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SOME DIFFERENCE GRAPHS

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Abstract. In this paper, we discuss difference labeling of various types of graphs, including the star graph, the generalized butterfly graph, the bistar graph, the umbrella graph, and the olive tree. Through our analysis, we demonstrate that these graphs are indeed difference graphs. Additionally, we present difference labeling for several types of snakes, such as the double triangular snake, the irregular triangular snake, the C_n- snake, and the alternate C_n- snake. A unique difference labeling for the complete graph K_3 is enabled, as well as a full signature characterization of the star graph is obtained.

Keywords: Difference graphs, Graph theory, Graph labeling.

AMS Subject Classification: 05C78

1. Introduction

The notion of the autograph was introduced by G. S. Bloom, P. Hell, and H. Taylor [1]. Harary [2] called the autograph a difference graph. S. Bloom, Hell, and Taylor [1] have shown that the following graphs are difference graphs: trees, $C_n; K_n; K_{n,n}; K_{n,n-1}$, star graph, pyramids, and n-prisms. Gervacio [3] proved that wheels W_n are difference graphs if and only if $n = 3; 4$, or 6. Sonntag [4] proved that cacti with girth at least 6 are difference graphs, and he conjectured that all cacti are difference graphs. Sugeng and Ryan [5] have provided difference labelings for cycles; fans; cycles with chords; graphs obtained by the one-point union of K_n and P_m; and graphs made from any number of copies of a given graph G that has a difference labeling by identifying one vertex of the first with a vertex of the second, a different vertex of the second with the third and so on, In [6], Seoud and Helmi provided a survey of all graphs of order at most 5 and showed that the following graphs are difference graphs: K_n for $n \geq 4$ with two deleted edges having no vertex in common; K_n for $n \geq 6$ with three deleted edges having no vertex in common; gear graphs G_n for $n \geq 3$; $P_m \times P_n$ for $m, n \geq 2$; triangular snakes; C_4-snakes; dragons; graphs consisting of two cycles of the same order joined by an edge, and graphs obtained by identifying the center of a star with a vertex of a cycle.

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In this paper, we proved that the complete graph with three vertices, the star graph, the butterfly graph, the bistar graph, the umbrella graph,the olive tree, the double triangular snake, irregular triangular snake, and alternate C_n-snake are difference graphs. Furthermore, we introduce a full signature characterization of the complete graph with three vertices, the star graph S_n and we induced a new possible signatures of the star graph S_n.

The paper is organized as follows, the next section is devoted to some basic concepts. Some difference labeling for some graphs are introduced in Section 3. In this paper we are following the basic definitions and notations for graph theory as in [7, 8, 9, 10].

2. Basic Concepts

In this section, we recall some basic definitions which are useful for this paper.

Definition 2.1. *[11] A complete graph K_n is simple undirected graph in which every pair of distinct vertices is connected by a unique edge.*

Definition 2.2. *[11] A graph G is called a complete bipartite graph $K_{m,n}$ if its vertices can be partitioned into two subsets V_1 and V_2 such that no edges has both end points in the same subset, and each vertex of $V_1(V_2)$ is connected with all vertices of $V_2(V_1)$.*

Definition 2.3. *[12] A star graph S_n is a complete bipartite graph with one internal node and n leaves.*

Definition 2.4. *[14] A generalized butterfly graph, BF_n, is obtained by inserting vertices into every wing with the assumption that the sum of inserted vertices to every wing is the same. Then it has $2n+1$ vertices and $4n-2$ edges. Let the vertex set of BF_n be $V(BF_n) = \{v_i \mid i = 0, 1, 2, \ldots, 2n\}$ and the edge set of BF_n be $E(BF_n) = \{(v_i, v_{i+1}) \mid i = 1, 2, \ldots, n-1, n+1, \ldots, 2n-1\} \cup \{(v_0, v_i) \mid i = 1, 2, \ldots, 2n\}$.*

Definition 2.5. *[13] A bistar graph $B_{m,n}$ is a graph derived by joining the centre node of two-star graph $K_{1,m}$ and $K_{1,n}$ by an edge.*

Definition 2.6. *[13] An umbrella graph $U_{m,n}$ is the graph obtained by joining a path P_n with the central vertex of a fan F_m.*

Definition 2.7. *[6] A triangular snake T_n is obtained from a path P_n by replacing each edge of the P_n by a cycle C_3.*

