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ON $\mathcal{I}_\lambda$- STATISTICAL CONVERGENCE OF SEQUENCES OF BI-COMPLEX NUMBERS

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Abstract. In this paper, we introduce the notion of $\mathcal{I}_\lambda$- statistical convergence of sequences as one of the extensions of $\mathcal{I}$-statistical convergence of bi-complex numbers. We investigate some fundamental properties of these notion and its relationship with $\mathcal{I}$-statistical convergence of bi-complex numbers. In the end we introduce and investigate the concept of $\mathcal{I}_\lambda$- statistical limit points, cluster points and establish some implication relations.

Keywords: Bi-complex number, ideal, filter, $\mathcal{I}$- convergence, $\mathcal{I}$- statistical convergence, $\mathcal{I}_\lambda$- statistical convergence.

AMS Subject Classification (2020): 40A35, 40G15.

1. Introduction

Statistical convergence, initially proposed by Fast [8] and Steinhaus [30] independently in 1951. Subsequently, additional exploration from the vantage point of sequence space was undertaken by Fridy [9], Salat [23], Mondal and Hossain [17] and several other scholars.

In 2000, statistical convergence was extended to $\lambda-$ statistical convergence by Mursaleen [18] involving a non-decreasing sequence of positove numbers $\lambda = \lambda_n$ satisfying $\lambda_1 = 1, \lambda_{n+1} - \lambda_n \leq 1$ and $\lambda_n \to \infty$ as $n \to \infty$. Later on, several works have been carried out in this direction by Colak and Bektas [4], Savas and Mohiuddin [25] and many others.

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On the other hand, Ideal convergence originally proposed by Kostyrko et al.[14] in 2001 as an extension of statistical convergence has garnered significant attention from researchers in subsequent years. Building upon this foundational work, Debnath and Hossain [6], Debnath and Rakshit [5], Choudhury and Debnath [3], Savas and Das [24], among others, have conducted extensive research in this area, exploring its applications and properties. Through their investigations, they have highlighted ideal convergence as a generalized form encompassing various established convergence concepts contributing to the advancement of mathematical analysis and its applications. For an extensive study on Ideal convergence, one may refer to [7, 10, 11, 12, 13, 15, 16, 19]

In 2011, Savas and Das [24] introduced the concept of $\mathcal{I}$-statistical convergence and $\mathcal{I}_\lambda$- statistical convergence. The generalized de la Vallee-Poussin mean is defined by $t_n(u_k) = \frac{1}{\lambda_n} \sum\limits_{k \in I_n} u_k$, where $I_n = [n - \lambda_n + 1, n]$. A sequence (u_k) is said to be $\mathcal{I} - [V, \lambda]$ summable [25] to u if $\mathcal{I} - \lim\limits_{n \to \infty} t_n(u_k) = u$.

In 1892, Segre [27] introduced the notion of bi-complex number, which constitute an algebra isomorphic to the literariness. In Price's book [20], the most comprehensive study of analysis in bi-complex numbers is available. For an extensive study on bi-complex number, one may refer to [21, 22, 26, 28, 29, 31]. In recent years, many important results were obtained in this area. Few of them, which pertain and lead to our work.

The exploration of convergence is a fundamental aspect of analysis playing a crucial role in various mathematical investigations. However, the study of convergence of sequences of bi-complex numbers remains relatively underdeveloped and has not yet received substantial attention. Despite its nascent stage, recent research indicates a notable analogy in the convergence behavior of sequences of bi-complex numbers.

Recently, Bera and Tripathy [1] made significant strides by introducing the concept of statistical convergence for sequences of bi-complex numbers. Their work marks a pivotal advancement in the study of convergence in this domain as they delved into various properties from both algebraic and topological viewpoints. This exploration may unveil new insights into the convergence properties of bi-complex numbers contributing to the broader landscape of mathematical analysis. Additionally, M. Mursaleen [18] introduced the notion of λ-statistical convergence. These two works serve as the primary motivation for our study on $\mathcal{I}_\lambda$- statistical convergence on sequences of bi-complex numbers.

Throughout the paper $\mathbb{C}_2$, $S_\lambda^{\mathcal{I}-st}$ and $S_{0\lambda}^{\mathcal{I}-st}$ represent the set of all bi-complex numbers, $\mathcal{I}_\lambda$- statistical convergence and $\mathcal{I}_\lambda$- statistical null sequence spaces respectively.

2. Definitions and Preliminaries

Definition 2.1 Segre [27] defined a bi-complex number as;
$\xi = z_1 + i_2 z_2 = t_1 + i_1 t_2 + i_2 t_3 + i_1 i_2 t_4$, where $z_1, z_2 \in \mathbb{C}$ (set of complex numbers) and $t_1, t_2, t_3, t_4 \in \mathbb{R}$ (set of real numbers) and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. Denote the set of all bi-complex numbers by $\mathbb{C}_2$; it is defined

as: $\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2 z_2 \, \forall \, z_1, z_2 \in \mathbb{C}\}$.

In the realm of bi-complex numbers, a number $\xi = t_1 + i_1 t_2 + i_2 t_3 + i_1 i_2 t_4$ is classified as a hyperbolic number if $t_2 = 0$ and $t_3 = 0$. These hyperbolic numbers are collectively denoted as H and the set comprising them is referred to as the H-plane.

