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# SOLVING A FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION WITH SINGULAR CONDITIONS

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ABSTRACT. In this paper the time-fractional Schrödinger equations with singular potentials are studied. Dirac function or even higher-order singularities are allowed. Using the Gronwall lemma and Laplace transforms, we give and prove the existence and uniqueness of the integral solution of the problem in Colombeau's algebra.

Keywords: Schrödinger equations, Caputo derivative, Generalised solution, Semigroup, Singular potential.

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## 1. Introduction

Fractional Schrödinger equations represent a captivating extension of the classical Schrödinger equation from quantum mechanics by incorporating fractional derivatives. These equations have gained significant attention due to their ability to describe various physical phenomena exhibiting anomalous diffusion, non-local interactions, and long-range memory effects. In many physical systems, particle movement deviates from the normal Brownian motion, exhibiting anomalous diffusion. This type of diffusion, characterized by non-Gaussian distributions and long-range temporal or spatial correlations, can be effectively modeled using fractional calculus. Fractional Schrödinger equations are particularly useful in describing such systems, where the standard Schrödinger equation falls short.
In contrast to the integer-order derivatives in the classical Schrödinger equation, fractional derivatives capture memory effects and non-local interactions, making fractional Schrödinger equations suitable for modeling systems with complex dynamics and anomalous diffusion. The fractional order in these equations can range between zero and one, capturing a broad spectrum of behaviors, from subdiffusion to superdiffusion.

In many physical systems, singular potentials provide a more accurate and realistic description of interactions. For instance, the Coulomb potential, which describes the

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electrostatic interaction between charged particles, has a singularity at the origin. Incorporating such potentials into quantum models allows for a precise representation of physical phenomena, especially at short distances or in high-energy scenarios.

In the first time A. Benmerrous and al[3] were able to prove the existence and uniqueness of the generalized solution under initial data are singular, and from this they open a great way to study the Schrödinger equations of the integer order, in their paper they deal with the following problem, taking the initial values as generalized functions.

$$\begin{cases} \frac{1}{i}\partial_s\psi(s,y) - \Delta\psi(s,y) + v(y)\psi(s,y) = 0 \\ v(y) = \delta(y), \quad \psi(0,y) = \delta(y). \end{cases}$$

Our objective is to enlarge this last work in the situation of the Caputo's time-fractional derivative with the exponent between 0 and 1. But before that, we will enter the fractional derivative into Colombeau algebra and after that we will discussing the notion of generalized semigroup through which we will give and demonstrate the existence and uniqueness of the solution of the following Cauchy problem

$$\begin{cases} D_s^\alpha\psi(s,y) - A\psi(s,y) = f\left(s,\psi(s,y)\right), \quad s \in [0,T] \\ \psi(0,y) = \psi_0 \end{cases},$$

where $D_s^\alpha$ is Caputo derivative of order $\alpha$, $0 < \alpha \leq 1$, and $A$ is the infinitesemal genrator of semigroup $(T(s))_{s\geq 0}$. If $A = \Delta$ we note that the latter is a generalization of our objective.

The paper is organized as follows: Section 2 introduces several concepts from Colombeau's algebra. Section 3, we will give and demonstrate the existence of Caputo derivative in Colombeau algebra. Section 4, introduce the concept of generalized fractional semigroup, in section 5 we gave and demonstrate the integral solution of a Cauchy problem and we will apply this in the section 6 (Schrödinger equation).

## 2. Preliminaries

We present below some symbols and definitions for future reference. [5, 3].

Consider $\mathcal{D}(\mathbb{R}^n)$ as the collection of all test functions $\phi : \mathbb{R}^n \to \mathbb{C}$ having compact support.

Let $q \in \mathbb{N}$, we define

$$\mathcal{A}_q(\mathbb{R}^n) = \left\{\phi \in \mathcal{D}\left(\mathbb{R}^n\right) / \int \phi(y)dy = 1 \quad and \int y^\beta\phi(y)dy = 0 \text{ with } 1 \leq \beta \leq q\right\}.$$

The members of the collection $\mathcal{A}_q$ are referred to as test functions.

It is evident that $\mathcal{A}_1$ contains $\mathcal{A}_2$ and so forth. Colombeau has demonstrated in his literature that the sets $\mathcal{A}_k$ are populated for every $k \in \mathbb{N}$ [3].

For any $\phi \in \mathcal{A}_q$, $\epsilon > 0$ we denote $\phi_\epsilon(y) = \frac{1}{\epsilon}\phi\left(\frac{y}{\epsilon}\right)$ with $\phi \in \mathcal{D}\left(\mathbb{R}^n\right)$, and $\check{\phi}(y) = \phi(-y)$.

We use the notation

$$\mathcal{E}\left(\mathbb{R}^n\right) = \left\{\varphi : \quad \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C}/\varphi(\phi, y) \text{ is } \mathcal{C}^\infty \text{ to } y\right\},$$

$$\varphi\left(\phi_\epsilon, y\right) = u_\epsilon(x) \quad \forall \phi \in \mathcal{A}_1,$$

$$\begin{aligned}\mathcal{E}_M\left(\mathbb{R}^n\right) = &\left\{\left(\varphi_\epsilon\right)_{\epsilon>0} \subset \mathcal{E}\left(\mathbb{R}^n\right) / \forall K \subset \mathbb{R}^n, \forall a \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that}\right.\\ &\left.\sup_{y \in K} \|D^\alpha \varphi_\epsilon(y)\| = \mathcal{O}\left(\epsilon^{-N}\right) \text{ as } \epsilon \to 0\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}\left(\mathbb{R}^n\right) = &\left\{\left(\varphi_\epsilon\right)_{\epsilon>0} \in \mathcal{E}\left(\mathbb{R}^n\right) / \forall K \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}, \forall p \in \mathbb{N} \text{ such that}\right.\\ &\left.\sup_{x \in K} \|D^\alpha \varphi_\epsilon(y)\| = \mathcal{O}\left(\epsilon^{p}\right) \text{ as } \epsilon \to 0\right\}.\end{aligned}$$

Then the Colombeau algebra is define by $\mathcal{G} = \mathcal{E}_M/\mathcal{N}$, where the elements of $\mathcal{E}_M$ exhibit moderation, while those of $\mathcal{N}$ are considered negligible.
The components of Colombeau algebras $\mathcal{G}$ consist of evenness classes stemming from regularizations. These regularizations manifest as sequences of smooth functions that satisfy specific asymptotic conditions concerning the regularization parameter $\epsilon$. Hence, for any given set X, we denote the ensemble of sequences $(u_\epsilon)_{\epsilon\in[0;1]}$ belonging to X as $X^{[0;1]}$. Such sequences are also termed as nets and are succinctly denoted as $u_\epsilon$.

