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NUMERICAL SOLUTION OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION BY COLLOCATION PELL-LUCAS POLYNOMIAL METHOD

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Abstract. The objective of this work is to solve the fractional integro-differential equation using the Pell-Lucas collocation method. In this approach, the fractional derivative is considered in the Caputo sense. The proposed approach for dealing with the problem is the collocation method based on the Pell-Lucas polynomials to obtain an approximation of the solution. The operational matrix of the Caputo derivative, based on the Pell-Lucas polynomials, is derived and utilized in the solution process. By discretizing the Fredholm integral term and the corresponding conditions in terms of the Pell-Lucas polynomials, the original problem is transformed into a system of algebraic equations, which can be solved numerically. The error analysis has been investigated and error estimation of the Pell-Lucas collocation method is studied for the considered problem. Some illustrative examples are investigated to show the accuracy and applicability of the proposed method.

Keywords: Pell-Lucas polynomials, Caputo derivative, Fractional integro-differential eqaution, Operational matrix, Collocation method.

AMS Subject Classification: 26A33, 45J05, 65L60

1. Introduction

The history of fractional calculus, in which derivatives and integrals of fractional order are defined and investigated, is as old as the classical calculus (see [9,11]). In the past decades, there has been a growing prevalence of using fractional order differential and integral operators in mathematical models [12,13]. Fractional operators are a generalization of classical calculus concerned with operations of integration and differentiation of non-integer orders [11]. The mathematical representation of numerous natural and engineering processes depends extensively on fractional calculus [14]. In [16], some of the applications of fundamental issues in statistical and continuum mechanics are investigated. The use of fractional calculus in the modeling of engineering and physical phenomena to solve

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real-world problems can be found in [17,18,39]. In [18], the Hilfer and Caputo fractional derivatives are explained physically in terms of the random motion of a particle traveling on the real line at Poisson paced times with finite velocity. A number of researchers have investigated differential equations with fractional operators in their research, particularly in [19,26].

The theory of integro-differential equations (IDEs) is a significant area of study within the theory of differential and integral equations. The study of IDEs is mostly driven by practical applications. Some scientific and engineering phenomena, like the theory of dynamical systems with automatic control [20], the kinetic theories of rarefied gases, plasma, radiation transfer, coagulation, the study of stochastic processes with jumps, and more specifically of Levy processes, which are expressed by IDEs [23], are modeled in a significant manner by the inclusion of integral terms in differential equations.

Additional comprehensive details and a literature pertaining to the theory of fractional integro-differential equations (FIDEs), including the Hilfer and Caputo fractional derivatives, can be found in [24]. In conventional models, several kinds of fractional integro-differential equations have been developed, and there has been considerable interest in creating numerical approaches for their solutions [25,26]. Numerous processes in the applied sciences, including physics, chemistry, economics, control theory, mechanics, biology, and engineering, are modeled by fractional integro-differential equations [27,33]. The study of FIDEs has attracted attention in recent years. In [30], several significant findings about the corresponding inequalities of fractional integro-differential equations have been shown. Fractional integro-differential equations are solved numerically using a variety of techniques, such as the collocation method [31], the Adomian decomposition method [32], the variational iteration method and homotopy perturbation method [29,34], wavelets [35,36], operational Tau method [37] and others.

We will approximate the solution of time fractional integro-differential equation in this paper using the Pell-Lucas polynomials as the basis in conjunction with collocation approach to solve numerically the proposed problem. To the best of our knowledge, there are still a few works to be done with Pell-Lucas polynomials for FIDEs. A family of polynomials known as Pell-Lucas polynomials is connected to both Lucas and Pell numbers [38]. Numerous authors have explored the properties and uses of these polynomials, developing strategies for solving differential equations (see, for instance, [28,38]) and the references therein. In this study, we use generalized Lucas polynomials to solve the fractional integro-differential equation with appropriate conditions. As utilizing the use of operational matrices is an effective strategy for handling various differential equations, we derive the operational matrices for fractional derivatives and integral terms using Pell-Lucas polynomials. Because of the significance of the spectral approach, we use the collocation spectral method in our research in order to evaluate numerical solutions. The major contribution of this study is to develop a numerical method for solving the fractional integro-differential equation (FIDE) problem. In order to obtain the desired results, the Pell-Lucas operational matrices of derivatives in integer and fractional orders are introduced and employed together with collocation method. By implementing Pell-Lucas polynomials approximations and the operational matrices of fractional derivative and integral term the original problem has been reduced to a set of algebraic equations. The numerical solution of FIDE can be obtained by solving this system of algebraic equations at collocation points.

In this study, we consider the numerical solution of the following linear fractional integro-differential equation:

$$^{C}D_x^{\alpha}u(x) = g(x) + \int_0^1 K(x,t)u(t)dt, \quad 0 \le x \le 1, \tag{1}$$

with the following supplementary conditions:

$$u^{(i)}(0) = \delta_i, \quad n-1 < \alpha \leq n, \quad i = 0, 1, \dots, n-1, \quad n \in \mathbb{N}. \tag{2}$$

where ${}^{C}D_x^{\alpha} u(x)$ indicates the fractional derivative of order α of $u(x)$ in Caputo sense which is defined in the next section. The functions $g(x)$ and $K(x,t)$ are known given functions and $u(x)$ is the unknown function to be determined.

This paper is organized as follows. In Section 2, some definitions of fractional and integral operators are given. Some significant points about the Pell-Lucas polynomials are mentioned in Section 3. Operational matrices of derivatives for Pell-Lucas polynomials has been provided in Section 4. In Section 5, the resultant linear system by method of the solution has been introduced. Additionally, the error analysis of the proposed technique has been studied in Section 6. In Section 7, some numerical examples are discussed to show that the efficiency of the proposed method. Conclusions are drawn in Section 8.

