4\rho+r}{\rho^2+r\rho+1} + \frac{s\rho^2+2\rho+s}{s\rho+1} + 1\right) \tag{13}$$

$$= \frac{\begin{array}{r}-s\rho^5 - (2+s+2sr)\rho^4 - (1+3r+5s+sr)\rho^3 \\ - (5+r+s+sr)\rho^2 - \rho + 1\end{array}}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}. \tag{14}$$

(ii) Next, in view of (12), we have

$$\begin{aligned}|\tau(z)| &\leq \left|\tau(z) - \frac{1}{1-\rho^2}\right| + \frac{1}{1-\rho^2} \\ &\leq \frac{\rho}{1-\rho^2}\left(\frac{r\rho^2+4\rho+r}{\rho^2+r\rho+1} + \frac{s\rho^2+2\rho+s}{s\rho+1} + 1\right) + \frac{1}{1-\rho^2} \\ &= \frac{\begin{array}{r}s\rho^5 + (2+s+2sr)\rho^4 + (1+3r+7s+sr)\rho^3 + (7+r+s+3sr)\rho^2 \\ + 2(1+r+s)\rho + 1\end{array}}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}.\end{aligned} \tag{15}$$

Also, in view of (14), we have

$|\tau(z)| \geq \operatorname{Re}(\tau(z))$

$$\geq \frac{-s\rho^5-(2+s+2sr)\rho^4-(1+3r+5s+sr)\rho^3-(5+r+s+sr)\rho^2-\rho+1}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}.$$

(iii) From expression (8), Lemma 2.1 and Lemma 2.2, we get

$$|f(z)|=\frac{|z||p(z)|}{|1+z||h(z)|}\leq\frac{|z||p(z)|}{|1+z|\operatorname{Re}(h(z))}\leq\frac{2\rho(1+r\rho+\rho^2)(1+s\rho+\rho^2)}{(2+s\rho)(1-\rho)^2(1+\rho)}. \tag{16}$$

If $z=\rho e^{i\theta}$, then $\frac{\partial}{\partial\rho}\log\left|\frac{f(z)}{z}\right|=\operatorname{Re}(\tau(z))-1$ and using (13), we have

$$\begin{aligned}\frac{\partial}{\partial\rho}\log\left|\frac{f(z)}{z}\right| &\geq \frac{1}{1-\rho^2}-\frac{\rho}{1-\rho^2}\left(\frac{r\rho^2+4\rho+r}{\rho^2+r\rho+1}+\frac{s\rho^2+2\rho+s}{s\rho+1}+1\right)-1\\ &=\frac{-\rho}{1+\rho}-\frac{\rho}{1-\rho^2}\left(\frac{r\rho^2+4\rho+r}{\rho^2+r\rho+1}+\frac{s\rho^2+2\rho+s}{s\rho+1}\right)\\ &=1-\frac{1}{1+\rho}-\frac{\rho}{1-\rho^2}\left(\frac{r\rho^2+4\rho+r}{\rho^2+r\rho+1}+\frac{s\rho^2+2\rho+s}{s\rho+1}\right).\end{aligned}$$

On integrating from $u=0$ to $u=|z|=\rho<1$, we get

$$\log\left|\frac{f(z)}{z}\right|\geq\rho-\log(1+\rho)-\int_0^\rho\frac{u}{1-u^2}\left(\frac{ru^2+4u+r}{u^2+ru+1}+\frac{su^2+2u+s}{su+1}\right)du$$

so that

$$|f(z)|\geq\rho\left(\frac{e^\rho}{1+\rho}\exp(M(\rho))\right) \tag{17}$$

where $M(\rho)=-\int_0^\rho\frac{u}{1-u^2}\left(\frac{ru^2+4u+r}{u^2+ru+1}+\frac{su^2+2u+s}{su+1}\right)du.$

(iv) Next, we consider

$$|f'(z)|=|\tau(z)|\left|\frac{f(z)}{z}\right| \tag{18}$$

Using (15) and (16), we get the desired inequality (4). Again, using (14), (17) in (18), we get the following

$$|f'(z)|\geq\operatorname{Re}(\tau(z))\left|\frac{f(z)}{z}\right|\geq\frac{e^\rho}{1+\rho}\exp(M(\rho))N(\rho)$$

where

$$M(\rho)=-\int_0^\rho\frac{u}{1-u^2}\left(\frac{ru^2+4u+r}{u^2+ru+1}+\frac{su^2+2u+s}{su+1}\right)du$$

and

$$N(\rho)=\frac{-s\rho^5-(2+s+2sr)\rho^4-(1+3r+5s+sr)\rho^3-(5+r+s+sr)\rho^2-\rho+1}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}.$$

∎

Next theorem gives the growth and distortion estimates for the class $\mathcal{H}_m^2$.

