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ON RELATIVE UNIFORM CONVERGENCE OF FRACTIONAL DIFFERENCE SEQUENCE OF FUNCTION RELATED TO ℓ_p SPACE

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Abstract. In this article, we define the notion of relative uniform convergence of fractional difference sequence of the function space ${}_{ru}m_{\phi}(\Delta^{\alpha}, p)$, where $p \geq 0$. We established many attributes of ${}_{ru}m_{\phi}(\Delta^{\alpha}, p)$, including solidity, symmetry, completeness, convergence-free, sequence algebra, and convex characteristics. The relative uniform fractional difference of $p-$ absolutely summable, bounded, convergent, null sequence of function spaces was also introduced. These are represented by the notations $\ell_p(\Delta^{\alpha}_{ru})$, $\ell_{\infty}(\Delta^{\alpha}_{ru})$, $c(\Delta^{\alpha}_{ru})$, $c_0(\Delta^{\alpha}_{ru})$, and their relationship to the space ${}_{ru}m_{\phi}(\Delta^{\alpha}, p)$ is reviewed.

Keywords: Relative uniform convergence, Fractional difference sequence, Sequence space, Completeness, Convexity.

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1. Introduction

The spaces of all sequences of functions are represented by ω_f and null sequence of functions by $\varnothing_f$. These notations have been used throughout the study.
After introducing the concept of difference sequence space, Kizmaz [14] investigated the difference sequence of function spaces $\ell_{\infty}(\Delta), c(\Delta)$, and $c_0(\Delta)$ in the following ways:

$$X(\Delta) = \{(x_n) \in \omega : (\Delta x_n) \in X\},$$

where $X = \ell_{\infty}, c, c_0$ and $\Delta x_n = x_n - x_{n+1},$ for all $n \in \mathbb{N}$.

The above spaces are Banach spaces with the normed defined by,

$$||x||_{\Delta} = |x_1| + \sup_n |\Delta x_n|.$$

Et and Colak [9] expanded on the idea by adding the spaces $\ell_{\infty}(\Delta^k), c(\Delta^k),$ and $c_0(\Delta^k)$, where $(\Delta^k x_n) = (\Delta^{k-1} x_n - \Delta^{k-1} x_{n+1}),$ for all $n, k \in \mathbb{N}$.

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Later, Et and Colak [9] modified these sequence spaces to the case of integral order m using the operator Δ^m, and that can be expressed in the following manner:

$$\Delta^m x_n = \sum_{k=0}^{m} (-1)^k \frac{m!}{k!(m-k)!} x_{n+k}.$$

Recently, Baliarsingh [1] generalized the above difference operator by introducing the fractional difference operator Δ^α, where

$$\Delta^\alpha x_n = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} x_{n+k}, \text{ for all } n \in \mathbb{N},$$

and Γ denotes the gamma function.

Some generalized concepts on fractional difference operators will be used in this article. The idea of a sequence of functions convergent uniformly relative to a scale function was initially suggested by Moore [15].
Chittenden [3] provided the following definition of the relative uniform convergence:
The value of the inequality

$$|f(x) - f_n(x)| \leq \varepsilon |\mu(x)|$$

holds uniformly in x on the compact domain D for each small positive constant ε. This means that for every $n \geq n_\varepsilon$, there exists an integer n_ε.
The function $\mu(x)$ is the scale function defined on the same compact domain D.
Numerous additional scholars, including Demirci *et. al.* [4], Demirci and Orhan [5], Sahin and Dirik [23], Devi and Tripathy ([6], [7], [8]), as well as others, looked deeper into the idea.
One may refer to the Kamthan and Gupta [13] for more information on the fundamentals of sequence spaces and summability theory. Some other researchers such as Paikray *et al.* [22], Jene *et al.* [11], [12] are worked on summability theory.
In 1960, Sargent [24] proposed the $m(\phi)$ space, which is closely connected to the ℓ_p space. He looked at a few $m(\phi)$ space characteristics. Subsequently, the space $m(\phi, p), p \geq 1$ was created by Tripathy and Sen [26], who examined some of its fundamental features. This space generalises the space $m(\phi)$.
A few geometric characteristics on the convexity of the space ${}_{ru}m_\phi(\Delta^\alpha, p)$ were also established. We established the norm of the space ${}_{ru}m_\phi(\Delta^\alpha, p)$ in this article, which are normed, linear, Banach spaces having an extra property of convexity of the norm. The notion of modular spaces was first presented by the Japanese mathematician Nakano [20]. Numerous other researchers, including Tripathy and Esi ([25]), Musielak and Orlicz ([17], [18]), Musielak [16], Musielak and Wasak [19], and others, also studied modular space.

In view of relative uniform convergence of a sequence of functions, several works have been done by the many researchers for Difference operator under usual convergence. However, we have considered the fractional order difference operator to establish certain new results based on topological and geometrical properties.

2. Definitions & Background

Definition 2.1. *A sequence of functions specified inside a compact domain D of real numbers is called fractional difference convergent of order α relatively uniformly convergent* *on D or **RUFD-Convergent** if for a given $\tilde{\mathcal{F}} > 0$, there exists an integer $n_0 = n_0(\tilde{\mathcal{F}})$ such that the inequality*

$$|\tilde{\zeta}(x) - \Delta^\alpha \zeta_n(x)| < \tilde{\mathcal{F}}|\mu(x)|,$$

where

$$\Delta^\alpha \zeta_n(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} \zeta_{n+k}(x), \text{ for all } n \in \mathbb{N},\ x \in D.$$

The space of all **RUFD-Convergent** denoted by ${}_{ru}\omega(\Delta^\alpha)$.