**Definition 1.**

*A function $f \in \mathcal{G}(\mathbb{R})$ is considered to have an 'associated distribution', denoted as $f \approx u$, if for every representative $f(\varphi_\epsilon, y)$ of $f$ and $\psi(y) \in \mathcal{D}(\mathbb{R})$, there exists a natural number $q$ such that for any $\varphi(y) \in \mathcal{A}_q(\mathbb{R})$, we have:*

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}} f(\varphi_\epsilon, y)\psi(y)dy = \langle u, \psi\rangle.$$

## 3. Caputo derivative in Colombeau algebra

A fractional integral in Caputo sense is defined by: [15]

$$I^\alpha f(r) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-s)^{\alpha-1} f(s)ds \quad \alpha \in \mathbb{R}^+.$$

In the Caputo meaning, the fractional derivative of order $\alpha > 0$ is defined as: [15]

$$D^\alpha f(r) = \frac{1}{\Gamma(m-\alpha)} \int_0^r \frac{f^{(m)}(s)ds}{(r-s)^{\alpha+1-m}} \quad , \quad m-1 < \alpha < m.$$

Let $(f_\varepsilon)$ be a representative of $f$ in $\mathcal{G}([0,+\infty[)$, then:

$$\begin{aligned}D^\alpha f_\epsilon(r) &= \frac{1}{\Gamma(1-\alpha)} \int_0^r \frac{f'(s)}{(r-s)^\alpha} ds \quad 0<\alpha<1\\ \sup_{t\in[0,T]} \|D^\alpha f_\varepsilon(r)\| &\le \frac{1}{\Gamma(1-\alpha)} \sup_{r\in[0,T]} \|\int_0^r \frac{f'(s)ds}{(r-s)^\alpha}\|\\ &\le \frac{1}{\Gamma(1-\alpha)} \left\|f'\right\|_{L^\infty([0,T])} \sup_{t\in[0,T]} \int_0^r \frac{ds}{(r-s)^\alpha} ds\\ &\le \frac{1}{\Gamma(1-\alpha)} \epsilon^{-N} \frac{T^{1-\alpha}}{1-\alpha} \le C_{\alpha,T}\epsilon^{-N}.\end{aligned}$$

Generally, for $\alpha \in (m-1, m)$.

$$\begin{aligned}\sup_{r\in[0,T]}\|D^\alpha f_\epsilon(r)\| &\le \frac{1}{\Gamma(m-\alpha)}\sup_{r\in[0,T]}\int_0^r \frac{\|f^{(m)}(s)\|}{(r-s)^{\alpha+1-m}}ds\\ &\le \frac{1}{\Gamma(m-\alpha)}\left\|f^{(m)}\right\|_{L^\infty([0,T])}\sup_{r\in[0,T]}\int_0^r\frac{1}{(r-s)^{\alpha+1-m}}ds\\ &\le \frac{1}{\Gamma(m-\alpha)}\epsilon^{-N}\frac{T^{m-\alpha}}{m-\alpha}\le C_{\alpha,T}\epsilon^{-N}.\end{aligned}$$

The constant $C_{\alpha,T}$ depends on two factors $\alpha$ and $T$.

**Proposition 1.**

*Let $(\omega_\epsilon(t))_\epsilon$ be a representative of $\omega(t)\in\mathcal{G}([0,+\infty))$. The regularized Caputo $\alpha^{th}$ fractional derivative of $(\omega_\epsilon(t))_\epsilon$, $\alpha>0$, is defined by*

$$i_{frac}:\begin{cases}\mathcal{G}([0,+\infty))\to\mathcal{G}([0,+\infty))\\ \omega\to\left(\widetilde{D^\alpha}\omega_\epsilon\right)_{\epsilon>0}:=\left(D^\alpha\omega*\varphi_\epsilon\right)_{\epsilon>0}.\end{cases}$$

**Proposition 2.** *Let $\epsilon>0$*

$$\left(\left(\widetilde{D^\alpha}\omega_\epsilon\right)\right)\approx\left(\left(D^\alpha\omega_\epsilon\right)\right).$$

*Proof.* Let: $u_\varepsilon\in G([0,+\infty))$.

We have,

$$\begin{aligned}\|\tilde{D}^\alpha u_\varepsilon(t)\| &= \|D^\alpha u_\varepsilon*\varphi_\varepsilon(t)\|\\ &= \|\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{u_\epsilon^{(2)}(s)ds}{(t-s)^{\alpha-1}}*\varphi_\epsilon(t)\|\\ &\leqslant \|\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{u_\epsilon^{(2)}(s)}{(t-s)^\alpha-1}ds\|\times\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\\ &\leqslant \|D^\alpha U_\varepsilon(t)\|\times\|\varphi_\varepsilon\|L^\infty(\mathbb{R}^n).\end{aligned}$$

Then:

$$\begin{aligned}\|\widetilde{D^\alpha}u_\varepsilon(t)-D^\alpha u_\varepsilon(t)\| \leqslant&\|D^\alpha u_\varepsilon(t)\|\left(\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)}-1\right)\\ \leqslant&\frac{1}{\Gamma(2-\alpha)}\sup_{t\in[0,T]}\|\int_0^t\frac{u_\varepsilon^{(2)}(\tau)}{(t-\tau)^{\alpha-1}}d\tau\|\times\left(\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)}-1\right)\end{aligned}$$

$\leqslant \frac{1}{\Gamma(2-\alpha)}\sup_{t\in[0,T]}\|u_\varepsilon^{(2)}(t)\|\times\frac{T^{2-\alpha}}{2-\alpha}\times\left(\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)}-1\right)$
$\leqslant C_{T,\alpha}\varepsilon^{2-\alpha}\overset{\varepsilon\longrightarrow 0}{\longrightarrow}0.$

We utilize the regularization for $\alpha\in(0,1)$

$$\tilde{\mathcal{D}}^\alpha u_\epsilon(y)=\mathcal{D}^\alpha u_\epsilon*\phi_\epsilon(y).$$

The form of convolution is provided by :

$$\tilde{\mathcal{D}}^\alpha u_\epsilon(y)=\int_{\mathbb{R}}\mathcal{D}^\alpha u_\epsilon(s)\phi_\epsilon(y-\tau)d\tau.$$

We state that $|\tilde{\mathcal{D}}^\alpha u_\varepsilon(y)-\mathcal{D}^\alpha u_\varepsilon(y)|\approx 0$.