Theorem 2.2. *Let $q=|1+3m|\leq 2$ and let $|z|=\rho<1$ and $\tau(z)=zf'(z)/f(z)$. For the class $\mathcal{H}_m^2$, the following inequalities hold:*

(i)

$$\operatorname{Re}(\tau(z)) \geq \frac{-(1+q)\rho^3-(3+q)\rho^2-\rho+1}{(1-\rho^2)(\rho^2+q\rho+1)}.$$

(ii)

$$|\tau(z)| \leq \frac{-\rho^5-q\rho^4+q\rho^3+(5+q)\rho^2+(2q+1)\rho+1}{(1-\rho^2)(\rho^2+q\rho+1)}$$

and

$$|\tau(z)| \geq \frac{-(1+q)\rho^3-(3+q)\rho^2-\rho+1}{(1-\rho^2)(\rho^2+q\rho+1)}.$$

(iii)

$$|f(z)| \leq \frac{\rho(1+q\rho+\rho^2)}{(1-\rho)^2(1+\rho)}.$$

Also,

$$|f(z)| \geq \rho\left(\frac{e^\rho}{1+\rho}\exp(M(\rho))\right)$$

where

$$M(\rho) = -\int_0^\rho \frac{u}{1-u^2}\left(\frac{qu^2+4u+q}{u^2+qu+1}\right)du.$$

(iv)

$$|f'(z)| \leq \frac{-\rho^5-q\rho^4+q\rho^3+(5+q)\rho^2+(2q+1)\rho+1}{(1-\rho)^2(1+\rho)^3}$$

and

$$|f'(z)| \geq \rho\left(\frac{e^\rho}{1+\rho}\exp(M(\rho))N(\rho)\right)$$

where $M(\rho) = -\int_0^\rho \frac{u}{1-u^2}\left(\frac{qu^2+4u+q}{u^2+qu+1}\right)du$ *and* $N(\rho) = \frac{-(1+q)\rho^3-(3+q)\rho^2-\rho+1}{(1-\rho^2)(\rho^2+q\rho+1)}$.

Proof. Let $f \in \mathcal{H}_m^2$ such that

$$\operatorname{Re}\left(\frac{1+z}{z}f(z)\right) > 0 \ (z \in \mathbb{D}). \tag{19}$$

We take into consideration the function h on $\mathbb{D}$ defined as

$$h(z) = \left(\frac{1+z}{z}\right)f(z) = 1+(1+3m)z+\cdots. \tag{20}$$

In view of (19) and (20), it can be seen that $h \in \mathcal{P}_{(1+3m)/2}$. Also,

$$f(z) = \frac{zh(z)}{1+z}. \tag{21}$$

It is easy to see that

$$\tau(z) = \frac{zh'(z)}{h(z)} + \frac{1}{1+z}. \tag{22}$$

For $\alpha = 0$ in Lemma 2.3, we get

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{\rho}{1-\rho^2}\left(\frac{q\rho^2+4\rho+q}{\rho^2+q\rho+1}\right). \tag{23}$$

Since the bilinear transformation $\dfrac{1}{1+z}$ maps $|z| \leq \rho$ onto

$$\left|\frac{1}{1+z} - \frac{1}{1-\rho^2}\right| \leq \frac{\rho}{1-\rho^2}. \tag{24}$$

Then using (23) and (24) in (22), we obtain

$$\left|\tau(z) - \frac{1}{1-\rho^2}\right| \leq \frac{\rho}{1-\rho^2}\left(\frac{q\rho^2+4\rho+q}{\rho^2+q\rho+1}+1\right). \tag{25}$$