2. Fractional and integral operators

In this section some basic definitions to fractional calculus that will be required in subsequence sections are introduced.

Definition 2.1. *([21]) A real function $u(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $u(x) = x^p u_1(x)$, where $u_1(x) \in C[0,1)$.*

Definition 2.2. *([21]) A function $u(x)$, $x > 0$, is said to be in the space C_μ^m, $m \in \mathbb{N} \cup \{0\}$, if $u^{(m)} \in C_\mu$.*

Definition 2.3. *The left sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $u \in C_\mu, \mu \geq -1$, is defined as [9]:*

$$I^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} u(\tau) d\tau, & \alpha > 0,\ x > 0, \\ u(x), & \alpha = 0. \end{cases} \tag{3}$$

Definition 2.4. *Let $u \in C_{-1}^m, m \in \mathbb{N} \cup \{0\}$, then the Caputo fractional derivative of $u(x)$ is defined as [3,4]:*

$${}^{C}D_x^{\alpha} u(x) = \begin{cases} I^{m-\alpha} u^{(m)}(x), & m-1 < \alpha \leq m,\ \ m \in \mathbb{N}, \\ \frac{d^\alpha u(x)}{dx^\alpha}, & \alpha = m. \end{cases} \tag{4}$$

Hence, we have the following propertie:

$${}^{C}D_x^{\alpha} x^\gamma = \begin{cases} 0, & \gamma \in \mathbb{N}_0,\ \gamma < \lceil \alpha \rceil, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, & \gamma \in \mathbb{N}_0,\ \gamma \geq \lceil \alpha \rceil, \end{cases}$$

where $\lceil \alpha \rceil$ denoted the smallest integer greater than or equal to α and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

3. The Pell-Lucas polynomials

Polynomials play a key role as an important tools in numerical methods for solving practical problems. Due to the characteristics of the Pell-Lucas polynomials, these polynomials are of interest for finding approximate solutions to problems of fractional and integer order differential equations [5]. Some interesting properties and results for Pell-Lucas polynomials are presented in [1,2].

The Pell-Lucas polynomials $\phi_m(x)$ are defined by

$$\phi_m(x) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m\Gamma(m-k)}{\Gamma(k+1)\Gamma(m-2k+1)} (2x)^{m-2k}, \tag{5}$$

where $\lfloor \frac{m}{2} \rfloor$ denotes greatest integer less that or equal to $\frac{m}{2}$. Also, Pell-Lucas polynomials, $\phi_m(x)$, can be produced by adopting the subsequent recurrence relation

$$\phi_{m+1}(x) = 2x\phi_m(x) + \phi_{m-1}(x), \quad m \geq 2,\ x \in \mathbb{R},$$

with the first two known Pell-Lucas polynomials $\phi_0(x) = 2$, $\phi_1(x) = 2x$ [10,15].
The derivative of the Pell-Lucas polynomials $\phi_m(x)$, which is very important for this study, is defined recursively by:

$$\phi'_m(x) = 2x\phi'_{m-1}(x) + \phi'_{m-2}(x) + 2\phi_{m-1}(x), \quad m \geq 2, \tag{6}$$

where $\phi'_0(x) = 0$ and $\phi'_1(x) = 2$.

For more important features to the Pell-Lucas polynomials, Horadam and mahon Bro, Horadam et al [1,2], can be examined.

Theorem 3.1. *The power function x^k can be expressed in terms of the Pell-Lucas polynomials according to the following:*

$$x^k = \frac{1}{2^k} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^i \Gamma(k+1)\xi_{k-2i}}{\Gamma(i+1)\Gamma(k-i+1)} \phi_{k-2i}(x), \tag{7}$$

where

$$\xi_{k-2i} = \begin{cases} \frac{1}{2}, & i = \frac{k}{2}, \\ 1, & i < \frac{k}{2}. \end{cases} \tag{8}$$

Proof. Equation (7) can be easily proved with the aid of [6]. □

Theorem 3.2. *The first derivative of $\phi_m(x)$ in terms of Pell-Lucas polynomials can be represented as follows:*

$$\frac{d}{dt}\phi_m(t) = 2m \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \xi_{m-2i-1} \phi_{m-2i-1}(t), \qquad m > 2. \tag{9}$$

Proof. With the help of [6], equation (9) can be obtained. □

The representation of Pell-Lucas polynomial in equation (5) in terms of power functions x^i can be rewritten as follows:

$$\phi_m(x) = m \sum_{i=0}^{m} \frac{2^{i+1}\Gamma(\frac{m+i+2}{2})\delta_{m+i}}{(m+i)\Gamma(i+1)\Gamma\left(\frac{m-i+2}{2}\right)} x^i, \quad m \geq 1, \tag{10}$$

where

$$\delta_z = \begin{cases} 1, & z \text{ even}, \\ 0, & z \text{ odd}. \end{cases} \tag{11}$$

The equivalent representation of equation (7) in terms of Pell-Lucas polynomials is by:

$$x^m = \frac{\Gamma(m+1)}{2^m} \sum_{i=0,\ (i+m)\ \text{even}}^{m} \frac{(-1)^{\frac{m-i}{2}} \xi_i}{\Gamma\left(\frac{m+i+2}{2}\right)\Gamma\left(\frac{m-i+2}{2}\right)} \phi_i(x), \qquad m \geq 1. \tag{12}$$

For more details about Pell-Lucas polynomials, one can see [1,7,8].