Definition 2.8. *[13] An alternate triangular snake $A(T_n)$ is obtained from a path P_n by replacing each alternate edge of P_n by a cycle C_3.*

Definition 2.9. *[13] A double triangular snake DT_n consists of two triangular snakes that have a common path.*

Definition 2.10. *[13] An irregular triangular snake IT_n for $n \geq 4$ is the graph obtained by the path P_n with vertex set $V(IT_n) = V(P_n) \cup \{v_i : 1 \leq i \leq n-2\}$ and edge set $E(IT_n) = E(P_n) \cup \{u_i v_i, v_i u_{i+2} : 1 \leq i \leq n-2\}$.*

Definition 2.11. *[13] A C_n–snake is the graph obtained from a path P_n by replacing each edge of the P_n by a cycle C_n.*

Definition 2.12. *[13] An alternate C_n-snake $A(C_n - snake)$ is the graph obtained from a path P_n by replacing every alternate edge of the P_n by a cycle C_n.*

Definition 2.13. *[13] An Olive tree (T_k) is a rooted tree consisting of k branches where the i^{th} branch is a path of length i.*

3. New Results

In this section, we will recall the definition of a difference graph and an important proposition, its proof due to Bloom *et al.* in [1], and then will prove our main results.

Definition 3.1. *[2] A graph $G(V, E)$ is called a difference graph if there is a bijective map f from V to a set of positive integers S such that $xy \in E$ if and only if $|f(x) - f(y)| \in S$, and S is said to be the signature of G.*

Proposition 3.1. *[1]*

i) *Vertex label values s and $2s$ belong to adjacent vertices (first type),*
ii) *Vertex label values r and t belong to vertices adjacent to a vertex labeled $r + t$ (second type),*
iii) *No other adjacency occur in difference graphs.*

Proof. (i) and (ii) are obvious. To prove (iii), note that if the vertices with labels r and $r + t$ are adjacent, then $|(r + t) - r| = t$ belongs to S. Hence, either $t = r$ and we have an edge of the first kind, or $t \neq r$ and $|(r + t) - r| = t \in S$, i.e., we have an edge of the second kind. □

Corollary 3.1. *Let S be the signature of a difference graph G, then*

i) *For all $s \in S$, $s \in \{2a, \frac{a}{2}, a + b, |a - b|\}$ for some $a, b \in S$.*
ii) *The minimum label must be $\frac{a}{2}$ or $|a - b|$ for some a and $b \in S$.*
iii) *The maximum label must be $2a$ or $a + b$ for some $a, b \in S$.*
iv) *The degree of the vertex with the maximum label s is odd if and only if it is adjacent to a vertex labeled by $\frac{s}{2}$.*

Proof.

i) The proof of this case is straightforward, using Proposition 3.1.
ii) Let c be the minimum label in G and let the vertex labeled by c be adjacent to the vertex labeled by a, hence $a - c \in S$. Therefore, either $a - c = c$ i.e. $c = \frac{a}{2}$, or $a - c = b$ for some $b \in S$, which implies that $c = a - b$.
iii) Let c be the maximum label in the G and let the vertex labeled by c be adjacent to the vertex labeled by a, hence $c - a \in S$. Therefore, either $c - a = a$ i.e. $c = 2a$, or $c - a = b$ for some $b \in S$, which implies that $c = a + b$.
iv) We have two types of labelings in the Difference graph. In case of the first type, the vertex with maximum label is adjacent to a vertex labeled by $\frac{s}{2}$, therefore it shares 1 in the degree of the vertex with maximum label. So, the statement is done. In case of the existence of second type, since the vertex with maximum label is adjacent to two vertices such that the sum of their labels is s, it shares multiples of 2 in the degree of the vertex with maximum label.