(i) In view of (25) the minimum value of $\mathrm{Re}(\tau(z))$ is given as

$$\mathrm{Re}\,(\tau(z)) \geq \frac{1}{1-\rho^2} - \frac{\rho}{1-\rho^2}\left(\frac{q\rho^2+4\rho+q}{\rho^2+q\rho+1}+1\right) \tag{26}$$

$$= \frac{-(1+q)\rho^3-(3+q)\rho^2-\rho+1}{(1-\rho^2)(\rho^2+q\rho+1)}. \tag{27}$$

(ii) Next, in view of the inequality (26), we have

$$\begin{aligned} |\tau(z)| &\leq \left|\tau(z) - \frac{1}{1-\rho^2}\right| + \frac{1}{1-\rho^2} \\ &\leq \frac{\rho}{1-\rho^2}\left(\frac{q\rho^2+4\rho+q}{\rho^2+q\rho+1}+1\right) + \frac{1}{1-\rho^2} \\ &= \frac{-\rho^5-q\rho^4+q\rho^3+(5+q)\rho^2+(2q+1)\rho+1}{(1-\rho^2)(\rho^2+q\rho+1)}. \end{aligned} \tag{28}$$

Also, in view of (27), we have

$$|\tau(z)| \geq \mathrm{Re}\,(\tau(z)) \geq \frac{-(1+q)\rho^3-(3+q)\rho^2-\rho+1}{(1-\rho^2)(\rho^2+q\rho+1)}.$$

(iii) It follows from the expression (21) and Lemma 2.1 that

$$|f(z)| = \frac{|z||h(z)|}{|1+z|} \leq \frac{\rho(1+q\rho+\rho^2)}{(1-\rho^2)(1-\rho)} = \frac{\rho(1+q\rho+\rho^2)}{(1-\rho)^2(1+\rho)} \tag{29}$$

For the function f_2 given by (3), we have

$$f_2(z) = \frac{z(1+(1+3m)z+z^2)}{(1+z)(1-z^2)}.$$

Let $b' = -(1+3b) > 0$ and $z = -\rho'$. Then

$$|f_2(z)| = \frac{\rho'(1+b'\rho'+\rho'^2)}{(1-\rho')^2(1+\rho')}$$

which proves the sharpness.

If $z = \rho e^{i\theta}$, then $\frac{\partial}{\partial\rho}\log\left|\frac{f(z)}{z}\right| = \mathrm{Re}\,(\tau(z)) - 1$ and using (26), we get

$$\begin{aligned} \frac{\partial}{\partial\rho}\log\left|\frac{f(z)}{z}\right| &\geq \frac{1}{1-\rho^2} - \frac{\rho}{1-\rho^2}\left(\frac{q\rho^2+4\rho+q}{\rho^2+q\rho+1}+1\right) - 1 \\ &= \frac{-\rho}{1+\rho} - \frac{\rho}{1-\rho^2}\left(\frac{q\rho^2+4\rho+q}{\rho^2+q\rho+1}\right) \\ &= 1 - \frac{1}{1+\rho} - \frac{\rho}{1-\rho^2}\left(\frac{q\rho^2+4\rho+q}{\rho^2+q\rho+1}\right). \end{aligned}$$

On integrating from $u = 0$ to $u = |z| = \rho < 1$, we get

$$\log\left|\frac{f(z)}{z}\right| \geq \rho - \log(1+\rho) - \int_0^{\rho} \frac{u}{1-u^2}\left(\frac{qu^2+4u+q}{u^2+qu+1}\right) du$$

so that

$$|f(z)| \geq \rho\left(\frac{e^{\rho}}{1+\rho}\exp(M(\rho))\right) \tag{30}$$

where $M(\rho) = -\int_0^{\rho} \frac{u}{1-u^2}\left(\frac{qu^2+4u+q}{u^2+qu+1}\right) du$.

(iv) Next, we have

$$|f'(z)| = |\tau(z)|\left|\frac{f(z)}{z}\right|. \tag{31}$$

Using (28) and (29) in (31), we have

$$|f'(z)| \leq \frac{-\rho^5 - q\rho^4 + q\rho^3 + (5+q)\rho^2 + (2q+1)\rho + 1}{(1-\rho)^2(1+\rho)^3}.$$

Again using (27), (30) in (31), we get

$$|f'(z)| \geq \mathrm{Re}\,(\tau(z))\left|\frac{f(z)}{z}\right| \geq \frac{e^{\rho}}{1+\rho}\exp(M(\rho))N(\rho)$$

where $M(\rho) = -\int_0^{\rho} \frac{u}{1-u^2}\left(\frac{qu^2+4u+q}{u^2+qu+1}\right) du$ and $N(\rho) = \frac{-(1+q)\rho^3-(3+q)\rho^2-\rho+1}{(1-\rho^2)(\rho^2+q\rho+1)}$.