$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) - \mathcal{D}^{\alpha} u_{\varepsilon}(y) |=| \mathcal{D}^{\alpha} u_{\varepsilon} * \phi_{\varepsilon}(y) - \mathcal{D}^{\alpha} u_{\varepsilon}(y) |$$
$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) - \mathcal{D}^{\alpha} u_{\varepsilon}(y) |=| \mathcal{D}^{\alpha} u_{\varepsilon} * \phi_{\varepsilon}(y) - \mathcal{D}^{\alpha} u_{\varepsilon} * \delta(y) |$$
$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) - \mathcal{D}^{\alpha} u_{\varepsilon}(y) |=| \mathcal{D}^{\alpha} u_{\varepsilon} * (\phi_{\varepsilon}(y) - \delta(y)) |$$
$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) - \mathcal{D}^{\alpha} u_{\varepsilon}(y) |=| \int_{\mathbb{R}} \mathcal{D}^{\alpha} u_{\epsilon}(y-\tau) \left(\phi_{\epsilon}(\tau) - \delta(s)\right) d\tau |$$
$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) - \mathcal{D}^{\alpha} u_{\varepsilon}(y) |= \int_{\mathbb{R}} | \mathcal{D}^{\alpha} u_{\varepsilon}(y-\tau) || \phi_{\varepsilon}(\tau) - \delta(\tau) | d\tau \longrightarrow 0.$$

Because of $\lim_{\epsilon \longrightarrow 0} | \phi_{\epsilon}(\tau) - \delta(\tau) |= 0$, consequently

$$\tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) \approx \mathcal{D}^{\alpha} u_{\varepsilon}(y)$$

By using assumption that $\phi_{\epsilon}(y)$ has compact support on $K_0$, the following computations can be made utilizing Holder inequalities:

$$\tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) = \mathcal{D}^{\alpha} u_{\epsilon} * \phi_{\epsilon}(y) = \int_{\mathbb{R}} \mathcal{D}^{\alpha} u_{\varepsilon}(y-\tau) \phi_{\epsilon}(\tau) d\tau$$
$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) |=| \int_{\mathbb{R}} \mathcal{D}^{\alpha} u_{\varepsilon}(y-\tau) \phi_{\epsilon}(\tau) d\tau |=| \int_{K_0} \mathcal{D}^{\alpha} u_{\varepsilon}(y-\tau) \phi_{\epsilon}(\tau) ds |$$
$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) |= \int_{K_0} | \mathcal{D}^{\alpha} u_{\varepsilon}(y-\tau) || \phi_{\varepsilon}(\tau) | d\tau$$
$$\sup_{y \in K} | \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) |= \sup_{y \in K} \left\{ \int_{K_0} | \mathcal{D}^{\alpha} u_{\epsilon}(y-\tau) || \phi_{\epsilon}(\tau) | d\tau \right\}.$$

So,

$$\sup_{y \in K_0} | \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) |\leq \sup_{y \in K_0} | \mathcal{D}^{\alpha} u_{\varepsilon}(y) | \int_{K_0} | \phi_{\varepsilon}(\tau) | d\tau$$
$$\sup_{y \in K} | \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) |\leq C_1 \varepsilon^p.$$

And

$$\frac{d}{dy} \left( \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) \right) = \frac{d}{dy} \left( \mathcal{D}^{\alpha} u_{\varepsilon} \right) * \phi_{\varepsilon}(y) = \mathcal{D}^{\alpha} u_{\varepsilon} * \frac{d}{dy} \left( \phi_{\varepsilon}(y) \right).$$

Then,

$$\sup_{y \in K} | \frac{d}{dy} \left( \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) \right) |\leq \sup_{y \in K_0} | \mathcal{D}^{\alpha} u_{\varepsilon}(y) | \int_{K_0} | \frac{d}{dy} \left( \phi_{\varepsilon}(\tau) \right) | d\tau \leq C_2 \varepsilon^p.$$

A similar approach is used to demonstrate moderateness for higher derivatives.

$$\sup_{y \in K} | \partial^n \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(y) |\leq C_{\epsilon} \varepsilon^p$$

□

By the same principle we define "Generalized caputo semigroup".

## 4. Generalized Caputo semigroup

Let's denote a Banach space as $(E, \|.\|)$, where $E$ is the space and $\|.\|$ is the norm. $\mathcal{L}(E)$ denotes the set of linear continuous mappings from $X$ to $E$.

**Definition 2.**

*We establish* $\mathrm{E}_M^S(\mathbb{R}^+, \mathcal{L}(E))$ *as the set of mappings* $(S_\epsilon)_\epsilon$ *from* $\mathbb{R}^+$ *to* $\mathcal{L}(E)$*, where* $0 < \epsilon < 1$*, satisfying* $S_\epsilon(0) = I$ *for all, for all* $T > 0$*, there exists* $n \in \mathbb{N}$ *such that:*

$$\sup_{0<s<T} \|S_\epsilon(s)\| = \mathcal{O}\left(\epsilon^n\right) \qquad as \qquad \epsilon \longrightarrow 0. \tag{1}$$

**Definition 3.**

*Define* $N^S\left([0, +\infty), \mathcal{L}(E)\right)$ *as the set of mappings* $(N_\epsilon)_\epsilon$ *from* $\mathbb{R}^+$ *to* $\mathcal{L}(E)$*, where* $0 < \epsilon < 1$*. For any* $C > 0$*,* $q \in \mathbb{N}$ *the following characteristics hold:*

$$\sup_{s\in[0,C]} \|N_\epsilon(s)\| = \mathcal{O}\left(\epsilon^q\right) \qquad as \qquad \epsilon \longrightarrow 0. \tag{2}$$

*There exist* $r > 0$ *and* $b \in \mathbb{R}$ *such that:*

$$\sup_{s<r} \left\| \frac{N_\epsilon(s)}{s} \right\| = \mathcal{O}\left(\epsilon^b\right). \tag{3}$$

*Moreover, there exists* $(H_\epsilon)_\epsilon$ *include in* $\mathcal{L}(E)$ *and* $0 < r < 1$ *such that:*

$$\lim_{s\to 0} \frac{N_\epsilon(s)}{s} y = H_\epsilon y, \quad y \in E, \quad \epsilon < r. \tag{4}$$

*For any* $r > 0$*,*

$$\|H_\epsilon\| = \mathcal{O}\left(\epsilon^r\right) \qquad as \qquad \epsilon \longrightarrow 0. \tag{5}$$

**Proposition 3.**

*1)* $E_M^S([0, +\infty),\, \mathcal{L}(E))$ *is an algebra.*

*2)* $N^S([0, +\infty),\, \mathcal{L}(E))$ *is an ideal of* $E_M^S([0, +\infty), \mathcal{L}(E))$*.*

*Proof.* Let $(L_\epsilon(s))_\epsilon \subset E_M^S(\mathbb{R}^+, \mathcal{L}(E))$ and $(M_\epsilon(s))_\epsilon \subset N^S(\mathbb{R}^+, \mathcal{L}(E))$.
We will focus solely on proving the second statement:

$$(L_\epsilon(s)M_\epsilon(s))_\epsilon), (M_\epsilon(s)_\epsilon L_\epsilon(s)) \in N^S([0, +\infty), \mathcal{L}(E)).$$

The composition is denoted by $L(s)M(s)$.
Let $0 < \epsilon < 1$. Using (2) and (4), for $y \in \mathbb{R}$ and $\forall z \in \mathbb{R}$,

$$\|L_\epsilon(s)M_\epsilon(s)\| \le \|L_\epsilon(s)\|.\|M_\epsilon(s)\| = \mathcal{O}(\epsilon^{y+z})$$