□

Theorem 3.1. *The complete graph K_3 is a difference graph with a unique signature form $S = \{3a, 2a, a\}$, where a is a positive integer.*

Proof. Let u_1 be the vertex with the maximum label A. Then, Corollary 3.1 implies that the label of the vertices u_2 and u_3 must be $\{a, A - a\}$. Without loss of generality, one can assume that $A - a > a$. Therefore, $A - a - a = A - 2a \in S$. Consequently, $A - 2a = a$, which gives $A = 3a$. Then, the labels of u_1, u_2 and u_3 will be $\{a, 2a, 3a\}$. □

Theorem 3.2. *The star graph S_n is a difference graph. Moreover, S is a signature of the star graph S_n if and only if it has one of the forms:*

i) If n is even, then

$$S = \left\{a, b_1, a - b_1, b_2, a - b_2, \ldots, b_{\frac{n}{2}}, a - b_{\frac{n}{2}}\right\}$$

with $a = \max\{s : s \in S\}$ is the label of the vertex of degree n and the difference between any two elements in $S - \{a\}$ does not belong to S, or

$$S = \left\{4a, 2a, a, b_1, 2a - b_1, b_2, 2a - b_2, \ldots, b_{\frac{n-2}{2}}, 2a - b_{\frac{n-2}{2}}\right\}$$

with $2a$ is the label of the vertex of degree n and the difference between any two elements of $S - \{2a\}$ does not belong to S.

ii) If n is odd, then

$$S = \left\{2a, a, b_1, 2a - b_1, b_2, 2a - b_2, \ldots, b_{\frac{n-1}{2}}, 2a - b_{\frac{n-1}{2}}\right\}$$

with $2a = \max\{s : s \in S\}$ is the label of the vertex of degree n and the difference between any two elements of $S - \{2a\}$ does not belong to S, or

$$S = \left\{2a, a, b_1, a - b_1, b_2, a - b_2, \ldots, b_{\frac{n-1}{2}}, a - b_{\frac{n-1}{2}}\right\}$$

with a is the label of the vertex of degree n and the difference between any two elements of $S - \{a\}$ does not belong to S.

Proof. In [1], Bloom *et al.* proved that the star graph is a difference graph. More precisely, in the case that n is even, they get the following signature

$$S = \{2n\} \cup \left\{2m - 1 : m \leq \frac{n}{2}\right\} \cup \left\{2n - (2m - 1) : m \leq \frac{n}{2}\right\},$$

and $S \cup \{4n\}$ in the case that n is odd.

The backward direction is clear. It remains for us to prove the forward direction of the given characterization. Let S be a signature of the star graph S_n and let the set of vertices of S_n be $\{u_0, u_1, u_2, \ldots, u_n\}$, where u_0 is the vertex of degree n.

i) Let n be an even integer. In the case that the label of the vertex u_0 is the maximum label, say a, Corollary 3.1 implies that the vertices $\{u_1, u_2, \ldots, u_n\}$ must be labeled by $S - \{a\} = \left\{b_1, a - b_1, b_2, a - b_2, \ldots, b_{\frac{n}{2}}, a - b_{\frac{n}{2}}\right\}$, where the difference between any two elements in $S - \{a\}$ does not belong to S. On the other hand, without loss of generality, assume that u_1 be the vertex with maximum label A, then Corollary 3.1 implies that u_0 must be labeled by $\frac{A}{2}$. Since no vertex from $u_2, u_3, \ldots, u_n$ can be labeled bigger than the label of u_0 (if that happened its label must be A which is rejected), therefore u_0 will be the vertex with maximum label of the vertices $\{u_0, u_2, u_3, \ldots, u_n\}$. Since the number of elements of the set $\{u_2, u_3, \ldots, u_n\}$ is odd, therefore Corollary 3.1 implies that the labels of the vertices $\{u_2, u_3, \ldots, u_n\}$ will be $\left\{\frac{A}{4}, b_1, \frac{A}{2} - b_1, b_2, \frac{A}{2} - b_2, \ldots, b_{\frac{n-2}{2}}, \frac{A}{2} - b_{\frac{n-2}{2}}\right\}$, where the difference between any two elements in $S - \left\{\frac{A}{2}\right\}$ does not belong to S. Replacing A by $4a$, we get the required form of S.

ii) The proof when n is odd is very similar to the proof of part (i).

□

Note, the signatures of star graph introduced by Bloom [1] satisfy the first case when n is even and the second case when n is odd in Theorem 3.2.

The following corollary introduces possible signatures - other than the Bloom's ones - induced by the signature characterization of the star graph S_n in Theorem 3.2.