∎

3. Various Starlikeness

This section gives various starlikeness for the functions belonging to the subclasses $\mathcal{H}^1_{m,n}$ and $\mathcal{H}^2_m$, respectively.

Theorem 3.1. *Let $r = |1+3n| \leq 2$ and $s = |3n-4m| \leq 2$ and let $\tau(z) = zf'(z)/f(z)$. For the class $\mathcal{H}^1_{m,n}$, the following radius results hold:*

(i) Let $\xi_1(\rho) = (1-\alpha)s\rho^5 + ((2-\alpha)(1+sr)+s)\rho^4 + (1+(3-\alpha)r+5s+sr)\rho^3 + (5+r+s+(1+\alpha)sr)\rho^2 + (1+\alpha(r+s))\rho + \alpha - 1$. Then $R_{\mathcal{S}^*(\alpha)}(\mathcal{H}^1_{m,n})$ is the root $\rho_1 \in (0,1)$ of the equation $\xi_1(\rho) = 0$.

(ii) Let $\xi_2(\rho) = s\rho^5 + (2+s+2sr)\rho^4 + (1+3r+(6-\sin(\pi\gamma/2))s+sr)\rho^3 + (6-\sin(\pi\gamma/2)+r+s+(2-\sin(\pi\gamma/2))st)\rho^2 + (1+(1-\sin(\pi\gamma/2))(r+s))\rho - \sin(\pi\gamma/2)$. Then $R_{\mathcal{S}^*_{\gamma}}(\mathcal{H}^1_{m,n})$ is the root $\rho_2 \in (0,1)$ of the equation $\xi_2(\rho) = 0$.

(iii) Let $\xi_3(\rho) = s\rho^5 + (3+2s+3sr)\rho^4 + (2+5r+10s+2sr)\rho^3 + (10+2(r+s)+3sr)\rho^2 + (2+r+s)\rho - 1$. The $R_{\mathcal{S}_p}(\mathcal{H}^1_{m,n})$ is the root $\rho_3 \in (0, 1/\sqrt{3})$ of the equation $\xi_3(\rho) = 0$.

(iv) Let $\xi_4(\rho) = (1+\sqrt{2})\rho^5 + (\sqrt{2}(1+\sqrt{2})(1+sr)+s)\rho^4 + (1+(3+\sqrt{2})r+7s+sr))\rho^3 + (7+r+s+(3-\sqrt{2})sr)\rho^2 + (1+\sqrt{2}(\sqrt{2}-1)(r+s))\rho + 1 - \sqrt{2}$. Then $R_{\mathcal{S}^*_L}(\mathcal{H}^1_{m,n})$ is the root $\rho_4 \in (0,1)$ of the equation $\xi_4(\rho) = 0$.

(v) Let $\xi_5(\rho) = (e-1)s\rho^5 + (2e-1+es+(2e-1)sr)\rho^4 + (e+(3e-1)r+5es+esr)\rho^3 + (5e+e(r+s)+(e+1)sr)\rho^2 + (e+r+s)\rho + 1 - e$. Then $R_{\mathcal{S}^*_e}(\mathcal{H}^1_{m,n})$ is the root $\rho_5 \in (0, ((1-2e+e^2)/(1+e^2))^{1/2})$ of the equation $\xi_5(\rho) = 0$.

(vi) *Let* $\xi_6(\rho) = (2+\sin 1)s\rho^5 + ((3+\sin 1)(1+sr)+s)\rho^4 + (1+(4+\sin 1)r+7s+sr)\rho^3 + (7+r+s+(2-\sin 1)sr)\rho^2 + (1+(1-\sin 1)(r+s))\rho - \sin 1$. *Then* $R_{S^*_{\sin}}(\mathcal{H}^1_{m,n})$ *is the root* $\rho_6 \in (0,1)$ *of the equation* $\xi_6(\rho) = 0$.