4. Operational matrices of derivatives for Pell-Lucas polynomials

In this section, we look into the operational matrices of Pell-Lucas polynomials for both the integer and fractional orders of derivatives.

4.1. **Operational matrices of derivatives of integer order.** A function $u(x) \in L_2(0,1)$, in terms of Pell-Lucas polynomials can be approximated as follows:

$$u(x) = \sum_{i=0}^{\infty} a_i \phi_i(x), \tag{13}$$

where $a_i, i = 0, 1, 2, \dots$ are the series unknown coefficients. The truncated series up to the first N-terms, equation (13) can be rewritten as follows:

$$u(x) \simeq u_N(x) = \sum_{i=0}^{N} a_i \phi_i(x) = \mathbf{A}_N^T \Phi_N(x), \tag{14}$$

where the unknown coefficients vector $\mathbf{A}_N^T$ and vector of Pell-Lucas polynomials $\Phi_N(x)$ are defined by:

$$\mathbf{A}_N^T = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}, \tag{15}$$

and

$$\Phi_N(x) = \begin{bmatrix} \phi_0(x) & \phi_1(x) & \cdots & \phi_N(x) \end{bmatrix}^T. \tag{16}$$

Theorem 4.1. *The operational matrix of first order of the vector $\Phi_N(x)$ can be defined as follows:*

$$\frac{d}{dx}\Phi_N(x) = W^{(1)}\Phi_N(x), \tag{17}$$

where $W^{(1)} = (w_{l,m}^{(1)})$ is the $(N+1) \times (N+1)$ operational matrix of first order derivative. The elements of operational matrix $W^{(1)}$ can be represented explicitly by

$$w_{0\le l,m\le N}^{(1)} = \begin{cases} 2l(-1)^{\frac{l-m+1}{2}}\xi_m, & l > m, (l+m) \text{ odd}, \\ 0, & l \le m, (l+m) \text{ even}. \end{cases} \tag{18}$$

The operational matrix of the mth order derivative of vector $\Phi_N(x)$ can be expressed as:

$$\frac{d^m}{dx^m}\Phi_N(x) = W^{(m)}\Phi_N(x) = (W^{(1)})^m \Phi_N(x), \tag{19}$$

where $n \in \mathbb{Z}^+$. To get more details, please see [2].

4.2. **The Operational matrix of derivative of fractional order.**

Theorem 4.2. *The Caputo fractional derivative of order α of $\Phi_N(x)$ can be represented as follows*

$${}^C D_x^{\alpha} \Phi_N(x) = x^{-\alpha} \boldsymbol{H}^{(\alpha)} \Phi_N(x), \tag{20}$$

where $\boldsymbol{H}^{(\alpha)} = (h_{r,s}^{\alpha})$ is a square matrix of $(N+1) \times (N+1)$ dimensional that represents the Pell-Lucas polynomials operational matrix of derivative in the Caputo sense of fractional order α and can be expressed as follows

$$\boldsymbol{H}^{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_\alpha(\lceil\alpha\rceil, 0) & \zeta_\alpha(\lceil\alpha\rceil, 1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_\alpha(r, 0) & \cdots & \zeta_\alpha(r, r) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_\alpha(N, 0) & \zeta_\alpha(N, 1) & \zeta_\alpha(N, r) & \cdots & \zeta_\alpha(N, N) \end{pmatrix}_{(N+1)\times(N+1)}, \tag{21}$$

where the elements of $\boldsymbol{H}^{(\alpha)}$ are given by

$$h^{\alpha}_{r,s} = \begin{cases} \zeta_\alpha(r,s), & r \geq s, \quad r \geq \lceil \alpha \rceil, \quad s = 0,1,\ldots,r, \\ 0, & r < s, \quad r < \lceil \alpha \rceil, \end{cases} \tag{22}$$

where

$$\zeta_\alpha(r,s) = r \sum_{k=\lceil \alpha \rceil, k-s+2>0}^{r} \frac{(-1)^{\frac{k-s}{2}} \delta_{r+k}\delta_{s+k}\xi_s \Gamma(k+1)\Gamma\left(\frac{r+k}{2}\right)}{\Gamma\left(\frac{r-k+2}{2}\right)\Gamma\left(\frac{k-s+2}{2}\right)\Gamma\left(\frac{k+s+2}{2}\right)\Gamma(k-\alpha+1)}, \tag{23}$$

Proof. Using Pell-Lucas polynomial definition in equation (10) and definition δ_z in equation (11), the power function x^m can be written as follows:

$$x^m = \frac{\Gamma(m+1)}{2^m} \sum_{i=0}^{m} \frac{(-1)^{\frac{m-i}{2}} \xi_i \delta_{i+m}}{\Gamma\left(\frac{m+i+2}{2}\right)\Gamma\left(\frac{m-i+2}{2}\right)} \phi_i(x), \qquad m \geq 1, \tag{24}$$

The fractional derivative of $\phi_r(x)$ in the Caputo sense of fractional order α in equation (10) is defined as follows:

$${}^{C}D^{\alpha}_{x}\phi_r(x) = r \sum_{i=0}^{r} \frac{2^{i+1}\Gamma(\frac{r+i+2}{2})\delta_{r+i}}{(i+r)\Gamma(i+1)\Gamma(\frac{r-i+1}{2})} \; {}^{C}D^{\alpha}_{x} x^i, \tag{25}$$