Corollary 3.2. *the signature of the star graph S_n Can take the following forms;*

i) *If n is even, then $S = \{4(n-1), 2(n-1)\} \cup \{2m-1 : m \leq n-1\}$.*

ii) *If n is odd, then $S = \{2n\} \cup \{2m-1 : m \leq n\}$.*

Proof. It is straight forward application of Theorem 3.2. □

Example 3.1. *Difference labelings of the star graph S_8 are illustrated in Fig.1*

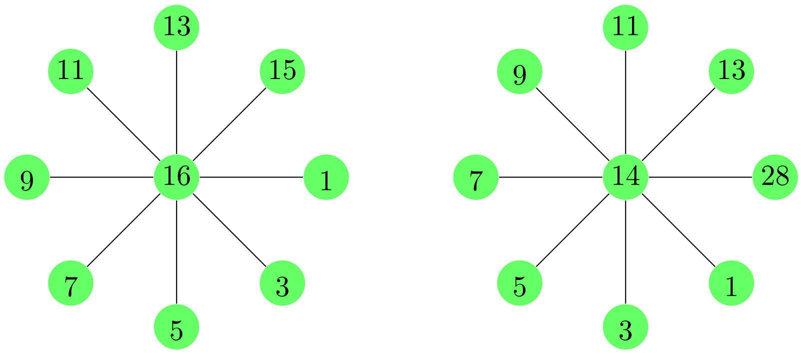

FIGURE 1. Difference labelings of S_8.

Theorem 3.3. *The generalized butterfly graph BF_n is a difference graph.*

Proof. Let the butterfly graph be described as indicated in Fig.2.

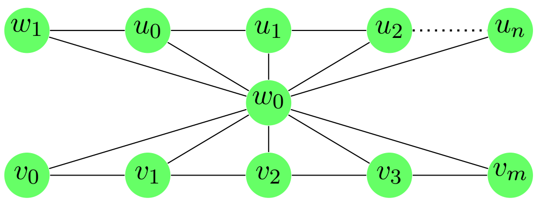

FIGURE 2. Generalized butterfly graph.

The required labeling function $f : V(BF_n) \longrightarrow S \subset \mathbb{N}$ is defined as follows:

$$\begin{aligned} f(w_0) &= 6, \\ f(w_1) &= 2, \\ f(u_i) &= 4+6i, \quad i = 0,1,2,\ldots,n, \\ f(v_j) &= 3+6j, \quad j = 0,1,2,\ldots,m. \end{aligned}$$

□

Example 3.2. *A difference labeling of the generalized butterfly graph BF_5 is illustrated in Fig.3.*

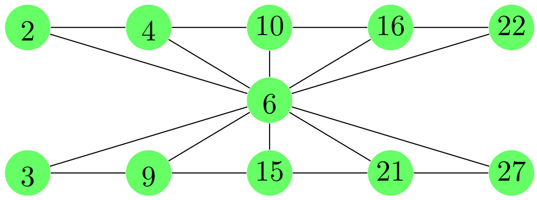

FIGURE 3. A difference labeling of the generalized butterfly graph BF_5.

Theorem 3.4. *The bistar graph $B_{m,n}$ is a difference graph.*

Proof. Let the bistar graph $B_{m,n}$ be described as indicated in Fig.4.

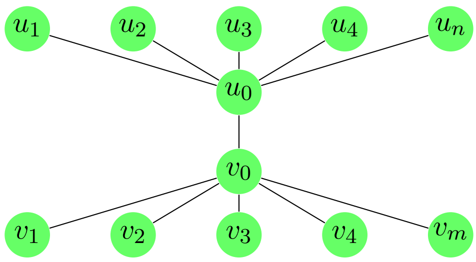

FIGURE 4. Bistar graph $B_{m,n}$.

The required labeling function $f : V(B_{m,n}) \longrightarrow S \subset \mathbb{N}$ is defined as follows:

$$\begin{aligned} f(u_0) &= 2n, \\ f(u_i) &= 2i-1, \quad i = 1, 2, \ldots, n, \\ f(v_j) &= 2n + j(2n+2), \quad j = 1, 2, \ldots, m, \\ f(v_0) &= f(u_0) + f(v_m) = 4n + 2m(n+1). \end{aligned}$$

□

Example 3.3. *A difference labeling of the bistar graph $B_{5,5}$ is illustrated in Fig.5.*

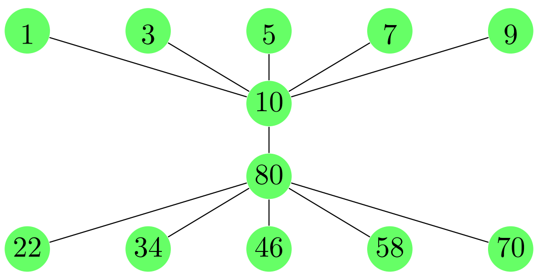

FIGURE 5. A difference labeling of $B_{5,5}$.