Equipped with coordinate-wise addition, real scalar multiplication and term-by-term multiplication the set $\mathbb{C}_2$ forms a commutative algebra with identity $1 = 1+i_1\cdot 0+i_2\cdot 0+i_1 i_2\cdot 0$. Within $\mathbb{C}_2$, there exist four idempotent elements, specifically 0, 1, $e_1 = \frac{1+i_1 i_2}{2}$, and $e_2 = \frac{1-i_1 i_2}{2}$.

It is obvious that $e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$. Every bi-complex number $\xi = z_1 + i_2 z_2$ has a unique idempotent representation as $\xi = T_1 e_1 + T_2 e_2$ where $T_1 = z_1 - i_1 z_2$ and $T_2 = z_1 + i_1 z_2$ are called the idempotent components of ξ.

On $\mathbb{C}_2$ the Euclidean norm $||\cdot||$ is defined as, $||\xi||_{\mathbb{C}_2} = \sqrt{t_1^2 + t_2^2 + t_3^2 + t_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|T_1|^2+|T_2|^2}{2}}$, where $\xi = z_1 + i_2 z_2 = t_1 + i_1 t_2 + i_2 t_3 + i_1 i_2 t_4 = T_1 u_1 + T_2 u_2$ with this norm $\mathbb{C}_2$ is a Banach space.

Definition 2.2 [2] A sequence of bi-complex number (ξ_n) is called statistically convergent to $l \in \mathbb{C}_2$ if for each $\varepsilon > 0$, $\delta(\{n \in \mathbb{N} : ||\xi_n - l||_{\mathbb{C}_2} \geq \varepsilon\}) = 0$.
Symbolically, we write stat-lim $\xi_k = l$.

Definition 2.3 [14] Consider a nonempty set X. A family of subsets $\mathcal{I} \subset \mathcal{P}(X)$ is called an ideal on X if it satisfies the following conditions:
(1) For every $X_1, X_2 \in \mathcal{I}$, the union $X_1 \cup X_2$ belongs to $\mathcal{I}$.
(2) For every $X_1 \in \mathcal{I}$ and every subset X_2 of X_1, X_2 is also in $\mathcal{I}$.
Further $\mathcal{I}$ is said to be admissible if $\forall x \in X$,{x} $\in \mathcal{I}$ and it is said to be non trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$.

Example 2.4 Below are some standard examples of ideals:
(1) The collection of all finite subsets of $\mathbb{N}$ constitutes a nontrivial admissible ideal on $\mathbb{N}$, denoted as $\mathcal{I}_f$.
(2) Indeed, the set comprising all subsets of $\mathbb{N}$ with natural density zero forms a nontrivial admissible ideal on $\mathbb{N}$. This particular ideal is denoted as $\mathcal{I}_\delta$.
(3) Let $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum\limits_{a \in A} a^{-1} < \infty\}$. Then $\mathcal{I}_c$ forms an ideal on N.
(4) Consider a partitioning of the natural numbers $\mathbb{N}$ into disjoint sets $D_1, D_2, D_3, \ldots$ such that $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ and $D_a \cap D_b = \emptyset$ for $a \neq b$. The set $\mathcal{I}$, comprising all subsets of $\mathbb{N}$ that have finite intersections with the sets D_p, constitutes an ideal on N.

Definition 2.5 [14] A family $\mathcal{F} \subset 2^X$ of subsets of a nonempty set X is called a filter in X if it satisfies the following conditions:
(1) The empty set $\emptyset$ does not belong to $\mathcal{F}$.

(2) For all $X_1, X_2 \in \mathcal{F}$, the intersection $X_1 \cap X_2$ is also in $\mathcal{F}$.
(3) For every $X_1 \in \mathcal{F}$ and every super set X_2 of X containing X_1, X_2 is also in $\mathcal{F}$.

Definition 2.6 [14] If $\mathcal{I}$ is a proper non-trivial ideal in Y, then
$\mathcal{F}(\mathcal{I}) = \{A \subset Y : \exists\, B \in \mathcal{I} : A = Y - B\}$ constitutes a filter in Y. This filter is commonly referred to as the filter associated with the ideal $\mathcal{I}$.

Definition 2.7 [14] Let $\mathcal{I} \subset P(\mathbb{N})$ denote a non-trivial ideal over $\mathbb{N}$. We define an $\mathcal{I}$-convergence for a real-valued sequence (ξ_n) towards l as follows, for every $\varepsilon > 0$, the set $H(\varepsilon) = \{n \in \mathbb{N} : |\xi_n - l| \geq \varepsilon\}$ must be an element of $\mathcal{I}$. Here, l is termed as the $\mathcal{I}$-limit of the sequence (ξ_n) and is denoted by $\mathcal{I} - \lim\limits_{k} \xi_k = l$.