Example 2.1. *Consider a sequence of functions $(\zeta_n(x))$ defined by*

$$\zeta_n(x) = \begin{cases} \frac{x}{1+n^2}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0, \end{cases}$$

then

$$\Delta^\alpha \zeta_n(x) = \begin{cases} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} \frac{x}{\{1+(n+k)^2\}}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0, \end{cases}$$

for $\alpha = \frac{1}{2}$.

$$\Delta^{\frac{1}{2}} \zeta_n(x) = \frac{(n^2+2n+3)x}{2(n^2+2n+2)(1+n^2)}.$$

It is obvious that $\Delta^{\frac{1}{2}}(\zeta_n(x))$ is uniformly convergent in relation to the scaling function $\mu(x)$, which is defined by

$$\mu(x) = \begin{cases} x, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Hence, the space $(\zeta_n) \in {}_{ru}\omega(\Delta^\alpha)$.

Some **RUFD-Convergent** sequence of function spaces are denoted by $X(\Delta^\alpha_{ru})$ and defined as follows,

$$X(\Delta^\alpha_{ru}) = \{(\zeta_n) \in {}_{ru}\omega(\Delta^\alpha) : (\Delta^\alpha_{ru}\zeta_n) \in \text{X}, \text{ relative uniform w. r. t. } \mu(x), \text{ for all } x \in D\}.$$

In this article, we introduced some **RUFD-Convergent** sequence of function spaces as follows:

$$\ell_p(\Delta^\alpha_{ru}) = \left\{ (\zeta_n) \in {}_{ru}\omega(\Delta^\alpha) : \sum_{n=1}^{\infty} |\Delta^\alpha \zeta_n(x)|^p \leq \tilde{\mathcal{F}}|\mu(x)| \right\},$$

$$\ell_\infty(\Delta^\alpha_{ru}) = \left\{ (\zeta_n) \in {}_{ru}\omega(\Delta^\alpha) : \sup_{n\geq 1} |\Delta^\alpha \zeta_n(x)| \leq \tilde{\mathcal{F}}|\mu(x)| \right\},$$

$$c(\Delta^\alpha_{ru}) = \left\{ (\zeta_n) \in {}_{ru}\omega(\Delta^\alpha) : \lim_{n\to\infty} \frac{|\Delta^\alpha \zeta_n(x)|}{|\mu(x)|} = L \right\},$$

$$c_0(\Delta^\alpha_{ru}) = \left\{ (\zeta_n) \in {}_{ru}\omega(\Delta^\alpha) : \lim_{n\to\infty} \frac{|\Delta^\alpha \zeta_n(x)|}{|\mu(x)|} = 0 \right\}.$$

The spaces mentioned above have a norm value,

$$||\zeta||_{\Delta^\alpha_{ru}} = \sup_{x\in D} |\zeta_1(x)\mu(x)| + \sup_{x\in D, n\in\mathbb{N}} |\Delta^\alpha \zeta_n(x)\mu(x)|.$$

Definition 2.2. *The relative uniform convergence of fractional difference $m(\phi)$ sequence of functions connected to ℓ_p space or* ***RUFD***$_{m(\phi)}$***-convergent*** *is represented by*

$$ {}_{ru}m_{\phi}(\Delta^{\alpha},p)=\left\{(\zeta_n)\in\ {}_{ru}\omega(\Delta^{\alpha}):\sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\zeta_n(x)|^p<\tilde{\mathcal{F}}|\mu(x)|\right\}. $$

The definition of the space ${}_{ru}m_{\phi}(\Delta^{\alpha},p)$ implies that,

1. *If $s=1$ and then ${}_{ru}m_{\phi}(\Delta^{\alpha},p)=\ {}_{ru}\ell_p(\Delta^{\alpha},p)$.*
2. *If $s=n$ and $p=1$ then ${}_{ru}m_{\phi}(\Delta^{\alpha})=\ {}_{ru}\ell_{\infty}(\Delta^{\alpha})$*

We presented a new function space sequence that we named relative uniform convergence of fractional difference. Sequence of functions $n(\phi)$ or ***RUFD***$_{n(\phi)}$***-convergent*** *is shown as*

$$ {}_{ru}n_{\phi}(\Delta^{\alpha},p)=\left\{(\zeta_n)\in\ {}_{ru}\omega(\Delta^{\alpha}):\sup_{\eta_n\in S(\zeta_n),\ x\in D}\left\{\sum_{n=1}^{\infty}|\Delta^{\alpha}\zeta_n(x)|\Delta\phi_n\right\}^p<\tilde{\mathcal{F}}|\mu(x)|\right\}, $$

where $S(\zeta_n)$ represent the rearrangement of (ζ_n).