Theorem 3.5. *The umbrella graph $U_{m,n}$ is a difference graph.*

Proof. Let the umbrella graph $U_{m,n}$ be described as indicated in Fig.6.

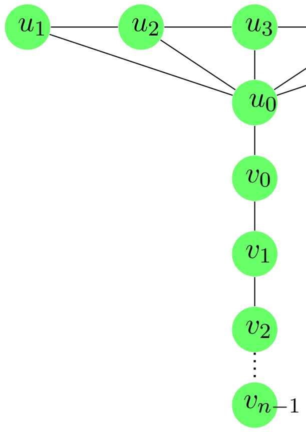

FIGURE 6. Umbrella graph $U_{m,n}$.

The required labeling function $f : V(U_{m,n}) \longrightarrow S \subset \mathbb{N}$ is defined as follows:

$$\begin{aligned} f(u_0) &= 2, \\ f(u_i) &= 2i-1, \quad i = 1, 2, \ldots, m, \\ f(v_0) &= f(u_0) + f(v_1) = 4m + 2, \\ f(v_j) &= 2^{j-1} \cdot 4m, \ j = 1, 2, \ldots, n-1. \end{aligned}$$

□

Example 3.4. *A difference labeling of the umbrella graph $U_{5,4}$ is illustrated in Fig.7*

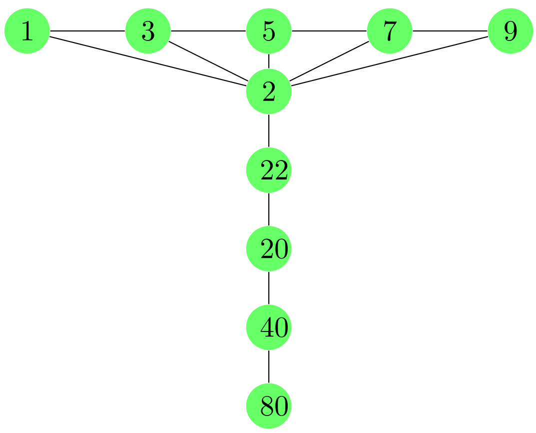

FIGURE 7. A difference labeling of $U_{5,4}$.

Theorem 3.6. *The double triangular snake DT_n is a difference graph.*

Proof. Let the double triangular snake DT_n be described as indicated in Fig.8.

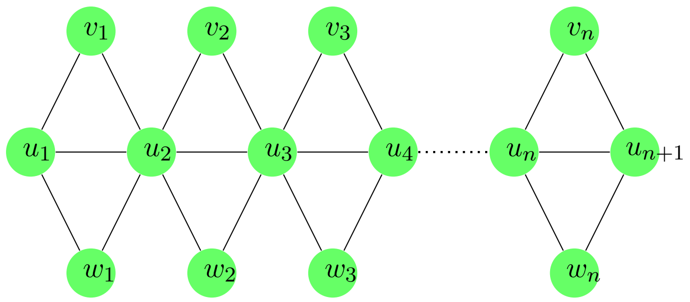

FIGURE 8. The double triangular snake DT_n.

The required labeling function $f : V(DT_n)) \longrightarrow S \subset \mathbb{N}$ is defined as follows

$$\begin{aligned} f(u_i) &= 3^{i-1} \cdot 2^{n-(i-1)} \ , i = 1, 2, \ldots, n+1, \\ f(v_i) &= 5 \cdot 3^{i-1} \cdot 2^{n-i} \ , i = 1, 2, \ldots, n, \\ f(w_i) &= 3^{i-1} \cdot 2^{n-i} \ , i = 1, 2, \ldots, n. \end{aligned}$$

□

Example 3.5. *A difference labeling of the double triangular snake DT_5 is illustrated in Fig.9*

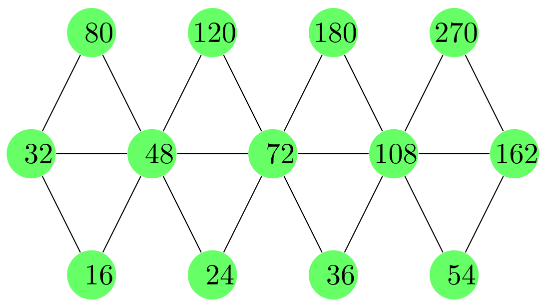

FIGURE 9. A difference labeling of DT_5.