Definition 2.8 [6] With respect to the Euclidean norm on $\mathbb{C}_2$, a sequence of bi-complex numbers (ξ_k) is deemed $\mathcal{I}$-convergent to $t \in \mathbb{C}_2$ if for each $\varepsilon_1 > 0$,
the set $F(\varepsilon_1) = \{k \in \mathbb{N} : ||\xi_k - t||_{\mathbb{C}_2} \geq \varepsilon_1\} \in \mathcal{I}$.
Symbolically, we write, $\xi_k \xrightarrow{\mathcal{I}-||\cdot||_{\mathbb{C}_2}} t$.

Definition 2.9 [6] Let $K \subseteq \mathbb{N}$ and $K_n = \{k \in K : k \in I_n\}$. Then, the $\mathcal{I}_\lambda$-density of K is defined by $d_\lambda^{\mathcal{I}}(K) = \mathcal{I} - \lim\limits_{n\to\infty} \dfrac{|K_n|}{\lambda_n}$, provided that the limit exists, where $I_n = [n - \lambda_n + 1, n], n \in \mathbb{N}$.

Definition 2.10 A real valued sequence (x_k) is deemed $\mathcal{I}_\lambda$ statistically convergent to l, if for every $\varepsilon > 0, \delta > 0$, the set $\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : |x_k - l| \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}$.
Equivalently, $d_\lambda^{\mathcal{I}}(\{k \in \mathtt{N} : |x_k - l| \geq \varepsilon\}) = 0$.
Symbolically, $x_k \xrightarrow{\mathcal{I}_\lambda^{st}} l$.
If (x_{m_k}) is a subsequence of the sequence (x_k) and
$d_\lambda^{\mathcal{I}}(\{m_1, m_2, m_3 ...\}) = 0$, then (x_{m_k}) is deemed an $\mathcal{I}_\lambda$-thin subsequence of (x_k). On the other hand if $d_\lambda^{\mathcal{I}}(\{m_1, m_2, m_3 ...\}) > 0$, then (x_{m_k}) is deemed an $\mathcal{I}_\lambda$-non-thin subsequence of (x_k).

Definition 2.11 [15] A number $u_0 \in \mathbb{R}$ is said to be an $\mathcal{I}_\lambda$-statistical limit point of a real-valued sequence (u_k), if there exists an $\mathcal{I}_\lambda$-non thin subsequence of (u_k) that converges to u_0.

Definition 2.12 [15] A number $u_0 \in \mathbb{R}$ is said to be an $\mathcal{I}_\lambda$-statistical cluster point of a real-valued sequence (u_k), if for every $\varepsilon > 0$, $d_\lambda^{\mathcal{I}}(\{k \in \mathbb{N} : |u_k - u_0| < \varepsilon\}) \neq 0$.

Definition 2.13 A sequence of bi-complex numbers (ξ_k) is said to be $\mathcal{I}$-statistical convergent to $p \in \mathbb{C}_2$ with respect to the Euclidean norm on $\mathbb{C}_2$ if for every $\varepsilon^{\mathbf{1}} > 0$ and $\delta^{\mathbf{1}} > 0$, the set $\{n_0 \in \mathbb{N} : \frac{1}{n_0}|k \leq n_0 : ||\xi_k - p||_{\mathbb{C}_2} \geq \varepsilon^{\mathbf{1}}| \geq \delta^{\mathbf{1}}\} \in \mathcal{I}$.
Symbolically, we write, $\xi_k \xrightarrow{\mathcal{I}st-||\cdot||_{\mathbb{C}_2}} p$.

Definition 2.14 A sequence space S is deemed solid (or normal) if $(t_k x_k) \in S$, whenever $(x_k) \in S$ and for all sequence (t_k) of scalars with $|t_k| \leq 1$, for all $k \in \mathbb{N}$.

Definition 2.15 A sequence space S is said to be symmetric if $(u_k) \in S$, implies $(u_{\Phi(k)}) \in S$ where Φ is a permutation of $\mathbb{N}$.

Definition 2.16 A sequence space S is called sequence algebra if $(u_k) * (v_k) = (u_k v_k) \in S$, whenever $(u_k), (v_k) \in S$.

Definition 2.17 A sequence space S is deemed convergence free if $(v_k) \in S$ whenever $(u_k) \in S$ and $u_k = 0$ implies $v_k = 0$.

Definition 2.18 Let $T = \{t_1 < t_2 < t_3 < ...\} \subseteq \mathbb{N}$ and S be a sequence space. A T-step space of S is a sequence space $\lambda_T^S = \{(u_{t_n}) \in w : (u_n) \in S\}$.
A canonical preimage of a sequence $(u_{k_n}) \in \lambda_T^S$ is a sequence $(v_n) \in w$
defined by $v_n = \begin{cases} u_n, & if\, n \in T; \\ 0, & otherwise. \end{cases}$

Definition 2.19 A canonical preimage of a step space λ_T^S is a set of canonical preimage of all elements in λ_T^S i.e v is in canonical preimage of λ_T^S iff v is canonical preimage of some $u \in \lambda_T^S$.

Definition 2.20 A sequence space S is deemed monotone if it contains the canonical preimages of its step spaces.

3. Main Results

Definition 3.1 Let (u_k) be a sequence in $\mathbb{C}_2$. Then, (u_k) is said to be $\mathcal{I} - [V, \lambda]_{\mathbb{C}_2}$ summable to $u \in \mathbb{C}_2$, if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $u_k \xrightarrow{\mathcal{I} - [V,\lambda]_{\mathbb{C}_2}} u$.