(vii) *Let* $\xi_7(\rho) = (3-2\sqrt{2})s\rho^5 + ((4-2\sqrt{2})(1+sr)+s)\rho^4 + (1+(5-2\sqrt{2})r+5s+sr)\rho^3 + (5+r+s+(2\sqrt{2}-1)sr)\rho^2 + (1+(2\sqrt{2}-2)(r+s))\rho + 2\sqrt{2}-3$. *Then* $R_{S^*_R}(\mathcal{H}^1_{m,n})$ *is the root* $\rho_7 \in (0, ((2-\sqrt{2})/2)^{1/2})$ *of the equation* $\xi_7(\rho) = 0$.

Proof. Let $f \in \mathcal{H}^1_{m,n}$ and $g \in \mathcal{A}_{3n}$ be functions such that

$$|\chi(z) - 1| < 1 \quad \text{and} \quad \operatorname{Re}\left(\frac{1+z}{z} g(z)\right) > 0 \ (z \in \mathbb{D})$$

where $\chi(z) = f(z)/g(z)$.

(i) Considering (14), we have

$$\begin{aligned}
&\operatorname{Re}(\tau(z)) \\
&\geq \frac{1 - s\rho^5 - (2+s+2sr)\rho^4 - (1+3r+5s+sr)\rho^3 - (5+r+s+sr)\rho^2 - \rho}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)} \\
&\geq \alpha
\end{aligned}$$

whenever $\xi_1(\rho) \leq 0$. For $0 \leq \alpha < 1$, $\xi_1(0) = \alpha - 1 < 0$ and $\xi_1(1) = 8+4r+7s+4sr > 0$. It follows from the intermediate value theorem that the equation $\xi_1(\rho) = 0$ has a root ρ_1 in $(0,1)$.

(ii) By [7, Lemma 3.1, p. 307], the function $f \in \mathcal{H}^1_{m,n}$ belongs in the class S^*_γ if

$$\frac{\begin{aligned}s\rho^5 + (2+s+2sr)\rho^4 + (1+3(r+2s)+sr)\rho^3 \\ + (6+r+s+2sr)\rho^2 + (1+r+s)\rho\end{aligned}}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)} \leq \frac{1}{1-\rho^2}\sin(\pi\gamma/2)$$

which gives $\xi_2(\rho) \leq 0$.

(iii) It is easy to see that $1/2 \leq 1/(1-\rho^2) \leq 3/2$ for $0 \leq \rho \leq 1/3$. In view of [21, Section 3, p. 321], the disk (12) lies in the region $\Omega_p = \{w : \operatorname{Re} w > |w-1|\}$, if

$$\frac{\begin{aligned}s\rho^5 + (2+s+2sr)\rho^4 + (1+3(r+2s)+sr)\rho^3 \\ + (6+r+s+2sr)\rho^2 + (1+r+s)\rho\end{aligned}}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)} \leq \frac{1}{1-\rho^2} - \frac{1}{2}$$

which on simplification gives $\xi_3(\rho) \leq 0$. It can be seen that $\xi_3(0) = -1 < 0$ and $\xi_3(1/\sqrt{3}) = 2/27(s(4(3+5\sqrt{3}) + 3(6+\sqrt{3})r) + 3(4(3+\sqrt{3}) + (3+4\sqrt{3})r)) > 0$. In view of the intermediate value theorem, we get that the desired S_p- radius lies in $(0, 1/\sqrt{3})$ for the class $\mathcal{H}^1_{m,n}$.

(iv) Using (12), we have

$$\begin{aligned}
|\tau(z) - 1| &\leq \left|\tau(z) - \frac{1}{1-\rho^2}\right| + \frac{\rho^2}{1-\rho^2} \\
&\leq \frac{\begin{aligned}2s\rho^5 + (3(1+sr)+s)\rho^4 + (1+4r+7s+sr)\rho^3 \\ + (7+r+s+2sr)\rho^2 + (1+r+s)\rho\end{aligned}}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}.
\end{aligned}$$

By [2, Lemma 2.2, p. 6559], the function $f \in \mathcal{H}^1_{m,n}$ belongs in the class S^*_L, whenever the following holds

$$\frac{\begin{aligned}2s\rho^5 + (3(1+sr)+s)\rho^4 + (1+4r+7s+sr)\rho^3 + (7+r+s+2sr)\rho^2 \\ + (1+r+s)\rho\end{aligned}}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)}$$

$$\leq \sqrt{2}-1$$

equivalently $\xi_4(\rho) \leq 0$. Note that $\xi_4(0) = 1-\sqrt{2} < 0$ and $\xi_4(1) = 12+6(sr+r+2s) > 0$. By intermediate value theorem the equation $\xi_4(\rho) = 0$ has a root $\rho_4 \in (0,1)$.