Definition 2.3. *A sequence space $Z_f\subset\omega_f$ is said to be convergence free if $(\eta_n)\in Z_f$, whenever $(\zeta_n)\in Z_f$ and $\zeta_n(x)=0\implies\eta(x)=0$, on $x\in D$.*

Example 2.2. *The space c_{00} is a convergence-free space.*

Definition 2.4. *A sequence space $Z_f\in\omega_f$ is said to be solid or normal if $(\zeta_n)\in Z_f$ implies $(\eta_n)\in Z_f$, for all (η_n) with $|\eta_n(x)|\leq|\zeta_n(x)|$, for all $n\in\mathbb{N}$ and all $x\in D$.*

Definition 2.5. *A sequence of functions space $Z_f\subset\omega_f$ is said to be a sequence algebra if there is defined a product $*$ on Z_f such that*

$$ (\zeta_n),(\eta_n)\in Z_f\implies(\zeta_n)*(\eta_n)\in Z_f. $$

Definition 2.6. *A subset $Z_f\subset\omega_f$ is considered as symmetric if $(\zeta_n)\in Z_f\implies(\zeta_{\pi(n)})\in Z_f$, where π is a $\mathbb{N}$ permutation (one may refer to [2]).*

Definition 2.7. *Let $K=\{k_1<k_2<k_3<...<k_n...\}\subset\mathbb{N}$ and $(\zeta_n)\in\omega_f$. Then the K-step space of the sequence of function space Z_f is defined by*

$$ \lambda_K^{Z_f}=\{\zeta_{k_i}(x)\in\omega_f:\zeta_n(x)\in Z_f\}. $$

Definition 2.8. *A canonical pre-image $(\eta_n(x))$ of a sequence $(\zeta_n)\in Z_f$, where K-step space $\lambda_K^{Z_f}$ is considered, is defined by*

$$ \eta_n(x)=\begin{cases}\zeta_n(x),n\in K;\\ 0,\ otherwise.\end{cases} $$

Definition 2.9. *A sequence of function space Z_f is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Definition 2.10. *If $Z_f\in\omega_f$ is sequence space, then*

1. *Z_f is called convex, if $(\zeta_n),(\eta_n)\in Z_f,\lambda_1+\lambda_2=1,\lambda_1\geq 0,\ \lambda_2\geq 0\implies\lambda_1\zeta_n+\lambda_2\eta_n\in Z_f$.*
2. *Z_f is called balanced if and only if $\zeta_n\in Z_f,\ |\lambda|\leq 1\implies\lambda\zeta_n\in Z_f$.*
3. *Z_f is a called absolutely convex if and only if $\zeta_n,\eta_n\in Z_f,\ |\lambda_1|+|\lambda_2|\leq 1\implies\lambda_1\zeta_n+\lambda_2\eta_n\in Z_f$.*

Definition 2.11. [20] *considered the definition of modular space as follows: Suppose X is a linear domain. Then $\Omega : X \to [0,\infty)$ is a modular function if*

(1) $\Omega(x) = 0$ *iff* $x = \Theta$.
(2) $\Omega(\delta_1 x) = \Omega(x)$ *for all scalar a with* $|\delta_1| = 1$.
(3) $\Omega(\delta_1 x + \delta_2 y) < \Omega(x) + \Omega(y)$, *for all* $x, y \in X$ *and* $\delta_1, \delta_2 \geq 0$, $|\delta_1| = |\delta_2| = 1$. *Further, the modular Ω is called convex if*
(4) $\Omega(\delta_1 x + \delta_2 y) \leq \delta_1 \Omega(x) + \delta_2 \Omega(y)$ *holds for all* $x, y \in X$ *and all* $\delta_1, \delta_2 \geq 0$ *with* $\delta_1 + \delta_2 = 1$.

3. Main Result

Theorem 3.1. *Assume that $p > 0$ and $\alpha > 0$ are real numbers, then the norm of the space ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is denoted by $||.||_{{}_{ru}m_\phi(\Delta^\alpha,\ p)}$ and defined by*

$$||\zeta||_{{}_{ru}m_\phi(\Delta^\alpha,\ p)} = \sup_{x\in D} |\zeta_1(x)||\mu(x)| + \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha \zeta_n(x)\mu(x)|^p \right\}^{\frac{1}{p}}, \tag{1}$$

for $1 \leq p < \infty$. The space ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is Banach space with this norm. The space ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is also complete p-normed space by the $p-$ norm

$$||\zeta||_{{}_{ru}m_\phi(\Delta^\alpha, p)} = \sup_{x\in D} |\zeta_1(x)\mu(x)|^p + \sup_{s\geq 1, \sigma\in\xi_s,\ x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha \zeta_n(x)\mu(x)|^p,\ \text{for } 0 < p < 1. \tag{2}$$

Proof. Let (ζ_n) and $(\eta_n) \in {}_{ru}m_\phi(\Delta^\alpha,\ p)$, then there exist $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2$ such that

$$\sup_{x\in D,\ s\geq 1,\ \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha \zeta_n(x)\mu_1(x)|^p \right\}^{\frac{1}{p}} < \tilde{\mathcal{F}}_1,\ 1 \leq p < \infty,$$

and

$$\sup_{x\in D,\ s\geq 1,\ \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha \eta_n(x)\mu_2(x)|^p \right\}^{\frac{1}{p}} < \tilde{\mathcal{F}}_2,\ 1 \leq p < \infty,$$

where $\mu_1(x)$ and $\mu_2(x)$ are two scale functions on a compact domain D respectively. Consider the two scalars δ_1, δ_2, then we have

$$\sup_{x\in D,\ s\geq 1,\ \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha (\delta_1\zeta_n(x) + \delta_2\eta_n(x))|^p \right\}^{\frac{1}{p}} < \max(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \max(|\mu_1(x)|, |\mu_2(x)|),$$

for $1 \leq p < \infty$.
Hence, $(\delta_1\zeta_n + \delta_2\eta_n) \in {}_{ru}m_\phi(\Delta^\alpha,\ p)$.
Therefore, ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is a linear space.
By using the norm (1), it is obvious that ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is a normed linear, for $1 \leq p < \infty$ and a $p-$normed space by the $p-$ normed (2), for $0 < p < 1$.
We now establish the completeness of ${}_{ru}m_\phi(\Delta^\alpha,\ p)$.
Let (ζ^i) be a Cauchy sequence, where $(\zeta^i) = (\zeta^i_n) = ((\zeta^i_1), (\zeta^i_2), ...) \in {}_{ru}m_\phi(\Delta^\alpha,\ p)$,