Theorem 3.7. *The irregular triangular snake IT_n is a difference graph.*

Proof. Assume that the irregular triangular snake IT_n is described as indicated in Fig.10. The required labeling function $f : V(IT_n)) \longrightarrow S \subset \mathbb{N}$ is defined as follows:

$$\begin{aligned} f(u_i) &= 2^i \ , i = 1, 2, \ldots, n, \\ f(v_j) &= f(u_j) + f(u_{j+2}) = 5 \cdot 2^j \ , i = 1, 3, 5, \ldots, n-3, \\ f(w_k) &= f(v_k) + f(v_{k+2}) = 5^2 \cdot 2^k, \ \ , i = 1, 2, \ldots, n-3, \\ f(w_{n-3}) &= f(v_{n-3}) + f(u_n). \end{aligned}$$

□

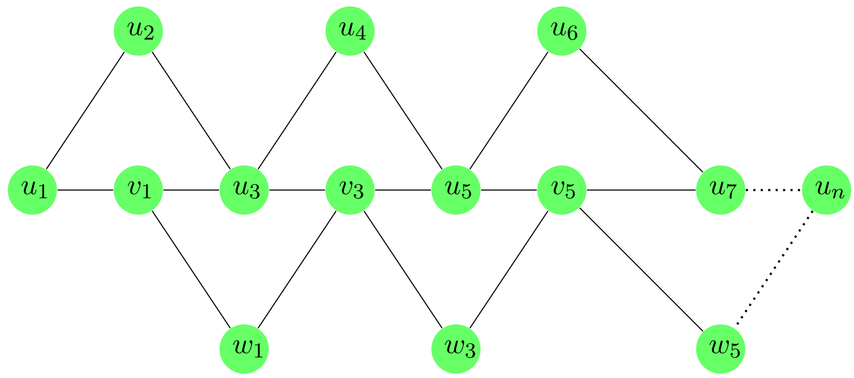

FIGURE 10. The irregular triangular snake IT_n.

Example 3.6. *A difference labeling of the irregular triangular snake IT_8 is illustrated in Fig.11*

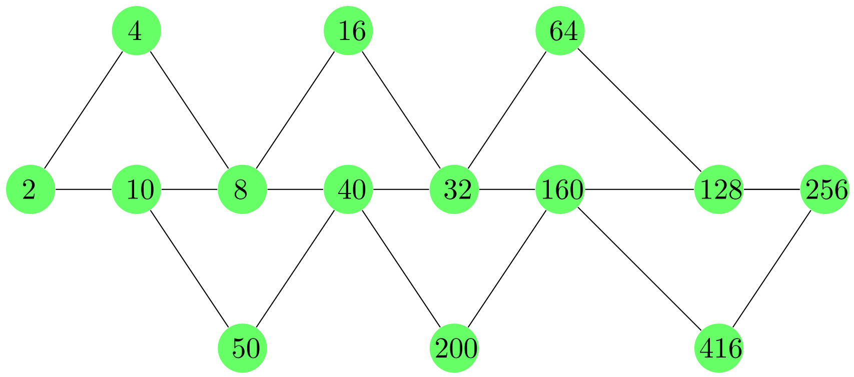

FIGURE 11. A difference labeling of IT_8.

Theorem 3.8. *The C_n-snake is difference graph.*

Proof. Let the C_n-snake be described as indicated in Fig.12

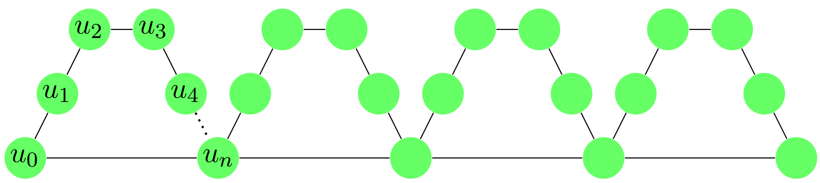

FIGURE 12. C_n-snake.