*when* $\epsilon \longrightarrow 0$.
The same holds for $\|M_\epsilon(s)L_\epsilon(s)\|$.
Additionally, by (2) and (4):
$\sup_{s<t_0} \|\frac{L_\epsilon(s)M_\epsilon(s)}{s}\| \le \sup_{s<t_0} \|L_\epsilon(s)\| \sup_{s<t_0} \|\frac{M_\epsilon(s)}{s}\| = \mathcal{O}(\epsilon^a)$ , as $\epsilon \longrightarrow 0$, for $t_0 > 0$, $a \in \mathbb{R}$.
Also,

$$\sup_{s<t_0} \|\frac{L_\epsilon(s)M_\epsilon(s)}{s}\| = \mathcal{O}(\epsilon^a) \text{ , as } \epsilon \to 0.$$

For $t_0 > 0$, $a \in \mathbb{R}$.
Let $\epsilon \in (0,1)$.

$$\begin{aligned}\left\|\frac{L_\varepsilon(s)M_\varepsilon(s)}{s}y - L_\varepsilon(0)H_\varepsilon y\right\| &= \left\|L_\varepsilon(s)\frac{M_\varepsilon(s)}{s}y - L_\varepsilon(s)H_\varepsilon y\right.\\ &\quad \left. + L_\varepsilon(s)H_\varepsilon y - L_\varepsilon(0)H_\varepsilon y\right\|\\ &\leq \|L_\varepsilon(s)\|\left\|\frac{M_\varepsilon(s)}{s}y - H_\varepsilon y\right\|\\ &\quad + \|L_\varepsilon(s)H_\varepsilon y - L_\varepsilon(0)H_\varepsilon y\|.\end{aligned}$$

Using (2) and (4) and the continuity of $L(s)(Hy)$ at 0, the final expression approaches zero as $s \longrightarrow 0$.
Therefore,

$$\begin{aligned}&\|\frac{M_\epsilon(s)L_\epsilon(s)}{s}y - H_\epsilon L_\epsilon(0)y\| =\\ &\qquad \|\frac{M_\epsilon(s)}{s}L_\epsilon(s)y - \frac{M_\epsilon(s)}{s}L_\epsilon(0)y\\ &\qquad + \frac{M_\epsilon(s)}{s}L_\epsilon(0)y - H_\epsilon L_\epsilon(0)y\|.\\ &\qquad \leq \left\|\frac{M_\epsilon(s)}{s}\right\| \|L_\epsilon(s)y - H_\epsilon(s)L_\epsilon(0)y\|\\ &\qquad + \left\|\frac{M_\epsilon(s)}{s}\left(L_\epsilon(0)y\right) - H_\epsilon\left(L_\epsilon(0)y\right)\right\|.\end{aligned}$$

Assertions (2), (3), and (4) denote that the last equation approaches 0 as $t \longrightarrow 0$. Consequently, (5) holds true in both cases.

□

**Definition 4.**
*The generalized semigroups are defined as:*
$G^S([0,+\infty),\mathcal{L}(E)) = E_M^S([0,+\infty),\mathcal{L}(E))/N^S([0,+\infty),\mathcal{L}(E))$.

**Definition 5.**
*A component $L$ in $G^S([0,+\infty),\mathcal{L}(E))$ is termed a generalized $C^0$-semigroup if there exists an indicative sequence $(L_\epsilon)_\epsilon$ of $L$, such that for the same $\epsilon_0$, $L_\epsilon$ forms a $C^0$-semigroup for all $\epsilon < \epsilon_0$.*

For sufficiently small $\epsilon$, we will only consider representatives $(L_\epsilon)_\epsilon$ of a Colombeau $C^0$-semigroup $L$ that are themselves $C^0$-semigroups.

**Proposition 4.**
*Let $(L_\epsilon)_\epsilon$ and $(\widetilde{L}_\epsilon)_\epsilon$ represent a Colombeau $C^0$-semigroup $L$, utilizing infinite generators $G_\epsilon, \epsilon < \epsilon_0$ respectively, where $\epsilon_0$ and $\tilde{\epsilon}_0$ are related according to definition (5) to $(L_\epsilon)_\epsilon$ and $\left(\tilde{L}_\epsilon\right)_\epsilon$, respectively.*
*Then, $D\left(G_\epsilon\right) = D\left(\tilde{G}_\epsilon\right)$, $\forall \epsilon < \tilde{\epsilon}_0 = \min\{\epsilon_0, \tilde{\epsilon}_0\}$ and $G_\epsilon - \tilde{G}_\epsilon$ could be expanded to such a component of $\mathcal{L}(E)$ indicated by $G_\epsilon - \tilde{G}_\epsilon$.*

*In a similar manner, $\forall a \in \mathbb{R}$,*

$$\left\|G_\epsilon - \tilde{G}_\epsilon\right\| = \mathcal{O}\left(\epsilon^a\right), \quad as \quad \epsilon \to 0.$$

*Proof.*
Denote $(M_\epsilon)_\epsilon = \left(L_\epsilon - \tilde{L}_\epsilon\right)_\epsilon \in N^S\left([0,+\infty), \mathcal{L}(E)\right)$.
If $\epsilon < \bar{\epsilon}_0$ and $y \in E$, we have :

$$\frac{L_\epsilon(s)y - y}{s} - \frac{\tilde{L}_\epsilon(s)y - y}{s} = \frac{M_\epsilon(s)}{s}y$$

When $s \longrightarrow 0$, we have $D\left(G_\epsilon\right) = D\left(\tilde{G}_\epsilon\right)$.
Then

$$\begin{aligned}\left(G_\epsilon - \tilde{G}_\epsilon\right)y &= \lim_{s\to 0}\frac{L_\epsilon(s)y - y}{s} - \lim_{s\to 0}\frac{\tilde{L}_\epsilon(s)y - y}{s}\\ &= \lim_{s\to 0}\frac{M_\epsilon(s)}{s}y = H_\epsilon y, y \in D\left(G_\epsilon\right)\end{aligned} \tag{6}$$

Since $\overline{D(G)} = E$, and the properties (3),(4),(5) show that $\forall r \in \mathbb{R}$.

$$\left\|G_\epsilon - \tilde{G}_\epsilon\right\| = \mathcal{O}\left(\epsilon^r\right) \quad \text{as} \quad \epsilon \to 0.$$

□

## 5. Generalized solutions

We consider the following Cauchy problem

$$\begin{cases} D_s^\alpha \psi(s,y) - A\psi(s,y) = F(t, \psi(s,y)), \\ \psi(0,y) = \psi_0 \in D'. \end{cases} \tag{7}$$

Were $A$ represents an infinitesimal generator of a generalized Colombeau semigroup denoted as $(T(s))_{s\geq 0} = \left[\left((T_\epsilon(s))_{s\geq 0}\right)_\epsilon\right]$, where $\psi$ belongs to $(\mathcal{G}(\mathbb{R}))^n$ and $F$ belongs to $(\mathcal{G}(\mathbb{R}))^n$.