Now, by using equation (4), we write equation (25) for values greater than $\lceil \alpha \rceil$ as follows

$${}^{C}D^{\alpha}_{x}\phi_r(x) = r x^{-\alpha} \sum_{i=\lceil \alpha \rceil}^{r} \frac{2^{i+1}\Gamma(\frac{r+i+2}{2})\delta_{r+i}}{(i+r)\Gamma(\frac{r-i+1}{2})\Gamma(i-\alpha+1)} x^i, \tag{26}$$

By changing the index from i to k, we have

$${}^{C}D^{\alpha}_{x}\phi_r(x) = r x^{-\alpha} \sum_{k=\lceil \alpha \rceil}^{r} \frac{2^{k+1}\Gamma\left(\frac{r+k+2}{2}\right)\delta_{r+k}}{(r+k)\Gamma\left(\frac{r-k+2}{2}\right)\Gamma(k-\alpha+1)} x^k, \tag{27}$$

By replacing x^k of equation (24) in equation (27) and using the properties of the Gamma function, we will have

$${}^{C}D^{\alpha}_{x}\phi_r(x) = x^{-\alpha} \sum_{s=0}^{r} \left(r \sum_{k=\lceil \alpha \rceil}^{r} \frac{(-1)^{\frac{k-s}{2}} \delta_{r+k}\delta_{s+k}\xi_s \Gamma(k+1)\Gamma(\frac{r+k}{2})}{\Gamma(\frac{r-k+2}{2})\Gamma(\frac{k+s+2}{2})\Gamma(\frac{k-s+2}{2})\Gamma(k-\alpha+1)} \right) \phi_s(x), \tag{28}$$

for $k - s + 2 > 0$. Then equation (28) can be rewritten as

$${}^{C}D^{\alpha}_{x}\phi_r(x) = x^{-\alpha} \sum_{s=0}^{r} \zeta_\alpha(r,s)\phi_s(x), \tag{29}$$

where $\zeta_\alpha(r,s)$ is given in (23). Equation (29) can also be represented as the vector form

$${}^{C}D^{\alpha}_{x}\phi_r(x) = x^{-\alpha}[\zeta_\alpha(r,0), \zeta_\alpha(r,1), \ldots, \zeta_\alpha(r,r), 0, 0, \ldots, 0]\Phi(x), \quad \lceil \alpha \rceil \leq r \leq N, \tag{30}$$

Moreover, we can write

$${}^{C}D^{\alpha}_{x}\phi_r(x) = x^{-\alpha}[0, 0, \ldots, 0], \qquad 0 \leq r \leq \lceil \alpha \rceil - 1. \tag{31}$$

Combining equations (30) and (31), the desired result is obtained. □

Theorem 4.3. *The matrix representation of fractional derivative of order α of approximate solution in equation* (14) *in Caputo sense has the following matrix structure:*

$$ {}^{C}D_x^{\alpha} u_N(x) = x^{-\alpha}\Phi_N(x)\left(\boldsymbol{H}^{(\alpha)}\right)^T \boldsymbol{A}_N, \tag{32}$$

where, $\Phi_N(x)$ *and* $\boldsymbol{A}_N(x)$ *are as in equations* (15)-(16) *and* $\boldsymbol{H}^{(\alpha)}$ *in equation* (21).

Proof. By taking the Caputo fractional derivative of order α of approximate solution (15) and by equation (21) the desired result is obtained. □

Theorem 4.4. *The matrix representation of the expression* $K(x,t)$ *in equation* (1) *is given by*

$$[K(x,t)] = \boldsymbol{X}_N^T(x)\boldsymbol{K}\boldsymbol{X}_N(t), \qquad \boldsymbol{K} = [k^T_{p_1,p_2}], \qquad p_1,p_2 = 0,1,\dots,N, \tag{33}$$

where

$$k^T_{p_1,p_2} = \frac{1}{p_1!p_2!}\frac{\partial^{p_1+p_2}K(0,0)}{\partial x^{p_1}\partial t^{p_2}}; \qquad p_1,p_2 = 0,1,\dots,N, \tag{34}$$
$$\boldsymbol{X}_N^T(x) = \begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix}, \qquad \boldsymbol{X}_N(t) = \begin{bmatrix}1 & t & t^2 & \cdots & t^N\end{bmatrix}^T,$$

and the dimension of $\boldsymbol{X}_N^T(x)$ *is* $1\times(N+1)$.

Proof. The truncated Maclaurin series of the kernel function $K(x,t)$ can be represented as

$$K(x,t) = \sum_{p_1=0}^{N}\sum_{p_2=0}^{N} k^T_{p_1p_2}x^{p_1}t^{p_2}, \tag{35}$$

subsequently, the vector $\mathbf{X}_N^T(x)$ is multiplied by the matrix $\mathbf{K}$ from the right by using (34). Finally, by multiplying this expression by the vector $\mathbf{X}_N(t)$ from the right, the relation (33) is obtained. □

Theorem 4.5. *The matrix representation for Fredholm integral equation in equation* (1) *is expressed as follows:*

$$\int_0^1 \boldsymbol{K}(x,t)u(t)dt = \boldsymbol{X}_N^T(x)\boldsymbol{K}\boldsymbol{N}\boldsymbol{A}_N, \tag{36}$$

where

$$\boldsymbol{N} = \int_0^1 \boldsymbol{X}_N(t)\Phi_N(t)dt.$$

Here, $\boldsymbol{X}_N^T(x)$*,* $\boldsymbol{K}$*, and* $\boldsymbol{X}_N(t)$ *are as in Theorem 4.4 and* $\Phi_N(t)$ *is as in* (16).