Definition 3.2 Let (u_k) be a sequence in $\mathbb{C}_2$. Then (u_k) is deemed $\mathcal{I}_\lambda$-statistical convergent to $u \in \mathbb{C}_2$, if for each $\varepsilon > 0$ and $\delta > 0$,
the set $\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in \mathcal{I}_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}$.
Equivalently, $d_\lambda^{\mathcal{I}}(\{k \in \mathbb{N} : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}) = 0$.
Symbolically, $u_k \xrightarrow{\mathcal{I}_\lambda^{st} - ||\cdot||_{\mathbb{C}_2}} u$.

Theorem 3.3 Let (u_k) be a sequence in $\mathbb{C}_2$ such that $u_k \xrightarrow{\mathcal{I}_\lambda^{st} - ||\cdot||_{\mathbb{C}_2}} u$. Then u is uniquely determined.

Proof: If possible suppose $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$ and $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} v$ holds for some $u \neq v$ in $\mathbb{C}_2$. Then, for any $\varepsilon > 0$, we have $B_1, B_2 \in \mathcal{F}(\mathcal{I})$, where $B_1 = B_1(\varepsilon,\delta) = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta\}$ and $B_2 = B_2(\varepsilon,\delta) = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - v||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta\}$. Clearly, $B_1 \cap B_2 \in \mathcal{F}(\mathcal{I})$ and is non-empty. Choose $m \in B_1 \cap B_2$ and take $\varepsilon = ||\frac{u-v}{3}||_{\mathbb{C}_2} > 0$. Then, $\frac{1}{\lambda_m}|\{k \in I_m : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta$ and $\frac{1}{\lambda_m}|\{k \in I_m : ||u_k - v||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta$. Now choosing $\delta \in (0, \frac{1}{2})$, the set $B = \{k \in I_m : ||u_k - u||_{\mathbb{C}_2} < \varepsilon\} \cap \{k \in I_m : ||u_k - v||_{\mathbb{C}_2} < \varepsilon\} \neq \emptyset$. Choose $p \in B$. Then, $\varepsilon = ||\frac{u-v}{3}||_{\mathbb{C}_2} \leq \frac{1}{3}(||u_k - u||_{\mathbb{C}_2} + ||u_k - v||_{\mathbb{C}_2}) < \frac{1}{3}(\varepsilon + \varepsilon) = \frac{2\varepsilon}{3}$, which is a contradiction.

Theorem 3.4 Consider two sequences (u_k) and (v_k) in $\mathbb{C}_2$ such that $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$ and $v_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} v$ then,
(i) $u_k + v_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u + v$ and
(ii) $au_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} au,\ a \in \mathbb{R}$.

Proof : (i) Suppose $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$ and $v_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} v$. Then, for any $\varepsilon > 0$ and $\delta > 0$, we have $B_1, B_2 \in \mathcal{I}$, where $B_1 = B_1(\varepsilon,\delta) = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \frac{\varepsilon}{2}\}| \geq \frac{\delta}{2}\}$ and$B_2 = B_2(\varepsilon,\delta) = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||v_k - v||_{\mathbb{C}_2} \geq \frac{\varepsilon}{2}\}| \geq \frac{\delta}{2}\}$. Then, $(\mathbb{N} \setminus B_1) \cap (\mathbb{N} \setminus B_2) \in \mathcal{F}(\mathcal{I})$ and so $(\mathbb{N} \setminus B_1) \cap (\mathbb{N} \setminus B_2) \neq \emptyset$. Choose $n \in (\mathbb{N} \setminus B_1) \cap (\mathbb{N} \setminus B_2)$. Then, the following inequality $\frac{1}{\lambda_n}|\{k \in I_n : ||(u_k + v_k) - (u + v)||_{\mathbb{C}_2} \geq \varepsilon\}| \leq \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \frac{\varepsilon}{2}\}| + \frac{1}{\lambda_n}|\{k \in I_n : ||v_k - v||_{\mathbb{C}_2} \geq \frac{\varepsilon}{2}\}|$ holds.
This leads to the following inclusion,
$(\mathbb{N} \setminus B_1) \cap (\mathbb{N} \setminus B_2) \subseteq \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||(u_k + v_k) - (u + v)||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta\}$ (1).
Now, as $(\mathbb{N} \setminus B_1) \cap (\mathbb{N} \setminus B_2) \in \mathcal{F}(\mathcal{I})$. Therefore, right-hand side of (1) $\in \mathcal{F}(\mathcal{I})$, indicating that $u_k + v_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u + v$.

(ii) When $a = 0$, there's nothing that needs proving. Suppose $a \neq 0$. Then for each $\varepsilon > 0$, the following inequation,
$\frac{1}{\lambda_n}|\{k \in I_n : ||au_k - au||_{\mathbb{C}_2} \geq \varepsilon\}| = \frac{1}{\lambda_n}|\{k \in I_n : |a|||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}|$
$\leq \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \frac{\varepsilon}{|a|}\}|$, holds good and the result follows.