The expression presented in (7) in its representative format as indicated by

$$\begin{cases} D_s^\alpha \psi_\epsilon(s,y) - A_\epsilon\psi_\epsilon(s,y) = F_\epsilon\left(s, \psi_\epsilon(s,y)\right), \\ \psi_\epsilon(0,y) = \psi_{0\epsilon}. \end{cases}$$

We express the Cauchy problem within the framework of an integral equation

$$\begin{cases} \psi_\epsilon(s) = \psi_{0\epsilon} + \frac{1}{\Gamma(\alpha)}\int_0^t (s-z)^{\alpha-1}\left[A_\epsilon\psi_\epsilon(z) + F_\epsilon\left(z, \psi_\epsilon(z)\right)\right]dz, \\ \psi_\epsilon(0) = \psi_{0\epsilon}. \end{cases} \tag{8}$$

The demonstration of the theorem necessitates the utilization of the two lemmas provided.

**Lemma 1.** *If the problem (8) is satisfied, it implies the existence of a probability density function $\mathscr{B}_\alpha$ defined over the interval $(0,+\infty)$ such that*

$$\begin{cases} \psi_\epsilon(s) = \int_0^\infty \mathscr{B}_\alpha T_\epsilon\left(s^\alpha y\right)\psi_{0\epsilon}dy \\ +\alpha\int_0^s\int_0^\infty y(s-z)^{\alpha-1}\mathscr{B}_\alpha(y)T_\epsilon\left((s-z)^\alpha y\right)F_\epsilon\left(z, \psi_\epsilon(z)\right)dydz, \\ \psi_\epsilon(0) = \psi_{0\epsilon}. \end{cases}$$

*Proof.* By subjecting the initial equation in (8) to the Laplace transform, we have

$$\mathcal{L}\psi_\epsilon(\beta) = \frac{1}{\beta}\psi_{0\epsilon} + \frac{1}{\beta^\alpha}A_\epsilon\mathcal{L}\left(\psi_\epsilon\right)(\beta) + \frac{1}{\beta^\alpha}\mathcal{L}\left(F_\epsilon\left(., \psi_\epsilon(.)\right)\right)(\beta),$$

then

$$\begin{aligned}
\psi_\epsilon(\beta) &= \beta^{\alpha-1}\left(\beta^{\alpha-1}I + A_\epsilon\right)^{-1}\psi_{0\epsilon} + \left(\beta^{\alpha-1}I + A_\epsilon\right)^{-1} \\
&= \beta^{\alpha-1}\left(\beta^{\alpha}I + A_\epsilon\right)^{-1}\psi_{0\epsilon} + \left(\beta^{\alpha}I + A_\epsilon\right)^{-1}\mathcal{L}\left(e^{-\beta s}F_\epsilon\left(s,\psi_\epsilon(s)\right)\right)(\beta) \\
&= \beta^{\alpha-1}\int_0^\infty e^{-\beta^\alpha s}T_\epsilon(s)\psi_{0\epsilon}ds + \int_0^\infty e^{-\beta^\alpha s}T_\epsilon(s)\omega(\beta)ds,
\end{aligned}$$

Here, $I$ represents the operator of identity , $\omega(\beta)$ denotes the Laplace transform of $F_\epsilon\left(s,\psi_\epsilon(s)\right)$. Let's examine the probability density function provided in [10], which is defined as

$$\mathscr{B}_\alpha(y) = \frac{1}{\pi}\sum(-1)^{n-1}y^{-\alpha n-1}\frac{\Gamma}{\Gamma(n+1)}\sin(n\pi\alpha), \quad y\in(0,\infty),$$

The Laplace transform of which is expressed as $\int e^{-\beta y \mathscr{B}}(y)dy = e^{-\beta^\alpha}$, where $\alpha$ belongs to the interval (0,1).

Then

$$\begin{aligned}
\psi_{0\epsilon} &= \beta^{\alpha-1}\int_0^\infty e^{-\beta^\alpha s}T_\epsilon(s)\psi_{0\epsilon}ds \\
&= \int_0^\infty \alpha(\beta t)^{\alpha-1}e^{-(\beta t)^\alpha}T_\epsilon\left(t^\alpha\right)\psi_{0\epsilon}ds \\
&= \int_0^\infty \frac{-1}{\beta}\frac{d}{dt}\left[e^{-(\beta t)^\alpha}\right]T_\epsilon\left(t^\alpha\right)\psi_{0\epsilon}ds \\
&= \int_0^\infty\left[\int_0^\infty y_\alpha(y)e^{-(\beta t y)}T_\epsilon\left(t^\alpha\right)\psi_{0\epsilon}dy\right]dt \\
&= \int_0^\infty e^{-\beta t}\left[\int_0^\infty \mathscr{B}_\alpha(y)T_\epsilon\left(\frac{t^\alpha}{y^\alpha}\right)\psi_{0\epsilon}dy\right]dt.
\end{aligned}$$

Regarding the subsequent expression,

$$\begin{aligned}
\int_0^\infty e^{-\beta^\alpha}T_\epsilon(s)\omega(\beta)ds &= \int_0^\infty\left[\int_0^\infty \alpha t^{\alpha-1}e^{-(\beta t)^\alpha}T_\epsilon\left(t^\alpha\right)e^{-\beta s}F_\epsilon\left(s,\psi_\epsilon(s)\right)ds\right]dt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\mathscr{B}_\alpha(y)e^{-\beta t y}T_\epsilon\left(t^\alpha\right)e^{-\beta s}t^{-\alpha-1}F_\epsilon\left(s,\psi_\epsilon(s)\right)dydsdt \\
&= \int_0^\infty e^{-\beta t}\left[\alpha\int_0^t\int_0^\infty \mathscr{B}_\alpha(y)T_\epsilon\left(\frac{(t-s)^\alpha}{y^\alpha}\right)\right. \\
&\quad\left.\times\, F_\epsilon\left(s,\psi_\epsilon(s)\right)\frac{(t-s)^\alpha}{y^\alpha}dyds\right]dt.
\end{aligned}$$

Based on the final equalities, we have

$$\begin{aligned}
\mathcal{L}\psi_\epsilon(\beta) &= \int_0^\infty e^{-\beta t}\int_0^\infty \mathscr{B}_\alpha(y)T_\epsilon\left(t^\alpha y\right)\psi_\epsilon(t)dtdy \\
&\quad+\alpha\int_0^t\int_0^\infty y(t-s)^{\alpha-1}\mathscr{B}_\alpha(y)T_\epsilon\left((t-s)^\alpha y\right)F_\epsilon\left(s,\psi_\epsilon(s)\right)dydsdt.
\end{aligned}$$

Then

$$\begin{aligned}
\psi_\epsilon(t) &= \int_0^\infty \mathscr{B}_\alpha(y)T_\epsilon\left(t^\alpha y\right)\psi_{0\varepsilon}dy \\
&+\alpha\int_0^t\int_0^\infty y(t-s)^{\alpha-1}\mathscr{B}_\alpha(y)T_\epsilon\left((t-s)^\alpha y\right)F_\epsilon\left(s,\psi_\epsilon(s)\right)dyds.
\end{aligned}$$

Next, we establish a representative$(S_\epsilon^\alpha)_{t\in\mathbb{R}_+}$by

$$S_\epsilon^\alpha(t)\psi_\epsilon = \alpha\int_0^\infty y\mathscr{B}_\alpha(y)T_\epsilon\left(t^\alpha y\right)\psi_\epsilon dy \in E_M^S.$$

□

Ultimately, the integral solution to the Cauchy problem (7) is represented as

$$\psi_\epsilon(s) = S_\epsilon^\alpha(s)\psi_{0\epsilon} + \int_0^s (s-z)^{\alpha-1}T_\epsilon^\alpha(s-z)F_\epsilon\left(z,\psi_\epsilon(z)\right)dz.$$

**Remark 1.**
*For any given s within the interval [0, T] where T is greater than 0, the families of operators* $(S_\epsilon^\alpha(s))\, s\geq 0$ *and* $(T_\epsilon^\alpha(s))_{s\geq 0}$*, indexed by the variable s, are both linear and bounded for each* $\epsilon\in(0,1)$.