Proof. By utilizing the (33) and with and replacement of $u(t) \cong u_N(t) = \Phi_N^T(t)\mathbf{A}_N$ in equatoin (36), the matrix representation of $\int_0^1 \mathbf{K}(x,t)u(t)dt$ is defined as:

$$\int_0^1 \mathbf{X}_N^T(x)\mathbf{K}\mathbf{X}_N(t)\Phi_N(t)\mathbf{A}_N dt = \mathbf{X}_N^T(x)\mathbf{K}\left\{\int_0^1 \mathbf{X}_N(t)\Phi_N(t)dt\right\}\mathbf{A}_N,$$

So, by definition of term $\mathbf{N}$, we obtain the desired result. □

Theorem 4.6. *We assume that the approximate solution of the problem* (1) *and* (2) *is sought in the form* (15)*. Then, the following matrix representation is obtained.*

$$\{x^{-\alpha}\Phi_N(x)(\boldsymbol{H}^{(\alpha)})^T - \boldsymbol{X}_N^T(x)\boldsymbol{K}\boldsymbol{N}\}\boldsymbol{A}_N = \boldsymbol{G}(x), \tag{37}$$

where $\boldsymbol{Z} = x^{-\alpha}\Phi_N(x)(\boldsymbol{H}^{(\alpha)})^T - \boldsymbol{X}_N^T(x)\boldsymbol{K}\boldsymbol{N}$.

Proof. By substituting the matrix representations (32) and (36) in (1), the desired result is obtained. □

Theorem 4.7. *The matrix form of the conditions* (2) *is given by*

$$\Phi_N^T(0)(W^{(i)})^T \boldsymbol{A}_N = \delta_i, \quad i = 0, 1, \ldots, n-1, \quad n \in \mathbb{N}. \tag{38}$$

Here, $\boldsymbol{A}_N$ *and* $\Phi_N(x)$ *are in equations* (15) *and* (16).

Proof. By inserting $x = 0$ in equations (19) and (14), the desired result is easily obtained. □

5. Method of the Solution

The purpose of this section is to introducing Pell-Lucas collocation technique in uniform collocation points in order to find an approximate solution to the problem (1)-(2). The matrix structures constructed for the fractional derivative and integral part are used to create a system of linear algebraic equations.

We define the uniformly distributed collocation points as

$$x_i = a + \frac{b-a}{N} i, \qquad i = 0, 1, \ldots, N. \tag{39}$$

We assume that the matrix representation for the approximate solution of the problem (1) and (2) is as follows:

$$\mathbf{Z}\mathbf{A}_N = \mathbf{G}, \tag{40}$$

Substituting the collocation points (39) into the expression (40) yields the following set of algebraic equations.

$$\mathbf{Z}(x_i)\mathbf{A}_N = [g(x_i)], \tag{41}$$

Where $\mathbf{K}$ is defiend in equation (33), and

$$\mathbf{G} = [g(x_0), g(x_1), \cdots, g(x_N)]^T, \tag{42}$$

$$\mathbf{N} = \begin{pmatrix} \int_0^1 \mathbf{X}_0(t)\Phi_0(t)dt & \int_0^1 \mathbf{X}_0(t)\Phi_1(t)dt & \cdots & \int_0^1 \mathbf{X}_0(t)\Phi_N(t)dt \\ \int_0^1 \mathbf{X}_1(t)\Phi_0(t)dt & \int_0^1 \mathbf{X}_1(t)\Phi_1(t)dt & \cdots & \int_0^1 \mathbf{X}_1(t)\Phi_N(t)dt \\ \vdots & \vdots & \cdots & \vdots \\ \int_0^1 \mathbf{X}_N(t)\Phi_0(t)dt & \int_0^1 \mathbf{X}_N(t)\Phi_1(t)dt & \cdots & \int_0^1 \mathbf{X}_N(t)\Phi_N(t)dt \end{pmatrix}. \tag{43}$$

Theorem 5.1. *The conditions* (2) *is matrix structure can be represented as follows:*

$$\boldsymbol{U}\boldsymbol{A}_N = \boldsymbol{\Delta}, \tag{44}$$

where $\boldsymbol{U}$ *is a matrix of* $n \times (N+1)$ *dimensional which is defined by*

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_{n-1} \end{pmatrix}, \qquad \mathbf{U}_i = (v_{i,0} \quad v_{i,1} \ \cdots \ v_{i,N}), \quad i = 0, 1, \ldots, n-1, \tag{45}$$

and

$$\boldsymbol{\Delta} = (\delta_0 \ \ \delta_1 \ \ \ldots \ \ \delta_{n-1})^T. \tag{46}$$

Proof. By employing initial conditions (2), and $u_N(x) = \Phi_N^T(x)\mathbf{A}_N$ we get

$$\Phi_N^T(0)\mathbf{A}_N = \delta_0 \Rightarrow (\upsilon_{0,0} \quad \upsilon_{0,1} \ \cdots \ \upsilon_{0,N})\,\mathbf{A}_N = \delta_0,$$

$$\Phi_N^T(0)(W^{(1)})^T\mathbf{A}_N = \delta_1 \Rightarrow (\upsilon_{1,0} \quad \upsilon_{1,1} \ \cdots \ \upsilon_{1,N})\,\mathbf{A}_N = \delta_1,$$

$$\vdots$$

$$\Phi_N^T(0)(W^{(n-1)})^T\mathbf{A}_N = \delta_{n-1} \Rightarrow (\upsilon_{0,0} \quad \upsilon_{0,1} \ \cdots \ \upsilon_{0,N})\,\mathbf{A}_N = \delta_{n-1},$$

or

$$\mathbf{U}_i\mathbf{A}_N = \delta_i, \tag{47}$$

where

$$\mathbf{U}_i = (\upsilon_{i,0} \quad \upsilon_{i,1} \ \ldots \ \upsilon_{i,N}), \qquad i = 0, 1, \ldots, n-1. \tag{48}$$