Theorem 3.5 Suppose (u_k) is any sequence in C_2 . Then,
(i) if $u_k \xrightarrow{\mathcal{I}-[V,\lambda]} u$ then $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$.
(ii) if $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$ then $u_k \xrightarrow{\mathcal{I}-[V,\lambda]} u$ holds if (u_k) is bounded in $\mathbb{C}_2$.

Proof: (i) Let $\varepsilon > 0$ be arbitrary and $u_k \xrightarrow{\mathcal{I}-[V,\lambda]} u$. Then we have

$$\sum_{k \in \mathcal{I}_n} ||u_k - u||_{\mathbb{C}_2} \geq \sum_{\substack{k \in I_n \\ ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon}} ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}|$$

This implies that $\frac{1}{\varepsilon\lambda_n}\sum\limits_{k\in I_n}||u_k-u||_{\mathbb{C}_2} \geq \frac{1}{\lambda_n}|\,\{k\in I_n : ||u_k-u||_{\mathbb{C}_2}\geq\varepsilon\}\,|$.
Consequently, for any $\delta>0$, the following inclusion,

$$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}|\{k\in I_n : ||u_k-u||_{\mathbb{C}_2}\geq\varepsilon\}|\geq\delta\right\}$$
$$\subseteq\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}(\sum_{k\in I_n}||u_k-u||_{\mathbb{C}_2})\geq\varepsilon\delta\right\}\in\mathcal{I}$$

and the result follows.

(ii) Let $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$ and (u_k) is bounded in $\mathbb{C}_2$. Then, boundedness of (u_k) implies the existence of $B>0$ such that $||u_k-u||_{\mathbb{C}_2}\leq B\,\forall\,k\in\mathbb{N}$. Then, for given $\varepsilon>0$ we have,

$$\frac{1}{\lambda_n}\left(\sum_{k\in I_n}||u_k-u||_{\mathbb{C}_2}\right)$$
$$=\frac{1}{\lambda_n}\left(\sum_{\substack{k\in I_n\\ ||u_k-u||_{\mathbb{C}_2}\geq\frac{\varepsilon}{2}}}||u_k-u||_{\mathbb{C}_2}\right)+\frac{1}{\lambda_n}\left(\sum_{\substack{k\in I_n\\ ||u_k-u||_{\mathbb{C}_2}<\frac{\varepsilon}{2}}}||u_k-u||_{\mathbb{C}_2}\right)$$
$$\leq\frac{B}{\lambda_n}|\left\{k\in I_n : ||u_k-u||_{\mathbb{C}_2}<\frac{\varepsilon}{2}\right\}|+\frac{\varepsilon}{2}.$$

Let $C=\{n\in\mathbb{N} : \frac{1}{\lambda_n}|\{k\in I_n : ||u_k-u||_{\mathbb{C}_2}\geq\frac{\varepsilon}{2}\}|\geq\frac{\varepsilon}{B}\}$. Then, by hypothesis, $C\in I$ and for any $n\in\mathbb{N}\setminus C$,

$\frac{1}{\lambda_n}\left(\sum\limits_{k\in I_n}||u_k-u||_{\mathbb{C}_2}\right)<2\varepsilon$. Consequently, the inclusion,

$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\left(\sum\limits_{k\in I_n}||u_k-u||_{\mathbb{C}_2}\right)\geq 2\varepsilon\right\}\subseteq C$ holds and the result follows from the hereditary property of $\mathcal{I}$.

Theorem 3.6 Suppose (u_k) is any sequence in $\mathbb{C}_2$. If every sub sequence of (u_k) is $\mathcal{I}_\lambda^{st}$ convergent to $u\in\mathbb{C}_2$, then (u_k) is also $\mathcal{I}_\lambda^{st}$ convergent to $u\in\mathbb{C}_2$.

Proof: If possible suppose (u_k) is not $\mathcal{I}_\lambda^{st}$ convergent to u in spite of having all the subsequences $\mathcal{I}_\lambda^{st}$ convergent to u. Then, by definition, $\exists$ particular $\varepsilon>0$ and $\delta>0$ such that the set, $B=B(\varepsilon,\delta)=\{n\in\mathbb{N} : \frac{1}{\lambda_n}|\{k\in I_n : ||u_k-u||_{\mathbb{C}_2}\geq\varepsilon\}|\geq\delta\}\notin\mathcal{I}$. Now admissibility of $\mathcal{I}$ ensures that B contains an infinite number of elements. Put $B=\{k_1<k_2...<k_j<...\}$ and define $v_j=u_{k_j}, j\in\mathbb{N}$. Then, v_j is a subsequence of (u_k) that is not $\mathcal{I}_\lambda^{st}$ convergent to u, which contradicts our assumption.

Theorem 3.7 Suppose (u_k) is any sequence in $\mathbb{C}_2$. Then $\mathcal{I}_\lambda^{st}\supseteq\mathcal{I}^{st}$ provided that $\lim\limits_{n\to\infty} inf\,\dfrac{\lambda_n}{n}>0$.