**Theorem 1.**
*Suppose* $F\in(\mathcal{G}(\mathbb{R}))^n$, $\|\nabla F\|\leq C\|\ln(\epsilon)\|$ *and* $0<\epsilon<1$.
*Then the problem (7) possesses one solution in Colombeau algebra* $(\mathcal{G}(\mathbb{R}))^n$.

*Proof.*
**Existence**

We need to demonstrate that the integral solution $(\psi_\epsilon)$, provided in Lemma 1, belongs to $\mathcal{E}_M(\mathbb{R})$ for any $\epsilon\in(0,1)$ and $\alpha\in(0,1)$.

Initially, we obtain the estimation

$$\begin{aligned}\|\psi_\epsilon(s)\| &= \|S_\epsilon^\alpha(s)\psi_{0\epsilon} + \int_0^s (s-z)^{\alpha-1}T_\epsilon^\alpha(s-z)F_\epsilon\left(z,\psi_\epsilon(z)\right)dz\|,\\ &\leq \|S_\epsilon^\alpha(s)\psi_{0\epsilon}\| + \int_0^s \|(s-z)^{\alpha-1}T_\epsilon^\alpha(s-z)F_\epsilon\left(z,\psi_\epsilon(z)\right)\|dz,\\ &\leq \|S_\epsilon^\alpha(s)\psi_{0\epsilon}\| + \int_0^s (s-z)^{\alpha-1}\|T_\epsilon^\alpha(s-z)F_\epsilon\left(z,\psi_\epsilon(z)\right)\|dz.\end{aligned}$$

The first-order approximation of $F_\epsilon$ results in

$$F_\epsilon(s,\psi(s)) = F_\epsilon(s,0) + \|\nabla_\epsilon F_\epsilon\|\psi_\epsilon(s) + N_\epsilon(s),$$

Here, $N_\epsilon(s)$ represents the zero component.

According to Lemma 1 and the condition $(\psi_\epsilon)\in\mathcal{E}_M(\mathbb{R})$, there exist positive constants $a_1$, $a_2$, $b_1$, and $b_2$, such that

$$\begin{aligned}\|\psi_\epsilon(s)\| &\leq a_2\epsilon^{-b_2} + \int_0^s (s-z)^{\alpha-1}\frac{\alpha a_1\epsilon^{-b_1}}{\Gamma(1+\alpha)}\|F_\epsilon\left(z,\psi_\epsilon(z)\right)\|dz\\ &\leq a_2\epsilon^{-b_2} + \int_0^s (s-z)^{\alpha-1}\frac{\alpha a_1\epsilon^{-b_1}}{\Gamma(1+\alpha)}\|F_\epsilon(z,0) + \|\nabla_\epsilon F_\epsilon\|\psi_\epsilon(z) + N_\epsilon(z)\|dz.\end{aligned}$$

Applying Gronwall's lemma leads to

$$\|\psi_\epsilon(s)\| \leq \left(a_2\epsilon^{-b_2} + a_1\epsilon^{-b_1}\right)\exp(-\epsilon\ln\epsilon).$$

Therefore, there exist $\bar{c}$ and $\bar{N}$ in $\mathbb{R}^+$ / $\|\psi_\epsilon(s)\|\leq\bar{c}\epsilon^{-\bar{N}}$.

Then

$$\psi_\epsilon(s) \in \mathcal{E}_M.$$

To derive approximations for higher-order derivatives, simply differentiate the integral solution and employ similar inductive reasoning, presuming that the lower-order terms are recognized as manageable from earlier steps.

**Uniqueness**

Assuming the existence of two solutions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ to the regularization of problem 7, let $e_\epsilon$ denote their difference. We then have:

$$e_\epsilon(s) = \int_0^s (s-z)^{\alpha-1} T_\epsilon^\alpha(s-z)\left[F_\epsilon\left(z, \psi_{1,\epsilon}\right) - F_\epsilon\left(z, \psi_{2,\epsilon}\right)\right] dz.$$

Now, employing the approximation of $F_\epsilon$, we have:

$$\|e_\epsilon(s)\| \leq \int_0^s \frac{\epsilon^\alpha}{\alpha} \|T_\epsilon^\alpha(s-z)\left[\|\nabla F_\epsilon\|\left(\psi_{1,\epsilon}(z) - \psi_{2,\epsilon}(z)\right) + N_\epsilon(z)\right]\| dz,$$

By leveraging the bounded nature of the linear operator $T_\epsilon^\alpha(s)$ for $s \geq 0$, Gronwall's lemma, and the understanding that $\psi_{1,\epsilon}(z) - \psi_{2,\epsilon}(z)$, as well as the negligible part $N_\epsilon$, are both of the order $\mathcal{O}\left(\epsilon^N\right)$, it follows that for any $N \geq 0$, $\|e_\epsilon(s)\| = \mathcal{O}\left(\epsilon^N\right)$ as $\epsilon \to 0$. This assertion substantiates the uniqueness of the solution in $(\mathcal{G}(\mathbb{R}))^n$. □

## 6. Application to Schrödinger equation

Let consider the next fractional nonlinear Schrödinger equation.

$$\begin{cases} \frac{1}{i}\partial_s^\alpha \psi(s,y) - \Delta\psi(s,y) + v(y)\psi(s,y) = 0, \\ v(y) = \delta(y), \psi(0,y) = \delta(y), \quad y \in \mathbb{R}^n. \end{cases} \tag{9}$$

Here $A = -\Delta$ and $\delta$ is the Dirac function.