The matrix structure for the equation (2) can be represented as follows

$$\mathbf{U}\mathbf{A}_N = \mathbf{\Delta}, \tag{49}$$

The desired result is obtained. □

The conditional number can be calculated for the system of equation (41). Adding equation (47) to system (41) yields the generalized matrix system shown below.

$$[\tilde{\mathbf{Z}}; \tilde{\mathbf{G}}]. \tag{50}$$

Solving the system (50) yields the Pell-Lucas coefficient vector $\mathbf{A}_N$. Hence, the Pell-Lucas solution is obtained by substituting these coefficients in equation (13).

6. Error analysis

In this section, the error analysis of the Pell-Lucas collocation method is investigated. For the problems that do not have an analytical solution, the approach for estimating the problem's error is introduced.

Theorem 6.1. *(Upper Bound of Errors). Let $u(x)$ and $u_N(x)$ are the exact and the Pell–Lucas approximate solutions with Nth degree of the equation (1) in domain $[0, l]$,respectively. Let $u_N^{Mac}(x)$ be the truncated Maclaurin series of exact solution $u(x)$ in domain $[0, l]$. The upper bound of absolute error of the proposed method is defined as follows:*

$$\|u(x) - u_N(x)\|_\infty \leq k_N\|\overline{\boldsymbol{A}}_N\|_\infty + \boldsymbol{q}_N\|\boldsymbol{A}_N\|_\infty + \frac{1}{(N+1)!}\|\left(x\frac{\partial}{\partial x}\right)^{N+1} u(c_x)\|_\infty, \tag{51}$$

where $\overline{\boldsymbol{A}}_N$ shows the coefficient matrix of $u_N^{Mac}(x)$, $k_N = \|\boldsymbol{X}_N^T(x)\|_\infty$, $\boldsymbol{q}_N = \|\Phi_N^T(x)\|_\infty$ and $0 \leq c_x \leq l$.

Proof. By assuming the approximate solution in the form $u_N(x) = \Phi_N^T(x)\mathbf{A}_N$. The Maclaurin expansion with Nth degree of the exact solution is denoted by $u_N^{Mac}(x)$ and is represented as following

$$u_N^{Mac}(x) = \sum_{i=0}^{N} \frac{1}{i!}\frac{d^i}{dx^i}u(x)|_{x=0}x^i.$$

$u_N^{Mac}(x)$ can be expressed by

$$u_N^{Mac}(x) = \mathbf{X}_N^T(x)\overline{\mathbf{A}}_N, \tag{52}$$

Here, $\mathbf{X}_N^T(x)$ is a vector as follows:

$$\mathbf{X}_N^T(x) = \begin{bmatrix}1 & x & x^2 & \ldots & x^N\end{bmatrix} \tag{53}$$

Also, here $\overline{\mathbf{A}}_N$ the vector of MacLauren expansion coefficients. Now, by utilizing the triangle inequality and the Maclaurin expansion $u_N^{Mac}(x)$, we have

$$\begin{aligned}\|u(x)-u_N(x)\|_\infty &= \|u(x)-u_N^{Mac}(x)+u_N^{Mac}(x)-u_N(x)\|_\infty \\ &\leq \|u(x)-u_N^{Mac}(x)\|_\infty+\|u_N^{Mac}(x)-u_N(x)\|_\infty. \end{aligned}\tag{54}$$

We can write

$$\begin{aligned}\|u_N^{Mac}(x)-u_N(x)\|_\infty &= \|\mathbf{X}_N^T(x)\overline{\mathbf{A}}_N-\Phi_N^T(x)\mathbf{A}_N\|_\infty \\ &\leq \|\mathbf{X}_N^T(x)\|_\infty\|\overline{\mathbf{A}}_N\|_\infty+\|\Phi_N^T(x)\|_\infty\|\mathbf{A}_N\|_\infty, \end{aligned}\tag{55}$$

where

$$\|\mathbf{X}_N^T(x)\|_\infty = k_N := \begin{cases} 1, & \text{if} \quad l\leq 1, \\ l^N, & \text{if} \quad l>1, \end{cases}\tag{56}$$

$\mathbf{q}_N$ showes the values of $\|\Phi_N^T(x)\|_\infty$ in $[0,l]$. So, equation (55) can be rewritten as follows

$$\|u_N^{Mac}(x)-u_N(x)\|_\infty \leq k_N\|\overline{\mathbf{A}}_N\|_\infty+\mathbf{q}_N\|\mathbf{A}_N\|_\infty.\tag{57}$$

In the following, the MacLaurin expansion of the exact solution with Nth degree is expressed as

$$u(x)=u(0)+xu'(0)+\frac{1}{2!}\left(x\frac{d}{dx}\right)^2u(0)+\cdots+\frac{1}{N!}\left(x\frac{d}{dx}\right)^Nu(0),\tag{58}$$

and its remainder term is

$$\frac{1}{(N+1)!}\left(x\frac{d}{dx}\right)^{N+1}u(c_x), \qquad 0\leq c_x\leq l.\tag{59}$$

Hence, we can write

$$\|u(x)-u_N^{Mac}(x)\|_\infty=\frac{1}{(N+1)!}\|\left(x\frac{d}{dx}\right)^{N+1}u(c_x)\|_\infty, \qquad 0\leq x\leq l.\tag{60}$$