(v) It is easy to see that $1/e \leq 1/(1-\rho^2) \leq (e+e^{-1})/2$ for $\rho \leq ((1-2e+e^2)/(1+e^2))^{1/2}$. By [16, Lemma 2.2, p. 368], the function $f \in \mathcal{H}^1_{m,n}$ belongs to the class S^*_e if

$$\frac{s\rho^5+(2+s+2sr)\rho^4+(1+3(r+2s)+sr)\rho^3 + (6+r+s+2sr)\rho^2+(1+r+s)\rho}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)} \leq \frac{1}{1-\rho^2}-\frac{1}{e}$$

which implies that $\xi_5(\rho) \leq 0$. Note that $\xi_5(0) = -1.71828 < 0$ and $\xi_5(((1-2e+e^2)/(1+e^2))^{1/2}) = 5.79487+4.85068s+r(3.04381+2.42573s) > 0$. By the intermediate value theorem, we get that the desired S^*_e-radius lies in $\big(0,((1-2e+e^2)/(1+e^2))^{1/2}\big)$.

(vi) Using [5, Lemma 3.3, p.219], the function $f \in \mathcal{H}^1_{m,n}$ belongs to the class $S^*_{\sin}$ if

$$\frac{s\rho^5+(2+s+2sr)\rho^4+(1+3(r+2s)+sr)\rho^3 + (6+r+s+2sr)\rho^2+(1+r+s)\rho}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)} \leq \sin 1-\frac{\rho^2}{1-\rho^2}$$

equivalently $\xi_6(\rho) \leq 0$. Using the intermediate value theorem, we get the desired $S^*_{\sin}$- radius.

(vii) It is easy to see that $2(\sqrt{2}-1) \leq 1/(1-\rho^2) \leq \sqrt{2}$ for $\rho \leq ((2-\sqrt{2})/2)^{1/2}$. By [10, Lemma 2.2 p. 202], the function $f \in \mathcal{H}^1_{m,n}$ belongs to the class S^*_R if

$$\frac{s\rho^5+(2+s+2sr)\rho^4+(1+3(r+2s)+sr)\rho^3 + (6+r+s+2sr)\rho^2+(1+r+s)\rho}{(1-\rho^2)(\rho^2+r\rho+1)(s\rho+1)} \leq \frac{1}{1-\rho^2}+2-2\sqrt{2}$$

equivalently $\xi_7(\rho) \leq 0$. Note that $\xi_7(0) = -3+2\sqrt{2} < 0$ and $\xi_7(((2-\sqrt{2})/2)^{1/2}) = 1/2(4.18621+3.2551s+2.17091r+1.5891sr) > 0$. It follows from the intermediate value theorem that the equation $\xi_7(\rho) = 0$ has a root $\rho_7 \in (0,((2-\sqrt{2})/2)^{1/2})$.

∎

Remark 3.1. *For $m=-1$ and $n=-1$, parts (1)-(7) of Theorem 3.1 gives [20, Theorem 2.4] .*

Theorem 3.2. *Let $q = |1+3m| \leq 2$ and let $\tau(z) = zf'(z)/f(z)$. For the class $\mathcal{H}^2_m$, the following results hold:*

(i) Let $\xi_1(\rho) = -\alpha\rho^4+(1+(1-\alpha)q)\rho^3+(3+q)\rho^2+(1+\alpha q)\rho+\alpha-1$. Then $R_{S^*(\alpha)}(\mathcal{H}^2_m)$ is the root $\rho_1 \in (0,1)$ of the equation $\xi_1(\rho)=0$.