We will employ regularization for the Dirac measure

$$v_\epsilon(y) = \delta_\epsilon(y) = (\phi_\epsilon(y)) = \|\ln \epsilon\|^{r_3 m}\phi\left(y\|\ln\epsilon\|^{r_3}\right) \quad with \quad r_3 \in \mathbb{R}_*^+,$$

$$\int_{\mathbb{R}} \phi = 1 \text{ with } \phi(y) \geq 0.$$

For $\psi_{0,\epsilon}$, we have

$$\psi_{0,\epsilon}(y) = \|\ln\epsilon\|^{r_1}\phi\left(y\|\ln\epsilon\|^{r_1}\right), \quad r_1 > 0.$$

**Theorem 2.** *The regularized equation of (9) is given by*

$$\begin{cases} \frac{1}{i}\partial_s^\alpha \psi_\epsilon(s,y) - \Delta\psi_\epsilon(s,y) + v_\epsilon(y)\psi_\epsilon(s,y) = 0, \\ v_\epsilon(y) = \delta_\epsilon(y), \psi_{0,\epsilon}(y) = \delta_\epsilon(y). \end{cases} \tag{10}$$

*Consequently, the problem (10) have one solution in $\mathcal{G}\left(\mathbb{R}^+ \times \mathbb{R}^n\right)$.*

*Proof.*
**Existence**
In the following we will denote $\|.\|_{L^\infty(\mathbb{R}^n)} = \|.\|_\infty$ and $\|.\|_{L^1(\mathbb{R}^n)} = \|.\|_1$.

By section 5 the integral solution of the equation (10) is

$$\psi_\epsilon(s,y) = \int_{\mathbb{R}^n} S_\epsilon^\alpha(s, y-e)\psi_{0,\epsilon}(e)de + \int_0^s \int_{\mathbb{R}^n} S_\epsilon^\alpha(s-\tau, y-e)v_\epsilon(y)\psi_\epsilon(\tau, e)ded\tau,$$

where $S_\epsilon^\alpha(s,y)\psi_\epsilon = \int_0^\infty \mathscr{B}_\epsilon(\xi) S_n(s^\alpha \xi)\psi_\epsilon d\xi$ and the heat kernel is $S_n(t,y)$.

So

$$\begin{aligned}\|\psi_\epsilon(s,.)\|_\infty &\le \|S_\epsilon^\alpha(s,y-.)\|_1 \|\psi_{0,\epsilon}\|_\infty \\ &\quad + \int_0^s \|S_\epsilon^\alpha(s-\tau,y-.)\|_1 \|v_\epsilon(.)\|_\infty \|\psi_\epsilon(\tau,.)\|_\infty d\tau.\end{aligned}$$

Using Remark 1, there is $R$ such that $\|S_\epsilon^\alpha\| \le R$, we get

$$\begin{aligned}\|\psi_\epsilon(s,.)\|_\infty &\le R\|\psi_{0,\epsilon}\|_\infty \\ &\quad + R\|v_\epsilon(.)\|_\infty \int_0^t \|\psi_\epsilon(\tau,.)\|_\infty d\tau.\end{aligned}$$

From Gronwall inequality, it follows

$$\|\psi_\epsilon(s,.)\|_\infty \le R\|\ln\epsilon\|^{r_1} \exp\left(RT\|\ln\epsilon\|^{r_1}\right).$$

Then $\exists N>0$, in a manner that

$$\|\psi_\epsilon(s,.)\|_\infty \le R\epsilon^{-N}.$$

When we compute the first derivative with respect to $y_j$, where j is within the range of 1 to n, we arrive at:

$$\begin{aligned}\partial_{y_j}\psi_\epsilon(s,y) &= \int_{\mathbb{R}^n} S_\epsilon^\alpha(s,y-e)\partial_{e_j}\psi_{0,\epsilon}(e)de \\ &\quad + \int_0^t \int_{\mathbb{R}^n} S_\epsilon^\alpha(s-\tau,y-e)\left(\partial_{e_j}v_\epsilon(e)\psi_\epsilon(\tau,e) + v_\epsilon(e)\partial_{e_j}\psi_\epsilon(\tau,e)\right)ded\tau,\end{aligned}$$

so,

$$\begin{aligned}\|\partial_y\psi_\epsilon(s,.)\|_\infty &\le \|S_\epsilon^\alpha(s,y-))\|_1 \|\partial_{e_j}\psi_{0,\epsilon}\|_\infty \\ &\quad + \int_0^t \|S_\epsilon^\alpha(s-\tau,y-.)\|_1 \left(\|\partial_{e_1}v_\epsilon\|_\infty \|\psi_\epsilon\|_\infty\right. \\ &\quad \left. + \|v_\epsilon\|_\infty \|\partial_{e_i}\psi_\epsilon(\tau,.)\|_\infty\right) d\tau,\end{aligned}$$

which implies

$$\begin{aligned}&\left\|\partial_{y_j}\psi_\epsilon(s,.)\right\|_\infty \le R\|\ln\epsilon\|^{r_1(m+1)} + R\int_0^s \|\ln\epsilon\|^{r_2(m+1)} \|\psi_\epsilon\|_\infty \\ &+ \|\ln\epsilon\|^{r_2 m} \left\|\partial_{e_j}\psi_\epsilon(\tau,.)\right\|_\infty d\tau \\ &\le R\left(\|\ln\epsilon\|^{r_1(m+1)} + T\|\ln\epsilon\|^{r_2(m+1)} \|\psi_\epsilon\|_\infty\right) \\ &+ R\|\ln\epsilon\|^{r_2 m} \int_0^t \|\partial_{e_i}\psi_\epsilon(\tau,.)\|_\infty d\tau.\end{aligned}$$

Using Gronwall inequality, we have

$$\|\partial_y\psi_\epsilon(s,.)\|_\infty \le R\left(\|\ln\epsilon\|^{r_1(\mathrm{m}+1)} + T\|\ln\epsilon\|^{r_2(m+1)} \|\psi_\epsilon\|_\infty\right)\exp\left(RT\|\ln\epsilon\|^{r_2 m}\right),$$

The preceding action guarantees the existence of a positive number N such that

$$\left\|\partial_{y_j}\psi_\epsilon(s,.)\right\|_\infty \le R\epsilon^{-N}.$$

When considering the second derivative for $e_i, 1 < i < n$, we have

$$\partial_{e_i}\partial_y\psi_\varepsilon(s,y) = \int_{\mathbb{R}^n} S^\alpha_\varepsilon(t, y-e)\left(\partial_{e_i}\partial_{e_j}\psi_{0,\epsilon}(e)de\right.$$
$$+ \int_0^s \int_{\mathbb{R}^n} S^\alpha_\varepsilon(s-\tau, y-e)\left(\partial_{e_i}\partial_{e_j}v_\varepsilon(e)\psi_\epsilon(\tau,e)\right.$$
$$+ \partial_{e_j}v_\epsilon(e)\partial_{e_t}\psi_\epsilon(\tau,e)$$
$$+ \partial_{e_t}v_\epsilon(e)\partial_{e_j}\psi_\epsilon(\tau,e)$$
$$\left.+v_\epsilon(e)\partial_{e_i}\partial_{e,}\psi_\epsilon(\tau,e)\right) ded\tau,$$

thus,

$$\left\|\partial_{y_i}\partial_{y_j}\psi_\varepsilon(s,.)\right\|_\infty \leq \|S^\alpha_\epsilon(s,y-.)\|_1 \left\|\partial_{e_i}\partial_{e_j}\psi_{0,\epsilon}(.)\right\|_\infty$$
$$+ \int_0^s \|S^\alpha_\epsilon(s-\tau,y-.)\|_1 \left(\left\|\partial_{e_j}\partial_{ej}v_\epsilon(.)\right\|_\infty \|\psi_\epsilon\|_\infty\right.$$
$$+ \left\|\partial_{e_j}v_\epsilon(.)\right\|_\infty \left\|\partial_{e_j}\psi_\epsilon\right\|_\infty$$
$$\left.+ \|\partial_{e_i}v_\epsilon(.)\|_\infty \left\|\partial_{e_j}\psi_\epsilon\right\|_\infty + \|v_\epsilon(.)\|_\infty \left\|\partial_{e_i}\partial_{e_j}\psi_\epsilon(\tau,.)\right\|_\infty\right) d\tau.$$