Finally, equation (54) is rewritten as follows

$$\|u(x)-u_N(x)\|_\infty \leq k_N\|\overline{\mathbf{A}}_N\|_\infty+\mathbf{q}_N\|\mathbf{A}_N\|_\infty+\frac{1}{(N+1)!}\|\left(x\frac{d}{dx}\right)^{N+1}u(c_x)\|_\infty,$$

where $0\leq x\leq l$. Thus, the proof is completed. □

Theorem 6.2. *Suppose $u(x)$ and $u_N(x)$ are the exact and approximate solution of problem (1)-(2), respectively. For the approximate solution $u_N(x)$, the residual function $R_N(x)$ is defined as follows*

$$R_N(x)=L[u_N(x)]-g(x).\tag{61}$$

In this case, the following problem is satisfied by the error function $e_N(x)=u(x)-u_N(x)$

$$\begin{cases} {}^CD_x^\alpha(e_N)(x)-\int_0^1K(x,t)e_N(t)dt=-R_N(x), & \\ (e_N)^{(i)}(0)=0, & i=0,1,\dots,n-1. \end{cases}\tag{62}$$

Proof. Based on the definition of $u(x)$ and $u_N(x)$, equation (1) can be represented in operator form as follows:

$$L[u(x)]=g(x), \qquad L[u(x)]={}^CD_x^\alpha u(x)-\int_0^1K(x,t)u(t)dt.\tag{63}$$

The approximate solution $u_N(x)$, satisfies the equations (1)-(2), so we have

$$L[u_N(x)] = {}^C D_x^{\alpha} u_N(x) - \int_0^1 K(x,t) u_N(t) dt, \tag{64}$$

and

$$u_N^{(i)}(0) = \delta_i(x), \quad x \in [0, l]. \tag{65}$$

Subtracting equations (64)-(65) from equations (1)-(2), the desired problem is obtained as

$$\begin{cases} {}^C D_x^{\alpha}(e_N)(x) - \int_0^1 K(x,t) e_N(t) dt = -R_N(x), & \\ e_N^{(i)}(0) = 0, & i = 0, 1, \ldots, n-1. \end{cases} \tag{66}$$

and the proof is completed. □

7. Numerical Resutls

In this section some numerical examples have been investigated to study the efficiency of the proposed method to solve fractional integro-differential equation. The MATLAB codes for the suggested approach in previous sections are written in MATLAB (R2016A), and all codes are executed on a personal computer.

The exact and approximate solution of the proposed method are denoted by $u(x)$ and $u_N(x)$, respectively. Furthermore, the actual error and the estimated error function of the problem are represented by $\|u(x) - u_N(x)\|$ and $e_N(x)$. The L_2 and L_∞ error norms are two measure criteria to compare the exact and approximate solution which are defined by

$$L_2 = \|u(x) - u_N(x)\|_2 = \left(\int_a^b (u(x) - u_N(x))^2 dx \right)^{\frac{1}{2}},$$

and

$$L_\infty = \|u(x) - u_N(x)\|_\infty = max\{|u(x) - u_N(x)|\}, \qquad 0 \le x \le 1.$$

Example 7.1. *In this example, we consider the following fractional integro-differential equation*

$${}^C D_x^{\frac{5}{3}} u(x) - \int_0^1 (xt + x^2 t^2) u(t) dt = \frac{3\sqrt{3}\Gamma(\frac{2}{3}) x^{\frac{1}{3}}}{\pi} - \frac{x^2}{5} - \frac{x}{4}, \qquad 0 \le x \le 1. \tag{67}$$

subject to $u(0) = 0, u'(0) = 0$ with the exact solution $u(x) = x^2$ [21].

The numerical results for exact solution $u(x)$ and approximate solution $u_N(x)$ are displayed in the Table 1. These values are obtained for values $N = 4$ and $\alpha = \frac{5}{3}$. Good results have been obtained for value $N = 4$. The values of L_2 and L_∞ have been reported in 1. Figure 1 displays the exact and approximate solutions for Example 7.1.

Example 7.2. *Consider the following fractional integro-differential equation*

$${}^C D_x^{\frac{1}{2}} u(x) - \int_0^1 xt u(t) dt = \frac{\frac{8}{3} x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12}, \qquad 0 \le x \le 1. \tag{68}$$

with the analytical solution $u(x) = x^2 - x$. The initial condition $u(0) = 0$ can be extracted from the exact solution.

Similarly as in example 7.1, the numerical results for the exact solution $u(x)$ and approximate solution $u_N(x)$ are calculated and reported in Table 2. The actual error and estimated error for different values of N are listed in Table 2. The values of L_2 and L_∞ error norms for different values of $N = 4$ and $N = 7$ have been reported in Table 2. Figure 2 displays the exact and approximate solutions for Example 7.2.