Proof: For any $\varepsilon>0$, $\frac{1}{n}|\{k\leq n : ||u_k-u||_{\mathbb{C}_2}\geq\varepsilon\}|$
$\geq\frac{1}{n}|\{k\in I_n : ||u_k-u||_{\mathbb{C}_2}\geq\varepsilon\}|\geq\frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k\in I_n : ||u_k-u||_{\mathbb{C}_2}\geq\varepsilon\}|$.

Now if $\lim_{n\to\infty} inf\ \dfrac{\lambda_n}{n} = p$, then by definition, the set $\{n \in \mathbb{N}, \frac{\lambda_n}{n} < \frac{p}{2}\}$ contains a finite number of elements and consequently the following
inclusion holds for any $\delta > 0$, $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\right\}$
$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \frac{p\delta}{2}\right\} \cup \left\{n \in \mathbb{N} : \frac{\lambda_n}{n} < \frac{p}{2}\right\}$. Now, if $u_k \xrightarrow{\mathcal{I}^{st}-||\cdot||_{\mathbb{C}_2}}$ u holds, then the set on the right-hand side belongs to $\mathcal{I}$ due to the admissibility of $\mathcal{I}$ and as a consequence, the set on the left-hand side also belongs to $\mathcal{I}$. Hence, $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$.

Theorem 3.8 Suppose (u_k) is any sequence in $\mathbb{C}_2$. Then $\mathcal{I}_\lambda^{st} \subseteq \mathcal{I}^{st}$ provided that $\lim_{n\to\infty} \dfrac{\lambda_n}{n} = 1$.

Proof: Let $\delta > 0$ be given. Since $\lim_{n\to\infty} \dfrac{\lambda_n}{n} = 1$, we can have a, $m \in \mathbb{N}$ satisfying $|\frac{\lambda_n}{n} - 1| < \frac{\delta}{2}\ \forall n \geq m$. Now for any $\varepsilon > 0$,
$\frac{1}{n}|\{k \leq n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}|$
$= \frac{1}{n}|\{k \leq n - \lambda_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| + \frac{1}{n}\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}|$
$\leq \frac{n-\lambda_n}{n} + \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}|$
$\leq 1 - (1 - \frac{\delta}{2}) + \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}|$
$= \frac{\delta}{2} + \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}|$ holds for all $n \geq m$. Therefore, the following inclusion holds: $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\}$
$\subseteq \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \frac{\delta}{2}\} \cup \{1, 2, 3, ..., m\}$.
Now, if $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$ holds , then the set on the right-hand side belongs to $\mathcal{I}$ due to the admissibility of $\mathcal{I}$ and as a consequence, the set on the left-hand side also belongs to $\mathcal{I}$. Therefore, $u_k \xrightarrow{\mathcal{I}^{st}-||\cdot||_{\mathbb{C}_2}} u$. Hence $\mathcal{I}_\lambda^{st} \subseteq \mathcal{I}^{st}$.

Definition 3.9 Consider an admissible ideal $\mathcal{I}$ in $\mathbb{N}$ and a sequence (u_k) in $\mathbb{C}_2$. We define $u_0 \in \mathbb{C}_2$ to be an $\mathcal{I}_\lambda^{st}$- limit point of (u_k) if there exists a set,
$T = \{t_1 < t_2 < \ldots < t_k < \ldots\} \notin \mathcal{I}$ such that $u_{t_k} \xrightarrow{||\cdot||\mathbb{C}_2} u_0$. The set of all $\mathcal{I}_\lambda^{st}$-limit points for any sequence (u_k) is denoted by $\Lambda_{(u_k)}(\mathcal{I}_\lambda^{st})$.

Definition 3.10 Consider an admissible ideal $\mathcal{I}$ in $\mathbb{N}$ and a sequence (u_k) in $\mathbb{C}_2$. We define $u_0 \in \mathbb{C}_2$ to be an $\mathcal{I}_\lambda^{st}$- cluster point of (u_k) if for any $\varepsilon > 0$ and $\delta > 0$,
$\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u_0||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\} \notin \mathcal{I}$. The set of all $\mathcal{I}_\lambda^{st}$-cluster points for any sequence (u_k) is denoted by $\Gamma_{(u_k)}(\mathcal{I}_\lambda^{st})$.

Theorem 3.11 In $\mathbb{C}_2$ consider the sequence (u_k) such that $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$.
Then $\Lambda_{(u_k)}(\mathcal{I}_\lambda^{st}) = \{u\}$.

Proof: If possible, let us assume that $\Lambda_{(u_k)}(\mathcal{I}_\lambda^{st})$ contains an additional element v such that $v \neq u$. Therefore, according to the definition, there exists a set $T \subset \mathbb{N}$ with $T = \{t_1 <$