(ii) Let $\xi_2(\rho) = -\rho^4+(2+q)\rho^3+2(3+q)\rho^2+(2+q)\rho-1$. Then $R_{S_p}(\mathcal{H}^2_m)$ is the root $\rho_2 \in (0,1/\sqrt{3})$ of the equation $\xi_2(\rho)=0$.

(iii) Let $\xi_3(\rho) = \sqrt{2}\rho^4+(1+(1+\sqrt{2})q)\rho^3+(5+q)\rho^2+(1+\sqrt{2}(\sqrt{2}-1)q)\rho+1-\sqrt{2}$. Then $R_{S^*_L}(\mathcal{H}^2_m)$ is the root $\rho_3 \in (0,1)$ of the equation $\xi_3(\rho)=0$. The result is sharp.

(iv) Let $\xi_4(\rho) = -\rho^4+(e+(e-1)q)\rho^3+e(3+q)\rho^2+(e+q)\rho+1-e$. Then $R_{S^*_e}(\mathcal{H}^2_m)$ is the root $\rho_4 \in (0,((1-2e+e^2)/(1+e^2))^{1/2})$ of the equation $\xi_4(\rho)=0$.

Proof. Let the function $f \in \mathcal{H}^2_m$ be such that $\operatorname{Re}\left(\dfrac{1+z}{z}f(z)\right) > 0,\ (z \in \mathbb{D})$.

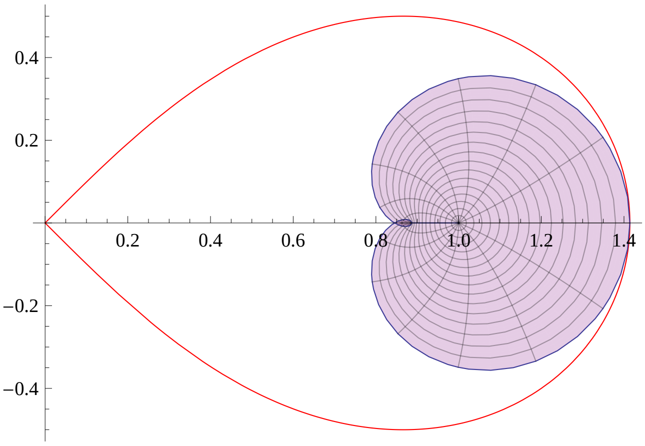

FIGURE 1. Sharpness for the class of function in Theorem 3.2 associated with the lemniscate for $m = -1/2, R_{S^*_L}(\mathcal{H}^2_m) = 0.1768$.

(i) In view of (27)

$$\operatorname{Re}(\tau(z)) \geq \frac{-(1+q)\rho^3 - (3+q)\rho^2 - \rho + 1}{(1-\rho^2)(\rho^2 + q\rho + 1)} \geq \alpha$$

whenever the following inequality holds

$$-\alpha\rho^4 + (1 + (1-\alpha)q)\rho^3 + (3+q)\rho^2 + (1+\alpha q)\rho + \alpha - 1 \leq 0$$

whenever $\xi_1(\rho) \leq 0$. For $0 \leq \alpha < 1$, $\xi_1(0) = \alpha - 1 < 0$ and $\xi_1(1) = 4 + 2q > 0$. It follows from the intermediate value theorem, the equation $\xi_1(\rho) = 0$ has a root ρ_1 in $(0, 1)$.

(ii) It is easy to verify that $1/2 \leq 1/(1-\rho^2) \leq 3/2$ for $0 \leq \rho \leq 1/3$. In view of [21, Section 3, p. 321], the disk (12) lies in the domain $\Omega_p = \{w : \operatorname{Re} w > |w-1|\}$ whenever the following inequality hold

$$\frac{(1+q)\rho^3 + (4+q)\rho^2 + (1+q)\rho}{(1-\rho^2)(\rho^2+q\rho+1)} \leq \frac{1}{1-\rho^2} - \frac{1}{2}$$

which on simplification gives $\xi_2(\rho) \leq 0$. It can be seen that $\xi_2(0) = -1 < 0$ and $\xi_2(1/\sqrt{3}) = 2/9(4 + 4\sqrt{3} + 2q + 2\sqrt{3}q) > 0$. In view of the intermediate value theorem, we get that the desired S_p- radius lies in $(0, 1/\sqrt{3})$ for the class $\mathcal{H}^2_m$.