The required labeling function $f : V(C_n - snake) \longrightarrow S \subset \mathbb{N}$ is defined as follows:

$$f(u_i) = 2^j \ (1 + 2^{n-2})^{\lfloor \frac{i}{n-1} \rfloor}, \qquad i = 0, 1, 2, \ldots,$$

where $j = rem(i, n-1)$ is the remainder of i when it divided by $n-1$. □

Example 3.7. *A difference labeling of the C_6- snake is illustrated in Fig.13.*

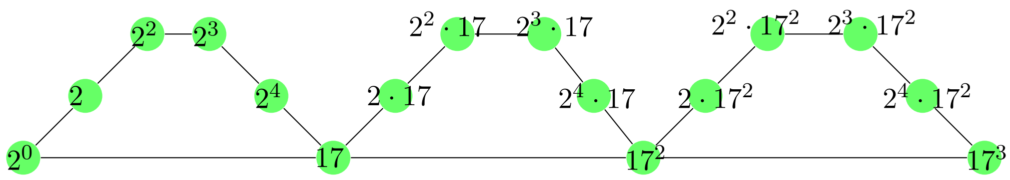

FIGURE 13. A difference labeling of C_6-snake.

Theorem 3.9. *The alternate C_n-snake $A(C_n - snake)$ is a difference graph.*

Proof. Let the alternate C_n-snake be described as indicated in Fig.14.

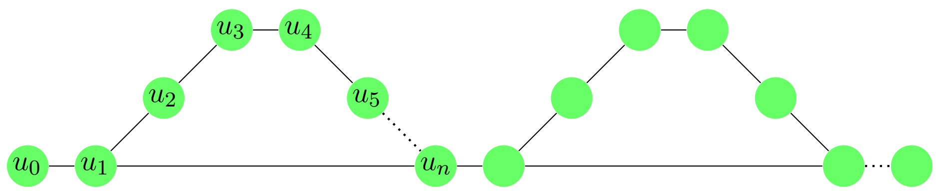

FIGURE 14. Alternate C_n-snake.

The required labeling function $f : V(A(C_n - snake)) \longrightarrow S \subset \mathbb{N}$ is defined as follows:

$$f(u_i) = 2^{j+\lfloor \frac{i}{n} \rfloor}\,(1 + 2^{n-2})^{\lfloor \frac{i}{n} \rfloor}, \quad i = 0, 1, 2, \ldots$$

where $j = rem(i, n)$ is the remainder of i when it divided by n. □

Example 3.8. *A difference labeling of the alternate C_5-snake $A(C_5-snake)$ is illustrated in Fig.15*

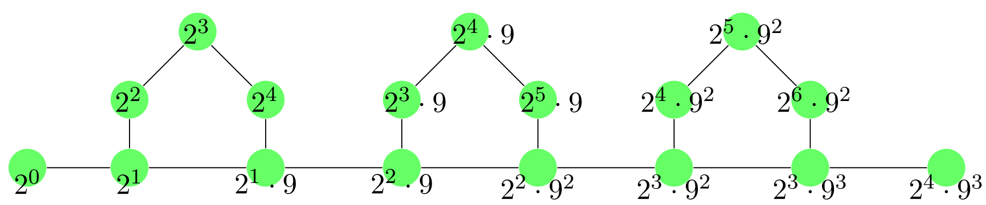

FIGURE 15. A difference labeling of $A(C_5 - snake)$.

Theorem 3.10. *The olive tree OT_k is a difference graph.*

Proof. Assume that the olive tree described as indicated in Fig.16. The required labeling function $f : V(OT_k) \longrightarrow S$ is defined as follows:

$$\begin{aligned} f(root) &= 3, \\ f(v_1) &= 6, \\ f(v_i) &= 3 + 10^{i-1},\ i = 2, 3, \ldots, n, \\ f(v_{i,j}) &= 2^{j-1} \cdot 10^{i-1},\ j = 1, 2, \ldots, i-1. \end{aligned}$$

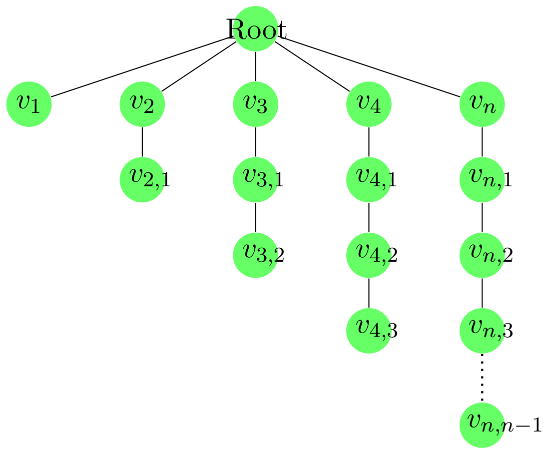

FIGURE 16. Olive tree OT_n.