We obtain

$$\left\|\partial_{e_i}\partial_{y_j}\psi_\epsilon(s,.)\right\|_\infty \leq R\left(\|\ln\epsilon\|^{r_1(m+2)} + \|\ln\epsilon\|^{r_2(m+1)}\|\psi_\epsilon\|_\infty\right.$$
$$\left.+ \|\ln\epsilon\|^{r_2(m+1)}\|\partial_{e_i}\psi_\epsilon\|_\infty + \|\ln\epsilon\|^{r_2(m+1)}\left\|\partial_{e_j}\psi_\epsilon\right\|_\infty\right)$$
$$+ R\|\ln\epsilon\|^{r_2 m}\int_0^s \left\|\partial_{e_i}\partial_{e_j}\psi_\epsilon(\tau,.)\right\|_\infty d\tau.$$

By Gronwall's inequality, we have

$$\left\|\partial_{e_j}\partial_{y_j}\psi_\epsilon(s,.)\right\|_\infty \leq R\left(\|\ln\epsilon\|^{r_1(m+2)} + \|\ln\epsilon\|^{r_2(m+1)}\|\psi_\epsilon\|_\infty\right.$$
$$+ \|\ln\epsilon\|^{r_2(m+1)}\|\partial_{e_i}\psi_\epsilon\|_\infty$$
$$\left.+\|\ln\epsilon\|^{r_2(m+1)}\|\partial_{ej}\psi_\epsilon\|_\infty\right)\exp\left(RT\|\ln\epsilon\|^{r_2 m}\right).$$

Afterward, there is a positive number N such that

$$\left\|\partial_{y_i}\partial_{y_j}\psi_\epsilon(s,.)\right\|_\infty \leq R\epsilon^{-N}.$$

**Uniqueness**

Let's assume there are two solutions $\psi_{1,\epsilon}(s,.)$ and $\psi_{2,\epsilon}(s,.)$ for problem 10. Consequently, we obtain

$$\frac{1}{i}\partial^\alpha_s\left(\psi_{1,\epsilon}(s,y) - \psi_{2,\epsilon}(s,y)\right) - \Delta\left(\psi_{1,\epsilon}(s,y) - \psi_{2,\epsilon}(s,y)\right)$$

$$+v_\epsilon(y)\left(\psi_{1,\epsilon}(s,y) - \psi_{2,\epsilon}(s,y)\right) = N_\epsilon(s,y),$$

$$\psi_{1,\epsilon}(0,y) - \psi_{2,\epsilon}(0,y) = N_{0,\epsilon}(y),$$

where $N_\epsilon(s,y) \in \mathcal{N}\left(\mathbb{R}^+ \times \mathbb{R}^n\right), N_{0,\epsilon}(y) \in \mathcal{N}\left(\mathbb{R}^n\right)$.
Then

$$\begin{aligned}\psi_{1,\epsilon}(s,y)-\psi_{2,\epsilon}(s,y) &= \int_{\mathbb{R}^n} S_\epsilon^\alpha(s,y-e)N_{0,\epsilon}(e)de \\ &+ \int_0^s \int_{\mathbb{R}^n} S_\epsilon^\alpha(s-\tau,y-e)v_\epsilon(e)\left(\psi_{1,\epsilon}(\tau,e)-\psi_{2,\epsilon}(\tau,e)\right)ded\tau \\ &+ \int_0^s \int_{\mathbb{R}^n} S_\epsilon^\alpha(s-\tau,y-e)N_\epsilon(\tau,e)ded\tau\end{aligned}$$

which leads to

$$\begin{aligned}\left\|\psi_{1,\epsilon}(s,.)-\psi_{2,\epsilon}(s,.)\right\|_\infty \le& \left\|S_\epsilon^\alpha(s,y-.)\right\|_1 \left\|N_{0,\epsilon}(.)\right\|_\infty + \left\|S_\epsilon^\alpha(s,y-.)\right\|_1 \\ &\times \int_0^s \left\|v_\epsilon(.)\right\|_\infty \left\|\psi_{1\epsilon}(\tau,.)-\psi_{2\epsilon}(\tau,.)\right\|_\infty d\tau \\ &+ \left\|S_\epsilon^\alpha(s,y-.)\right\|_1 \left\|N_\epsilon(\tau,.)\right\|_\infty .\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\psi_{1,\epsilon}(s,.)-\psi_{2,\epsilon}(s,.)\right\|_\infty \le& C\left(\left\|N_{0,\epsilon}(.)\right\|_\infty + \left\|N_\epsilon(\tau,.)\right\|_\infty\right) \\ &+ C\left\|v_\epsilon(.)\right\|_\infty \int_0^s \left\|\psi_{1,\epsilon}(\tau,.)-\psi_{2,\epsilon}(\tau,.)\right\|_\infty d\tau.\end{aligned}$$

By Gronwall's inequality, we have

$$\begin{aligned}\left\|\psi_{1,\epsilon}(s,.)-\psi_{2,\epsilon}(s,.)\right\|_\infty \le& C\left(\left\|N_{0,\epsilon}(\cdot)\right\|_\infty + \left\|N_\epsilon(\tau,.)\right\|_\infty\right) \\ &\times \exp\left(CT\left\|v_\epsilon(.)\right\|_\infty\right).\end{aligned}$$

Which prove that

$$\left\|\psi_{s,\epsilon}(t,.)-\psi_{2,\epsilon}(s,.)\right\|_\infty \le C\epsilon^q, \text{ for all q in } \mathbb{N}.$$

Consequently, it can be concluded that problem (10) possesses one solution within the space $X$. $\mathcal{G}\left(\mathbb{R}^+ \times \mathbb{R}^n\right)$. □

## 7. Conclusion

In conclusion, this paper investigates time-fractional Schrödinger equations with singular potentials, including Dirac functions and higher-order singularities. By employing the Gronwall lemma and Laplace transforms, the existence and uniqueness of the integral solution within Colombeau's algebra are established. Additionally, the existence of distribution solutions for certain classes of these equations is demonstrated. We aim to enhance the equation by integrating a numerical component, which will allow for more precise and detailed analysis. This development will enable us to solve complex quantum mechanical problems with greater accuracy and efficiency, paving the way for significant advancements in our research.

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