(iii) It follows from (25) that

$$\begin{aligned} |\tau(z) - 1| &\leq \left|\tau(z) - \frac{1}{1-\rho^2}\right| + \frac{\rho^2}{1-\rho^2} \\ &\leq \frac{\rho^4 + (1+2q)\rho^3 + (5+q)\rho^2 + (1+q)\rho}{(1-\rho^2)(\rho^2+q\rho+1)}. \end{aligned} \tag{32}$$

By [2, Lemma 2.2, p. 6559], the function $f \in \mathcal{H}^2_m$ belongs to the class S^*_L, whenever the following inequality holds

$$\frac{\rho^4 + (1+2q)\rho^3 + (5+q)\rho^2 + (1+q)\rho}{(1-\rho^2)(\rho^2+q\rho+1)} \leq \sqrt{2} - 1$$

alternatively, if the following inequality holds

$$\sqrt{2}\rho^4 + (1 + (1+\sqrt{2})q)\rho^3 + (5+q)\rho^2 + (1 + \sqrt{2}(\sqrt{2}-1)q)\rho + 1 - \sqrt{2} \leq 0.$$

Note that $\xi_3(0) = 1 - \sqrt{2} < 0$ and $\xi_3(1) = 4(2+q) > 0$. It follows from the intermediate value theorem that the equation $\xi_3(\rho) = 0$ has a root $\rho_3 \in (0,1)$. In view of the triangle inequality and (32) we get $|\tau(z)+1| \leq \sqrt{2}+1$ and hence

$$\left|(\tau(z))^2 - 1\right| = 1.$$

The number $\rho' = \rho_2$ satisfies

$$\frac{(1+q)\rho'^3 + (5+q)\rho'^2 + (1+2q)\rho' + 1}{(1-\rho'^2)(1+q\rho'+\rho'^2)} = \sqrt{2}.$$

Thus, for the function f_2 given in (3), we get

$$\begin{aligned}\left|(\tau(z))^2 - 1\right| &= \left|\left(\frac{-(1+b')z^3 + (5+b')z^2 - (1+2b')z + 1}{(1-z^2)(1-b'z+z^2)}\right)^2 - 1\right| \\ &= \left|\left(\frac{(1+b')\rho'^3 + (5+b')\rho'^2 + (1+2b')\rho' + 1}{(1-\rho'^2)(1+b'\rho'+\rho'^2)}\right)^2 - 1\right| \\ &= 1 \quad (z = -\rho' = -\rho_2).\end{aligned}$$

This illustrates the sharpness.

(iv) It is easy to see that $1/e \leq 1/(1-\rho^2) \leq (e+e^{-1})/2$ for $\rho \leq ((1-2e+e^2)/(1+e^2))^{1/2}$. By [16, Lemma 2.2, p. 368], the function $f \in \mathcal{H}_m^2$ belongs to the class S_e^* if

$$\frac{(1+q)\rho^3 + (4+q)\rho^2 + (1+q)\rho}{(1-\rho^2)(\rho^2+q\rho+1)} \leq \frac{1}{1-\rho^2} - \frac{1}{e}$$

which implies that $\xi_4(\rho) \leq 0$. Note that $\xi_4(0) = -1.71828 < 0$ and $\xi_4(((1-2e+e^2)/(1+e^2))^{1/2}) = 3.20809+1.9087q > 0$. By the intermediate value theorem, we get that the desired S_e^*-radius lies in $\big(0, ((1-2e+e^2)/(1+e^2))^{1/2}\big)$. ∎

Remark 3.2. *Considering $m=-1$ in parts (1)-(4) of Theorem 3.2, we get [20, Theorem 2.6] .*

4. Conclusion

It is observed that for the class $\mathcal{H}_{m,n}^1$ other radius estimates can be determined for the classes of starlike functions associated with the functions $1+(4/3)z+(2/3)z^2$; $\sqrt{2}-(\sqrt{2}-1)((1-z)/(1+2(\sqrt{2}-1)z))^{1/2}$; $z+(1+z^2)^{1/2}$; $1+z-(z^3/3)$ and $2/(1+e^{-z})$.

Acknowledgement. The authors would like to express their gratitude to the referee for many valuable suggestions and insights that helped to improve the quality and clarity of this manuscript.

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