for each $i \in \mathbb{N}$.
Then for considering $\tilde{\mathcal{F}} > 0$, there a positive number $n_0 \in \mathbb{N}$ such that

$$||\zeta^i - \zeta^j||_{{}_{ru}m_\phi(\Delta^\alpha,\ p)} = \sup_{x\in D} |\zeta_1^i(x) - \zeta_1^j(x)||\mu(x)| + \sup_{x\in D,\ s\geq 1,\ \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha(\zeta_n^i(x) - \zeta_n^j(x)||\mu(x)|^p \right\}^{\frac{1}{p}} < \tilde{\mathcal{F}},\ \text{for all } i, j > n_0. \tag{3}$$

implies $(\zeta_1^i)_{i=1}^\infty$ and $(\Delta^\alpha \zeta_n^i)_{i=1}^\infty$ are Cauchy sequences in D with respect to the scale function $\mu(x)$, for $x \in D$.
Let $(\zeta_1^i)_{i=1}^\infty$ converges to ζ_1 and $(\Delta^\alpha \zeta_n^i)_{i=1}^\infty$ converges to ζ_n for all $n \in \mathbb{N}$.
i.e.,

$$\lim_{i\to\infty} \zeta_1^i(x) = \zeta_1(x) \text{ and } \lim_{i\to\infty} \Delta^\alpha \zeta_n^i(x) = \zeta_n(x),\ \text{for all } n \in \mathbb{N},\ x \in D.$$

Taking limit as $j \to \infty$ in equation (3), we get

$$\sup_{x\in D} |\zeta_1^i(x) - \zeta_1(x)||\mu(x)| + \sup_{x\in D,\ s\geq 1,\ \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha(\zeta_n^i(x) - \zeta_n(x))\mu(x)|^p \right\}^{\frac{1}{p}} < \tilde{\mathcal{F}},$$

for each $n \geq n_0$.

$\implies ||\zeta^i - \zeta^j||_{{}_{ru}m_\phi(\Delta^\alpha,\ p)} < \tilde{\mathcal{F}},$ for all $n \geq n_0,\ x \in D$.
Hence, $(\zeta_n^i - \zeta_n) \in {}_{ru}m_\phi(\Delta^\alpha,\ p)$.
As ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is a linear space, we can conclude that (ζ_n^i), $(\zeta_n^i - \zeta_n) \in {}_{ru}m_\phi(\Delta^\alpha,\ p)$, it consequently follows that,

$$(\zeta_n(x)) = (\zeta_n^i(x) - (\zeta_n^i(x) - \zeta_n(x)),\ \text{for all } n \in \mathbb{N},\ x \in D.$$

Hence, ${}_{ru}m_\phi(\Delta^\alpha, p)$ is complete for $1 \leq p < \infty$.
In the same way, we can demonstrate that for $0 < p < 1$, ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is complete space $p-$normed by (2). □

The preceding theorem's consequence is the proof of the following outcomes.

Corollary 3.1. *The classes of function spaces ${}_{ru}n_\phi(\Delta^\alpha,\ p)$, $X(\Delta_{ru}^\alpha)$ are Banach spaces, where $X = \ell_p,\ \ell_\infty,\ c,\ c_0$.*

Theorem 3.2. *The space ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is a BK-space, for $1 \leq p < \infty$ and any rational number $\alpha = \frac{\alpha_1}{\alpha_2}$ where $\alpha_2 \geq \alpha_1$. But the space is not BK-space for $\alpha_1 < \alpha_2$.*

Proof. Let $\Delta_{ru}^\alpha(\zeta_n - \zeta) \to \varnothing_f$, as $n \to \infty$, then for a given $\tilde{\mathcal{F}} > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\Delta_{ru}^\alpha(\zeta_n(x) - \zeta(x)) < \tilde{\mathcal{F}},\ \text{for all } n \geq n_0,\ x \in D.$$

$$\sup_{x\in D} |\zeta_1^i(x) - \zeta_1(x)| + \sup_{x\in D,\ s\geq 1,\ \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha(\zeta_n^i(x) - \zeta_n(x))|^p \right\}^{\frac{1}{p}} < \tilde{\mathcal{F}}|\mu(x)|, \tag{4}$$

for all $i \geq n_0$. From equation (4) it follows that

$$|\zeta_1^i(x) - \zeta_1(x)| < \tilde{\mathcal{F}}|\mu(x)|,\ \text{for each } i \geq n_0,\ x \in D.$$

From equation (4), taking $s = 1$ and $n = 1$, we have

$$|(\zeta_1^i(x) - \zeta_1(x)) - \frac{\alpha_1}{\alpha_2}(\zeta_2^i(x) - \zeta_2(x))| < \tilde{\mathcal{F}}|\mu(x)|\phi_1,\ \text{for each } i \geq n_0,\ x \in D. \tag{5}$$

From equation (4) and equation (5) it follows that

$$|\zeta_2^i(x) - \zeta_2(x)| < \frac{\alpha_2}{\alpha_1}\tilde{\mathcal{F}}|\mu(x)|(\phi_1 - 1), \text{ for all } i \geq n_0, \ x \in D.$$

$$\implies \lim_{i\to\infty} \zeta_2^i = \zeta_2.$$

Proceeding in this way inductively, we have

$$\lim_{i\to\infty} \zeta_n^i = \zeta_n, \text{ for all } n \in \mathbb{N}.$$

Hence, ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ is a BK-space, for $1 \leq p < \infty$, with the condition that $\alpha_2 \geq \alpha_1$. □

For the next part of the theorem we consider an example as follows.