□

Example 3.9. *A difference labeling of the olive tree OT_5 labeling is illustrated in Fig.17.*

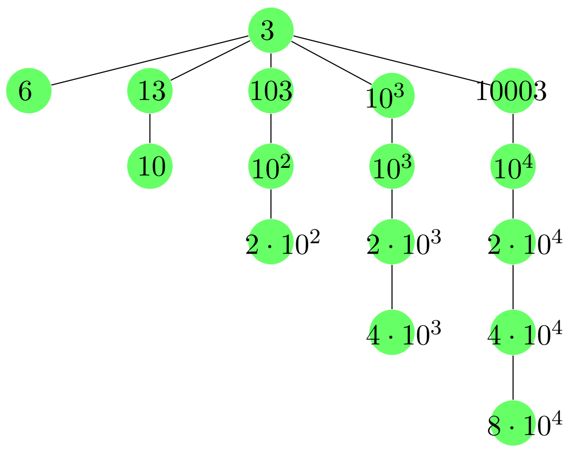

FIGURE 17. A difference labeling of OT_5.

Theorem 3.11. *The complete bipartite graph $K_{2,4}$ is not a difference graph.*

Proof. There are many cases which are too involved. The reader can contact the second author for details. □

4. CONCLUSIONS

In this paper, we have discussed the idea of difference labeling applied to different types of graphs, such as stars, butterflies, bistars, umbrellas, and olive trees. Additionally, we've introduced new difference labeling for various snake graphs like the double triangular snake, irregular triangular snake, C_n- snake, and alternate C_n-snake. We've also devised a unique difference labeling for the complete graph K_3 and demonstrated all forms of difference labeling of the star graph.

References

[1] Bloom, G. S., Hell, P., and Taylor, H., (1979), Collecting autographs: n-node graphs that have n-integer signatures, Annals N.Y. Acad. Sci., 319, 93–102.
[2] Harary, F., (1990), Sum graphs and difference graphs, Congr. Numer., 72, 101–108.
[3] Gervacio, S. V., (1983), Which wheels are proper autographs?, Sea Bull. Math., 7, 41–50.
[4] Sonntag, M., (2003), Difference labeling of cacti, Discuss. Math. Graph Theory, 23, 55–65.
[5] Sugeng, K. A., and Ryan, J., (2007), On several classes of monographs, Australas. J. Combin., 37, 277–284.
[6] Seoud, M. A., and Helmi, E. F., (2011), On difference graphs, J. Combin. Math. and Combin. Comput., 76, 189–199.
[7] Diestel, R., (2017), Graph Theory, GTM, Springer, Heidelberg, 173.
[8] Wallis, W. D., (2007), A beginner's guide to graph theory, Springer Science and Business Media.
[9] Saoub, K. R., (2017), A Tour Through Graph Theory, CRC Press, 2.
[10] Bickle, A., (2020), Fundamentals of Graph Theory, American Mathematical Soc.
[11] West, D. B., (2001), Introduction to Graph Theory, Prentice Hall, 2nd Edition, 6.
[12] Balakrishnan, R., and Ranganathan, K., (2000), A Textbook of Graph Theory, Universitext Springer.
[13] Gallian J. A., (2022), A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 25.
[14] Wahyuna, H. D., and Indriati, D., (2018), On the total edge irregularity strength of generalized butterfly graph, Journal of Physics: Conference Series, 1008.
[15] Talal Al-Hawary, Sumaya Al-Shalaldeh, Muhammad Akram, (2023), Certain matrices and energies of fuzzy graphs, TWMS JPAM V.14, N.1, pp.50-68.

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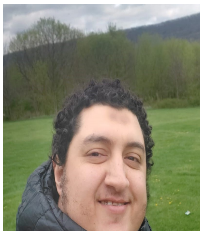

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