TABLE 1. Numerical results of the Example 7.1. for $N = 4$ and $\alpha = 5/3$

x	Exact Solution	Approximate Solution	Actual Error	Estimated Error
0.00	$00.0000e+000$	$-63.5605e-024$	$63.5605e-024$	$0.0000e+000$
0.10	$10.0000e-003$	$10.0000e-003$	$2.0240e-018$	$-5.7726e-018$
0.20	$40.0000e-003$	$40.0000e-003$	$8.1576e-018$	$-12.3398e-018$
0.30	$90.0000e-003$	$90.0000e-003$	$18.4961e-018$	$-19.5749e-018$
0.40	$160.0000e-003$	$160.0000e-003$	$33.1394e-018$	$-27.2958e-018$
0.50	$250.0000e-003$	$250.0000e-003$	$52.1918e-018$	$-35.2708e-018$
0.60	$360.0000e-003$	$360.0000e-003$	$75.7620e-018$	$-43.2220e-018$
0.70	$490.0000e-003$	$490.0000e-003$	$103.9631e-018$	$-50.8266e-018$
0.80	$640.0000e-003$	$640.0000e-003$	$136.9128e-018$	$-57.7184e-018$
0.90	$810.0000e-003$	$810.0000e-003$	$174.7329e-018$	$-63.4883e-018$
1.00	$1.0000e+000$	$1.0000e+000$	$217.5500e-018$	$-67.6850e-018$
L_2				$342.6191e-018$
L_∞				$217.5500e-018$

TABLE 2. Numerical results of the Example 7.2. for $N = 4$, $N = 7$ and $\alpha = 0.5$.

x	Exact Solution	Approximate Solution	Actual Error	Estimated Error $N = 4$	Estimated Error $N = 7$
0.00	$00.000e+000$	$-352.6483e-039$	$352.6483e-039$	$0.0000e+000$	$0.0000e+000$
0.10	$-90.000e-003$	$-90.0000e-003$	$9.8147e-018$	$236.4940e-021$	$257.7399e-021$
0.20	$-160.000e-003$	$-160.0000e-003$	$19.0828e-018$	$-1.3265e-018$	$-1.2951e-018$
0.30	$-210.000e-003$	$-210.0000e-003$	$27.7504e-018$	$-3.8972e-018$	$-3.8664e-018$
0.40	$-240.000e-003$	$-240.0000e-003$	$35.7711e-018$	$-7.1432e-018$	$-7.1157e-018$
0.50	$-250.000e-003$	$-250.0000e-003$	$43.1059e-018$	$-10.8435e-018$	$-10.8178e-018$
0.60	$-240.000e-003$	$-240.0000e-003$	$49.7234e-018$	$-14.8284e-018$	$-14.8029e-018$
0.70	$-210.000e-003$	$-210.0000e-003$	$55.5994e-018$	$-18.9588e-018$	$-18.9328e-018$
0.80	$-160.000e-003$	$-160.0000e-003$	$60.7174e-018$	$-23.1161e-018$	$-23.0905e-018$
0.90	$-90.000e-003$	$-90.0000e-003$	$65.0682e-018$	$-27.1973e-018$	$-27.1734e-018$
1.00	$000.000e+000$	$-68.6502e-018$	$68.6502e-018$	$-31.1126e-018$	$-31.0903e-018$
L_2				$150.2175e-018$	$150.1211e-018$
L_∞				$68.6502e-018$	$68.6240e-018$

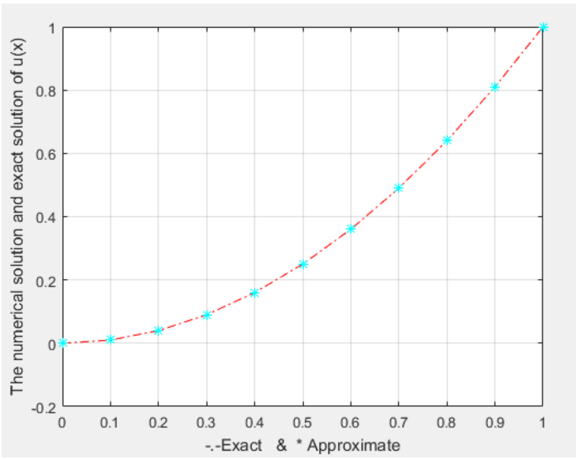

FIGURE 1. The graghs of the exact and approximate solutions in case of $N = 4, \alpha = \frac{5}{3}$, corresponded to Example 7.1.

8. CONCLUSIONS

In this study, a collocation method based on Pell-Lucas polynomials has been introduced for solving the fractional linear Fredholm integro-differential equation problem. The Pell-Lucas collocation method approximates the solution of the FIDE problem with suggested

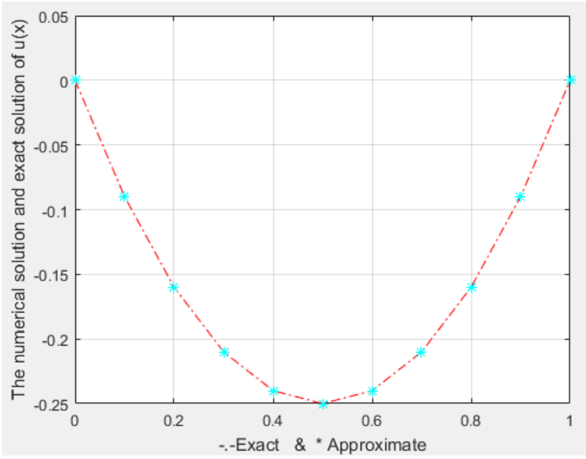

Figure 2. The graghs of the exact and approximate solutions in case of $N = 4, \alpha = 0.5$, corresponded to Example 7.2.

basis functions. The matrix representation of the Caputo fractional derivative, the matrix structure of Fredholm integral term based on the Pell-Lucas polynomials have been derived. The proposed approach transforms the original problem into a matrix structure by using matrix represention of the fractional derivative and Fredholm integral term with Pell-Lucas polynomial basis functions. In order to demonstrate the proposed method's efficiency and applicability, it was applied and studied on some numerical examples with various values of fractional orders and sample points. The agreement between numerical and exact solution is supported by numerical obtained results, which demonstrate the validity and accuracy of the method. Our research encourages the use of this method in the investigation of the existence problems and the stability of the solution for some problems.

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