Example 3.1. *Let,* $(\zeta_n) \in {}_{ru}m_\phi(\Delta^\alpha,\ p)$ *for* $\alpha = \frac{3}{2}$, *from (4) and (5) we get,*

$$|\zeta_2^i(x) - \zeta_2(x)| \leq \frac{2}{3}\tilde{\mathcal{F}}|\mu(x)|(\phi_1 - 1) - \frac{1}{4}|\zeta_3^i(x) - \zeta_3(x)|$$

$$\implies \lim_{i\to\infty} \zeta_2^i \neq \zeta_2 \implies \lim_{i\to\infty} \zeta_n^i \neq \zeta_n.$$

Hence, the sequence of function space ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ *is not BK-space when* $\alpha_1 > \alpha_2$.

Result 3.1. *The class of the sequence of functions* ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ *is not solid in general.*

The proposed result follows from the example that follows:

Example 3.2. *Let us have a look at a sequence of functions* $(\zeta_n(x))$, *which are defined as follows:*

$$\zeta_n(x) = \begin{cases} \frac{\sqrt{x}}{1+n^2}, & \textit{for } x \neq 0; \\ 0, & \textit{for } x = 0, \end{cases}$$

then

$$\Delta^\alpha \zeta_n(x) = \begin{cases} \sum_{i=0}^\infty \frac{(-1)^i \Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \frac{\sqrt{x}}{\{1+(n+i)^2\}}, & \textit{for } x \neq 0; \\ 0, & \textit{for } x = 0. \end{cases}$$

Let, $\phi_n = 1$, *for each* $n \in \mathbb{N}$, $p = 1$, $\alpha = \frac{1}{2}$,
therefore,

$$\Delta^{\frac{1}{2}} \zeta_n(x) = \frac{(n^2 + 2n + 3)\sqrt{x}}{2(n^2 + 2n + 2)(1 + n^2)} \textit{ for each } n \in \mathbb{N} \textit{ and } x \in D.$$

Subsequently, $\Delta^{\frac{1}{2}}(\zeta_n(x))$ *is uniformly convergent with regard to the scale function* $\mu(x)$, *which is defined as follows:*

$$\mu(x) = \begin{cases} \sqrt{x}, & \textit{for } x \neq 0; \\ 0, & \textit{for } x = 0. \end{cases}$$

Hence, $(\Delta^{\frac{1}{2}} \zeta_n) \in {}_{ru}m_\phi(\Delta^\alpha, p)$.
Let us consider another sequence of function $\lambda_n(x)$ *be defined by,*

$$\lambda_n(x) = \begin{cases} \frac{\sqrt{x}}{n}, & \textit{for } x \neq 0; \\ 0, & \textit{for } x = 0, \end{cases}$$

then for each $n \in \mathbb{N}$,

$$\Delta^{\frac{1}{2}} \lambda_n(x) = \begin{cases} \frac{\sqrt{x}}{2}\left\{\frac{1}{n} + \frac{1}{n(n+1)}\right\}, & \textit{for } x \neq 0; \\ 0, & \textit{for } x = 0. \end{cases}$$

Hence, $(\lambda^{\frac{1}{2}}\zeta_n) \in {}_{ru}m_\phi(\Delta^\alpha, p)$, *with respect to the scale function* $\mu(x)$ *as specified by,*

$$\mu(x) = \begin{cases} \sqrt{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Also, we have $|\Delta^{\frac{1}{2}}\lambda_n(x)| \leq 1$, *for all* $n \in \mathbb{N}$ *and* $x \in D$.
Then,

$$\Delta^{\frac{1}{2}}\lambda_n\Delta^{\frac{1}{2}}\zeta_n(x) = \frac{x}{4n(1+n^2)}\left\{1 + \frac{1}{1+n} + \frac{1}{n^2+2n+2} + \frac{1}{(n+1)(n^2+2n+2)}\right\} \sim \frac{x}{4n^6}.$$

With regards to the same scale function $\mu(x)$, *the previously mentioned sequence of functions is not uniformly convergent.*
Therefore, $(\Delta^{\frac{1}{2}}\lambda_n\Delta^{\frac{1}{2}}\zeta_n) \notin {}_{ru}m_\phi(\Delta^{\frac{1}{2}}, p)$.
Hence, ${}_{ru}m_\phi(\Delta^{\frac{1}{2}}, p)$ *is not solid.*
Therefore the class of sequences $(\Delta^\alpha\lambda_n\Delta^\alpha\zeta_n) \notin {}_{ru}m_\phi(\Delta^\alpha,\ p)$ *is not solid in general.*

Result 3.2. *The class of sequence of functions* ${}_{ru}m_\phi(\Delta^\alpha, p)$ *is not symmetric.*
Using the example below, the result is obvious.