$t_2 < \ldots < t_k < \ldots\} \notin \mathcal{I}$ such that $u_{t_k} \xrightarrow{st-||\cdot||_{\mathbb{C}_2}} u$. Let $G = G(\varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{n}|\{k \in I_n : ||u_k - v||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\}$ then, G is finite. So $(\mathbb{N} \setminus G) \in \mathcal{F}(\mathcal{I})$. Now, $u_k \xrightarrow{\mathcal{I}_\lambda^{st}-||\cdot||_{\mathbb{C}_2}} u$, so for any $\varepsilon > 0$ and $\delta > 0$, the set $F = F(\varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\} \in \mathcal{F}(\mathcal{I})$. Put $D = D(\varepsilon, \delta) = \{n \in T : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\}$. Since $(\mathbb{N} \setminus D) \supset F$, so $(\mathbb{N} \setminus D) \in \mathcal{F}(\mathcal{I})$. Thus, we have $(\mathbb{N} \setminus G) \cap (\mathbb{N} \setminus D) \in \mathcal{F}(\mathcal{I})$ and eventually $(\mathbb{N} \setminus G) \cap (\mathbb{N} \setminus D) \neq 0$. Let $p \in (\mathbb{N} \setminus G) \cap (\mathbb{N} \setminus D)$ and take $\varepsilon = ||\frac{u-v}{2}||_{\mathbb{C}_2}$ then we have $\frac{1}{\lambda_p}|\{k \in I_p : ||u_k - u||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta$ and $\frac{1}{\lambda_p}|\{k \in \mathcal{I}_p : ||u_k - v||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta$. By selecting δ to be sufficiently small, we can obtain an element say
$q \in \{k \in \mathcal{I}_p : ||u_k - u||_{\mathbb{C}_2} < \varepsilon\} \cap \{k \in I_p : ||u_k - v||_{\mathbb{C}_2} < \varepsilon\}$. But then, $\varepsilon = ||\frac{u-v}{2}||_{\mathbb{C}_2} \leq \dfrac{1}{2}(||u_k - u||_{\mathbb{C}_2} + ||u_k - v||_{\mathbb{C}_2}) < \dfrac{1}{2}(\varepsilon + \varepsilon) = \varepsilon$, which is a contradiction.

Theorem 3.12 For any sequence (u_k) in $\mathbb{C}_2$, $\Lambda_{(u_k)}(\mathcal{I}_\lambda^{st}) \subseteq \Gamma_{(u_k)}(\mathcal{I}_\lambda^{st})$.
Proof: Let $u_0 \in \Lambda_{(u_k)}(\mathcal{I}_\lambda^{st})$. Then, $\exists$ a set $T = \{t_1 < t_2 < ... < t_k < ...\} \notin \mathcal{I}$ such that $u_{t_k} \xrightarrow{st-||\cdot||_{\mathbb{C}_2}} u$. Thus, for all $\delta > 0$, $\exists\ n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\frac{1}{\lambda_n}|\{t_k \in I_n : ||u_k - u_0||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta$. Let $D = D(\varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u_0||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\}$. Then, $D \supseteq T \setminus \{t_1, t_2, t_3, ...t_{n_0}\}$. As $\mathcal{I}$ is admissible and $T \notin \mathcal{I}$. So $D \notin \mathcal{I}$.
Hence proved.

Theorem 3.13 Consider two sequences (u_k) and (v_k) in $\mathbb{C}_2$ such that $\{k \in \mathbb{N} : u_k \neq v_k\} \in \mathcal{I}$. Then, (i) $\Lambda_{(u_k)}(\mathcal{I}_\lambda^{st}) = \Lambda_{(v_k)}(\mathcal{I}_\lambda^{st})$ and
(ii) $\Gamma_{(u_k)}(\mathcal{I}_\lambda^{st}) = \Gamma_{(v_k)}(\mathcal{I}_\lambda^{st})$.

Proof : (i) Let $u_0 \in \Lambda_{(u_k)}(\mathcal{I}_\lambda^{st})$. Then, $\exists$ a set $T \subset \mathbb{N}$ with $T = \{t_1 < t_2 < ... < t_k < ...\} \notin \mathcal{I}$ such that $u_{t_k} \xrightarrow{st-||\cdot||_{\mathbb{C}_2}} u_0$. As $\{k \in T : u_k \neq v_k\} \subseteq \{k \in \mathbb{N} : u_k \neq v_k\}$ holds, so $N = \{k \in T : u_k = v_k\} \notin \mathcal{I}$ and $N \subseteq T$. Therefore, $v_{t_k} \xrightarrow{st-||\cdot||_{\mathbb{C}_2}} u_0$ holds and eventually we have, $\Lambda_{(u_k)}(\mathcal{I}_\lambda^{st}) \subseteq \Lambda_{(v_k)}(\mathcal{I}_\lambda^{st})$. Due to symmetry, $\Lambda_{(u_k)}(\mathcal{I}_\lambda^{st}) \supseteq \Lambda_{(v_k)}(\mathcal{I}_\lambda^{st})$.
Hence, we have $\Lambda_{(u_k)}(\mathcal{I}_\lambda^{st}) = \Lambda_{(v_k)}(\mathcal{I}_\lambda^{st})$.