Example 3.3. *Let us assume that,* $\phi_n = 1$ *for each* $n \in \mathbb{N}, p = 1, \alpha = \frac{3}{2}$.
Consider a sequence of functions (ζ_n) *defined by*

$$\zeta_n(x) = \frac{x}{n}, x \neq 0 \text{ and } \zeta_n(0) = 0, \text{ for each } n \in \mathbb{N}.$$

$$\Delta^{\frac{3}{2}}\zeta_n(x) = \frac{x(7n^2+3n-16)}{8n(n+1)(n+2)}, x \neq 0 \text{ and } \zeta_n(0) = 0, \text{ for all } n \in \mathbb{N}.$$

Following that, $(\Delta^{\frac{3}{2}}\zeta_n) \in {}_{ru}m_\phi(\Delta^{\frac{3}{2}}, p)$ *relative uniformly with respect to the scale function* $\mu(x) = x$ *for* $x \neq 0$ *and* $\mu(0) = 0$.
Now consider the rearrangement $(\Delta^{\frac{3}{2}}\eta_n)$ *of* $(\Delta^{\frac{3}{2}}\zeta_n)$ *defined as follows,*
$\Delta^{\frac{3}{2}}\eta_n(x) = (\Delta^{\frac{3}{2}}\zeta_2(x), \Delta^{\frac{3}{2}}\zeta_7(x), \Delta^{\frac{3}{2}}\zeta_{19}(x), \Delta^{\frac{3}{2}}\zeta_1(x), \Delta^{\frac{3}{2}}\zeta_{70}(x), ...)$.
Then, $(\Delta^{\frac{3}{2}}\eta_n) \notin {}_{ru}m_\phi(\Delta^{\frac{3}{2}},\ p)$.
Therefore, ${}_{ru}m_\phi(\Delta^{\frac{3}{2}},\ p)$ *is not symmetric.*
Hence, ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ *is not symmetric in genaral.*

Result 3.3. *There is no convergence-free class for a sequence of functions* ${}_{ru}m_\phi(\Delta^\alpha,\ p)$.
Using the example below, the result is obvious.

Example 3.4. *Let us consider* $\phi_n = 1$ *for all* $n \in \mathbb{N},\ p = 1,\ \alpha = 1$ *and examine the sequence of functions* (ζ_n) *that are defined as follows,*

$$\zeta_n(x) = \frac{x}{n}, \text{ for all } n \in \mathbb{N},\ x \in [1, 2].$$

Then, $\Delta\zeta_n(x) = \frac{x}{n^2+n} \sim \frac{x}{n^2}$ *, for each* $n \in \mathbb{N}, x \in [1, 2]$.
The above sequence of functions converges relatively uniformly with respect to the scale function $\mu(x) = x,$ *for* $x \in [1, 2]$.
Therefore, $(\Delta\zeta_n) \in {}_{ru}m_\phi(\Delta^\alpha, p)$.
We now examine the sequence of functions (η_n) *that are defined by,*

$$\eta_n(x) = n^2x, \text{ for } x \in [1, 2], \text{ for all } n \in \mathbb{N}.$$

Then $\Delta\eta_n(x) = -(2n+1)x,$ *for* $x \in [1,2],$ *for all* $n \in \mathbb{N}.$
But $(\Delta\eta_n) \notin {}_{ru}m_\phi(\Delta^\alpha, p)$ *with respect to the same scale function.*
Hence, the classes of sequences ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ *are not convergence-free in general.*

Theorem 3.3. $\ell_p(\Delta^\alpha_{ru}) \subseteq {}_{ru}m_\phi(\Delta^\alpha,\ p) \subseteq \ell_\infty(\Delta^\alpha_{ru}).$

Proof. For $\phi_n = 1$, for each $n \in \mathbb{N}$, $\ell_p(\Delta^\alpha_{ru}) = {}_{ru}m_\phi(\Delta^\alpha,\ p),$ for all $n \in \mathbb{N}$.
Therefore we can write

$$\ell_p(\Delta^\alpha_{ru}) \subseteq {}_{ru}m_\phi(\Delta^\alpha,\ p).$$

Let us now assume that $(\zeta_n) \in {}_{ru}m_\phi(\Delta^\alpha,\ p)$. Afterwards, we have

$$\sup_{s\geq 1,\ \sigma\in\ \xi_s,\ x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma} |\Delta^\alpha\zeta_n(x)|^p < \tilde{\mathcal{F}}|\mu(x)|, \text{ for some positive integer } \tilde{\mathcal{F}}.$$

Hence,

$$|\Delta^\alpha\zeta_n(x)|^p < \tilde{\mathcal{F}}|\mu(x)|\phi_1, \text{ for all } n \in \mathbb{N}.$$

Thus, $(\zeta_n) \in \ell_\infty(\Delta^\alpha_{ru})$
$\implies {}_{ru}m_\phi(\Delta^\alpha,\ p) \subseteq \ell_\infty(\Delta^\alpha_{ru}).$
Therefore, the theorem is justified. □

Result 3.4. *Sequence algebra does not apply to the class of sequence of function* ${}_{ru}m_\phi(\Delta^\alpha,\ p).$

The result is established by the example as follows.

Example 3.5. *Taking two sequence of real valued functions* (ζ_n) *and* (η_n) *from the space* ${}_{ru}m_\phi(\Delta^\alpha,\ p)$*that are defined on any compact domain* D *such that*

$$\zeta_n(x) = \frac{(-1)^n x}{n} \textit{ and } \eta_n(x) = (-1)^n x, \textit{ for all } n \in \mathbb{N}, \textit{ for } x \in D.$$

Taking, $\phi_n = 1$ *for all* $n \in \mathbb{N},\ p = 1,\ \alpha = \frac{1}{2}.$

$$\Delta^{\frac{1}{2}}\zeta_n(x) = \frac{(-1)^n x(n+2)}{2n(n+1)}, \textit{ for } x \in D, \textit{ for all } n \in \mathbb{N}.$$

Then, $(\Delta^{\frac{1}{2}}\zeta_n) \in {}_{ru}m_\phi(\Delta^{\frac{1}{2}},\ p)$ *and* $\Delta^{\frac{1}{2}}\eta_n(x) = \frac{(-1)^n x}{2}$ *with respect to the scale function* $\mu(x) = x,$ *for* $x \in D.$
But, $\Delta^{\frac{1}{2}}\zeta_n\Delta^{\frac{1}{2}}\eta_n(x) = \frac{x^2(n+2)}{2(n^2+n)}$ *for all* $n \in \mathbb{N}$ *is relatively uniform convergent with respect to the scale function* $\mu(x) = x^2,$ *for* $x \in D.$
i.e., $(\Delta^{\frac{1}{2}}\zeta_n), (\Delta^{\frac{1}{2}}\eta_n) \in {}_{ru}m_\phi(\Delta^{\frac{1}{2}},\ p)$ *but* $(\Delta^{\frac{1}{2}}\zeta_n\Delta^{\frac{1}{2}}\eta_n) \notin {}_{ru}m_\phi(\Delta^\alpha,\ p).$
Hence, ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ *is not a sequence algebra.*

Result 3.5. *The classes of a sequence of function* ${}_{ru}m_\phi(\Delta^\alpha,\ p)$ *are not monotone.*

The result is established by the example that follows.

Example 3.6. *Let us consider* $(\zeta_n) = x$*, for all* $n \in \mathbb{N}.$

Consider a sequence of function $(\gamma_n(x)) = \begin{cases} x, \textit{ for } n \textit{ is even;} \\ x^2, \textit{otherwise.} \end{cases}$

The sequence of function $(\zeta_n) \in {}_{ru}m_\phi(\Delta^\alpha, p)$*, w.r.t. the scale function* $\mu(x)$ *defined in a compact domain* D*. But* $(\gamma_n) \neq {}_{ru}m_\phi(\Delta^\alpha, p)$ *w.r.t. the same scale function.*

Proposition 3.1. *$m_\phi(ru,\ p) \subset\ _{ru}m_\phi(\Delta^\alpha,\ p)$ and the inclusion is strict. Where $m_\phi(ru,\ p)$ denotes the spaces of relative uniform convergence of $m(\phi)$ sequence of function.*

Proof. The first part of the inclusion can be established using standard technique. □

The following example is taken into consideration for the second portion of the proposition.

Example 3.7. *Let* $\zeta_n(x) = \begin{cases} \frac{x^n}{n^2}, & for\ x \neq 0; \\ 0, & for\ x = 0. \end{cases}$
Taking $s = 1, \phi_s = 1$ and $p = 1$ then $(\zeta_n) \in\ _{ru}m_\phi(\Delta^\alpha,\ p)$ with respect to the scale function $\mu(x) = x^{n+1}$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
But $(\zeta_n) \notin\ m_\phi(ru,\ p)$, with respect to the same scale funtion.
Hence, $_{ru}m_\phi(\Delta^\alpha,\ p) \nsubseteq m_\phi(ru,\ p)$.

4. Geometric Properties

Theorem 4.1. *The convex modular space $_{ru}m_\phi(\Delta^\alpha,\ p)$ has the modular*

$$\Omega(\zeta_n) = \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha \zeta_n(x)\mu(x)|^p \right\}^{\frac{1}{p}},\ for\ 1 \leq p < \infty.$$

Proof. Clearly, the space $_{ru}m_\phi(\Delta^\alpha,\ p)$ is a linear space.
Now we consider a function $\Omega :\ _{ru}m_\phi(\Delta^\alpha,\ p) \to [0, \infty)$.
To check for conditions of norm:

(1) Clearly, $\Omega(\zeta_n) = 0$ if an only of $\zeta_n = \Theta$.
(2) For any scalar δ_1 with $|\delta_1| = 1$,

$$\begin{aligned}\Omega(\delta_1\zeta_n(x)) &= \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\delta_1\Delta^\alpha \zeta_n(x)\mu(x)|^p \right\}^{\frac{1}{p}} \\ &= |\delta_1| \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha \zeta_n(x)\mu(x)|^p \right\}^{\frac{1}{p}} \\ &= \Omega(\zeta_n(x))\end{aligned}$$

(3) Let $(\zeta_n), (\eta_n) \in\ _{ru}m_\phi(\Delta^\alpha,\ p)$ and $\delta_1, \delta_2 \geq 0$ with $\delta_1 + \delta_2 = 1$. To prove the convexity of the function we have,

$$\begin{aligned}\Omega(\delta_1\zeta_n + \delta_2\eta_n) &= \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |(\delta_1\Delta^\alpha \zeta_n(x) + \delta_2\Delta^\alpha\eta_n)\mu(x)|^p \right\}^{\frac{1}{p}} \\ &\leq |\delta_1| \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha \zeta_n(x)\mu(x)|^p \right\}^{\frac{1}{p}} \\ &+ |\delta_2| \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^\alpha \eta_n(x)\mu(x)|^p \right\}^{\frac{1}{p}} \\ &\leq \delta_1\Omega(\zeta_n) + \delta_2\Omega(\eta_n).\end{aligned}$$

Hence the space ${}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$ is convex.

If in place of the above condition (*3*), the following condition (*4*) is hold,

(4) $\Omega(\delta_1\zeta_n + \delta_2\eta_n) \leq \delta_1^t\Omega(\zeta_n) + \delta_2^t\Omega(\eta_n)$, for $\delta_1, \delta_2 \geq 0, \delta_1^t + \delta_2^t = 1, t \in [0, 1)$, then the modular Ω is called $t-$convex with $t = 1$.