(ii) Let $u_0 \in \Gamma_{(u_k)}(\mathcal{I}_\lambda^{st})$. Then, by definition for each $\varepsilon > 0$, and $\delta > 0$, the set $A = A(\varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||u_k - u_0||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta \notin \mathcal{I}$. Let $L = L(\varepsilon, \delta) = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||v_k - u_0||_{\mathbb{C}_2} \geq \varepsilon\}| < \delta\}$. We assert that $L \notin \mathcal{I}$. As if $L \in \mathcal{I}$ then $(\mathbb{N} \setminus L) \in \mathcal{F}(\mathcal{I})$ and consequently, from the hypothesis, we deduce $(\mathbb{N} \setminus L) \cap \{k \in \mathbb{N} : u_k = v_k\} \in \mathcal{F}(\mathcal{I})$. Consequently $(\mathbb{N} \setminus A) \supseteq (\mathbb{N} \setminus L) \cap \{k \in \mathbb{N} : u_k = v_k\}$ which is a contradiction that $(\mathbb{N} \setminus A) \in \mathcal{F}(\mathcal{I})$. So, $L \notin \mathcal{I}$ i.e $u_0 \in \Gamma_{(v_k)}(\mathcal{I}_\lambda^{st})$. Thus, $\Gamma_{(u_k)}(\mathcal{I}_\theta^{st}) \subseteq \Gamma_{(v_k)}(\mathcal{I}_\lambda^{st})$. By symmetry $\Gamma_{(u_k)}(\mathcal{I}_\lambda^{st}) \supseteq \Gamma_{(v_k)}(\mathcal{I}_\lambda^{st})$. Hence, we have $\Gamma_{(u_k)}(\mathcal{I}_\lambda^{st}) = \Gamma_{(v_k)}(\mathcal{I}_\lambda^{st})$.

Proposition 3.1 The spaces $S_\lambda^{\mathcal{I}-st}$ and $S_{0\lambda}^{\mathcal{I}-st}$ are solid and monotone.

Proof: Suppose (t_k) be a sequence of scalars with $||t_k||_{\mathbb{C}_2} \leq 1$ for all $k \in \mathbb{N}$. Then, we have the space $(S_0^{\mathcal{I}-st})_\lambda$ is solid by the following relation

$\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||t_k x_k||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\} \subseteq \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : ||x_k||_{\mathbb{C}_2} \geq \varepsilon\}| \geq \delta\}$.
The space $S_{0\lambda}^{\mathcal{I}-st}$ is monotone. Since every solid space is monotone.
The remaining aspect of the result proceeds analogously.

Proposition 3.2 The spaces $S_{\lambda}^{\mathcal{I}-st}$ and $S_{0\lambda}^{\mathcal{I}-st}$ are not sequence algebra in general.

Proof: The validity of the result is substantiated by the subsequent example.

Example 3.1: Consider $\mathcal{I} = \mathcal{I}_d$. Let $\lambda_n = \begin{cases} e_1 + e_2, & if\ n = 1; \\ \frac{n}{2}, & otherwise, \end{cases}$
and let (u_k) and (v_k) be two sequences in $\mathbb{C}_2$ defined as,
$u_k = \begin{cases} i_1 + \frac{i_1 i_2}{k^2}, & if\ k\ is\ even; \\ i_1 i_2, & otherwise, \end{cases}$ and $v_k = \begin{cases} \frac{i_1 i_2}{k^2}, & if\ k\ is\ odd; \\ 0, & otherwise. \end{cases}$
Then, $(u_k), (v_k) \in S_{\lambda}^{\mathcal{I}-st}$ but $(u_k).(v_k) \notin S_{\lambda}^{\mathcal{I}-st}$.
Hence $S_{\lambda}^{\mathcal{I}-st}$ is not a sequence algebra.
The remaining aspect of the result proceeds analogously.

Proposition 3.3 The spaces $S_{\lambda}^{\mathcal{I}-st}$ and $S_{0\lambda}^{\mathcal{I}-st}$ are not symmetric in general.

Proof: The validity of the result is substantiated by the subsequent example.

Example 3.2: Consider $\mathcal{I} = \mathcal{I}_c$. Suppose $\lambda_n = \begin{cases} e_1 + e_2, & if\ n = 1; \\ \frac{n}{2}, & otherwise, \end{cases}$
and let (u_k) be a sequences in $\mathbb{C}_2$ defined as,
$u_k = \begin{cases} i_1 + i_2, & if\ k = n^2, n \in \mathbb{N}; \\ i_1 i_2 k, & otherwise. \end{cases}$
Then, $(u_k) \in S_{\lambda}^{\mathcal{I}-st}$. Let (v_k) be a rearrangement of (x_k), defined by $(v_k) = (u_1, u_2, u_4, u_3, u_9, u_5, u_{16}, u_6, ...)$. Then, $(v_k) \notin S_{\lambda}^{\mathcal{I}-st}$.
Hence $S_{\lambda}^{\mathcal{I}-st}$ is not symmetric.
The remaining aspect of the result proceeds analogously.

4. Conclusion

The main contribution of this paper is to provide the notion of $\mathcal{I}_\lambda$-statistical convergence of sequences of bi-complex number and study some of its properties and identify the relationships between newly introduced notion and its relationship with $\mathcal{I}$-statistical convergence of bi-complex numbers. In the end we introduce and investigate the concept of $\mathcal{I}_\lambda$- statistical limit points, cluster points and establish some implication relations. These ideas and results are expected to be a source for researchers in the area of convergence sequences of bi-complex number. Also, these concepts can be generalized and applied for further studies.

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