□

Theorem 4.2. *The classes of sequence of functions space ${}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$ is absolute 1-convex.*

Proof. Let $0 < r < 1$, $V = \{\zeta = (\zeta_n) \in \omega_f : ||\zeta||_{{}_{ru}m_{\phi}(\Delta^{\alpha},\ p)} \leq r\}$ is absolute 1-convex set for $(\zeta_n), (\eta_n) \in V$ and $|\lambda_1| + |\lambda_2| \leq 1$, then

$$||\lambda_1\zeta_n + \lambda_2\eta_n||_{{}_{ru}m_{\phi}(\Delta^{\alpha},\ p)} \leq (|\lambda_1| + |\lambda_2|)r \leq r.$$

□

Theorem 4.3. *The classes of sequence of functuions space ${}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$ is absolute convex.*

Proof. Let us assume that, $(\zeta_n), (\eta_n) \in\ {}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$ and $|\lambda_1| + |\lambda_2| \leq 1$. Then,

$$\begin{aligned}
&||\lambda_1\zeta_n(x) + \lambda_2\eta_n(x)||_{{}_{ru}m_{\phi}(\Delta^{\alpha},\ p)}\\
=&\sup_{x\in D}|\lambda_1\zeta_1(x) + \lambda_2\eta_1(x)||\mu(x)| + \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}(\lambda_1\zeta_n(x) + \lambda_2\eta_n(x))|^p\right\}^{\frac{1}{p}}\\
\leq&\sup_{x\in D}|\lambda_1\zeta_1(x)||\mu(x)| + \sup_{x\in D}|\lambda_2\eta_1(x)||\mu(x)| + |\lambda_1|\sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\zeta_n(x)|^p\right\}^{\frac{1}{p}}\\
&+|\lambda_2|\sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\eta_n(x)|^p\right\}^{\frac{1}{p}}\\
\leq&|\lambda_1|\sup_{x\in D}|\zeta_1(x)||\mu(x)| + |\lambda_2|\sup_{x\in D}|\eta_1(x)||\mu(x)| + (1-|\lambda_2|)\\
&\sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\zeta_n(x)|^p\right\}^{\frac{1}{p}} + |\lambda_2|\sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\eta_n(x)|^p\right\}^{\frac{1}{p}}\\
\leq&|\lambda_1|\sup_{x\in D}|\zeta_1(x)||\mu(x)| + |\lambda_2|\sup_{x\in D}|\eta_1(x)||\mu(x)| + (1+|\lambda_2|)\\
&\sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\zeta_n(x)|^p\right\}^{\frac{1}{p}} + |\lambda_2|\sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\eta_n(x)|^p\right\}^{\frac{1}{p}}\\
\leq&|\lambda_1|\sup_{x\in D}|\zeta_1(x)||\mu(x)| + |\lambda_2|\left\{\sup_{x\in D}|\eta_1(x)||\mu(x)| + \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\zeta_n(x)|^p\right\}^{\frac{1}{p}}\right.\\
&\left.+\sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\eta_n(x)|^p\right\}^{\frac{1}{p}}\right\} + \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta^{\alpha}\zeta_n(x)|^p\right\}^{\frac{1}{p}}\\
<&\infty.
\end{aligned}$$

Hence the classes of sequence of function space ${}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$ is absolute convex. $\square$

Theorem 4.4. *The space* ${}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$ *is balanced.*

Proof. Let $(\zeta_n) \in {}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$ and $|\lambda| \leq 1$, then

$$
\begin{aligned}
&||\lambda\zeta_n||_{{}_{ru}m_{\phi}(\Delta^{\alpha},\ p)} \\
&= \sup_{x\in D} |\lambda\zeta_n(x)||\mu(x)| + \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^{\alpha}(\lambda\zeta_n(x))|^p \right\}^{\frac{1}{p}} \\
&= |\lambda| \sup_{x\in D} |\zeta_n(x)||\mu(x)| + |\lambda| \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^{\alpha}(\lambda\zeta_n(x))|^p \right\}^{\frac{1}{p}} \\
&\leq \sup_{x\in D} |\zeta_n(x)||\mu(x)| + \sup_{s\geq 1,\ \sigma\in\xi_s,\ x\in D} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^{\alpha}(\lambda\zeta_n(x))|^p \right\}^{\frac{1}{p}} \\
&< \tilde{\mathcal{F}} < \infty.
\end{aligned}
$$

Hence, the classes of sequence of function space ${}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$ is balanced. $\square$

5. Conclusions

In this work, we first obtained two new notions of relative uniform convergence of fractional difference sequences of functions related to ℓ_p-space w.r.t. a scale function $\mu(x)$ that are defined by ${}_{ru}m_{\phi}(\Delta^{\alpha},\ p)$, ${}_{ru}n_{\phi}(\Delta^{\alpha},\ p)$. The purpose of this paper is to study some of its properties and identify the relationships with the spaces $\ell_p(\Delta^{\alpha}_{ru}),\ \ell_{\infty}(\Delta^{\alpha}_{ru}),\ c(\Delta^{\alpha}_{ru}),\ c_0(\Delta^{\alpha}_{ru})$. This sequence of function space is specifically applicable for positive fractional difference sequences of functions. Unfortunately, our notion does not give a guarantee of convergence for negative fractional difference sequences of functions. This is the first paper on this topic and is expected to attract researchers for further investigations and applications.

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