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ON THE CLASSIFICATION OF PYTHAGOREAN FUZZY SUBGROUPS OF ABELIAN GROUPS

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Abstract. Pythagorean fuzzy set is one of the most used tool for depicting uncertainty. A divisible subgroup is among the most significant categories of subgroups of an abelian group. The number of Pythagorean fuzzy subgroups in any group is infinite without a suitable equivalence constraint on the Pythagorean fuzzy sets. To get a meaningful categorization, a sufficient equivalent condition on the collection of all Pythagorean fuzzy subgroups needs to be defined. In this paper, the concept of Pythagorean fuzzy divisible subgroups of a group is introduced. An equivalence relation on Pythagorean fuzzy sets is defined. Several properties of this equivalence relation on Pythagorean fuzzy subgroups are explained. Pythagorean fuzzy subgroups related to their maximal chains are introduced. All possible Pythagorean fuzzy subgroups of finite abelian groups are investigated.

Keywords: Pythagorean fuzzy set, Pythagorean fuzzy divisible subgroup, equivalence relation on PFS, maximal chains of PFSG, counting of PFSG.

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1. Introduction

Analysis of the subgroups of finite groups in a fuzzy environment is a critical issue. The discussion of fuzzy subgroups has advanced significance in past years. To deal with uncertainty in practical problems, Zadeh [31] created fuzzy set (FS) in 1965. Rosenfeld [21] established the idea of a fuzzy subgroup (FSG) in 1971. In 1979, FSG was redefined with the t-norm by Sherwood and Anthony [7]. Das [13] first mentioned the terminology of a fuzzy level subgroup. Level subgroups in the fuzzy environment and the union property of FSGs were explained by Dixit et al. [14] in 1990. Alkhamees [6] studied fuzzy cyclic p-subgroups as well as fuzzy cyclic subgroups. Sidky and Mishref introduced divisible fuzzy subgroups in 1990. In 2021, Ejegwa et al. [15] described divisible and pure fuzzy multigroups. In 2015, Tarnauceanu [26] classified fuzzy normal subgroups of finite groups. In 2016, Onasanya [20] reviewed some anti-fuzzy properties of fuzzy subgroups. In 2018,

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Addis [2] developed fuzzy homomorphism theorems on groups. Solairaju [23] and Mahalakshmi described hesitant intuitionistic fuzzy soft groups in 2018. Abuhijleh et al. [1] worked on complex fuzzy groups in 2021. Alolaiyan et al. [4] studied algebraic structure of (α, β)-complex fuzzy subgroups. Alolaiyan et al. [5] developed bipolar fuzzy subrings in 2021. In 2021, Talafha et al. [24] studied fuzzy fundamental groups and fuzzy folding of fuzzy Minkowski space.

On the set that includes all fuzzy subsets, the FSGs of a group can be categorised by proper equivalence relations on that set. Numerous papers have discussed how to analyze FSGs for specific instances of groups in terms of appropriate equivalence relations. In 1998, Zhang and Zou [32] determined the count of FSGs in a group that has the same equivalence classes. They specified the count of FSGs of cyclic groups. Murali and Makamba [16] described the circumstances by which the level subsets of FSs can explain the equivalent relations of FSs in 2001. Additionally, they suggested that only finite groups should utilize the equivalence approach to analyze FSGs. Murali and Makamba [17, 18], identified the quantity of FSGs in cyclic groups of order without square. In 2004, Volf [27] discussed chains of subgroups for counting fuzzy subgroups. Tarnauceanu and Benta [25] investigated the count of FSGs of Abelian groups of finite order in 2008.

1986, Atanassov [8] invented the intuitionistic fuzzy set (IFS). Yamak et al.[30] studied divisible and pure intuitionistic fuzzy subgroups in 2008. In 2013, Pythagorean fuzzy set (PFS) was introduced by Yager[28]. In many decision-making issues, the PFS is very effective. This idea is completely suited to mathematically illustrate uncertainty. Also to create a formalized method to deal with imprecision in practical scenarios. In 2021, Bhunia et al. [10] introduced Pythagorean fuzzy subgroups (PFSG) of any groups. Bhunia and Ghorai [9, 11] explained (α, β)-PFSs in 2021. In 2024, a number of findings regarding Pythagorean fuzzy subgroups are offered in [12]. In 2018, Naz et al. [19] proposed a novel approach to decision-making problems using Pythagorean fuzzy set. In 2019, Akram and Naz [3] applied complex Pythagorean fuzzy set to decision-making problems.

Divisible groups are a key concept in group theory, characterized by the property that for every element in the group and every positive integer, there exists an element whose multiple equals the given element. They play a crucial role in the classification of abelian groups and connect deeply with algebraic geometry, particularly in the study of elliptic curves and Galois representations. Here, we introduce divisible subgroups in Pythagorean fuzzy environment and study various properties of them. Without a proper equivalence condition on PFSs of a set, the number of PFSGs of any group is uncountable. An adequate equivalence condition must be defined on the collection of all PFSGs for obtaining a meaningful classification. To overcome this situation, we have introduced a proper equivalence condition on the collection of all PFSs of any set. With the help of proper equivalence relations, we investigate the total number of PFSGs of finite abelian groups.

The structure of this paper is designed in the following way: in Section 3, Pythagorean fuzzy divisible subgroups are described. An equivalence relation on PFSGs is defined in Section 4. In Section 5, results related to the count of PFSGs of abelian groups are given. Conclusion statements are produced in Section 6.

2. Preliminaries

We review some key terminologies in this section that will be crucial in developing subsequent sections.

Definition 2.1. [31] *Let U be a crisp set. Then $\mu : U \to [0, 1]$ is called a fuzzy subset of U. Here, $\mu(u)$ is called degree of membership of $u \in U$.*

Definition 2.2. [8] *Let U be a crisp set. An IFS I on U is defined by $I = \{(u, \mu(u), \nu(u))|u \in U\}$, where $\mu(u) \in [0,1]$ and $\nu(u) \in [0,1]$ are the degree of membership and non-membership of $u \in U$ respectively, which satisfy the condition $0 \leq \mu(u) + \nu(u) \leq 1$, $\forall u \in u$.*

Definition 2.3. [21] *A FS μ is a FSG of a group U if*
(i) $\mu(u_1 \cdot u_2) \geq \mu(u_1) \wedge \mu(u_2)$, $\forall u_1, u_2 \in U$,
(ii) $\mu(u^{-1}) \geq \mu(u)$, $\forall u \in U$.

Definition 2.4. [22] *Let μ be a FSG of U. Then μ is a fuzzy divisible subgroup of U if $\forall x_a \subseteq \mu$ with $a > 0$, and $\forall n \in \mathbb{N}$, there exists $y_a \subseteq \mu$ such that $n(y_a) = x_a$.*

Definition 2.5. [28] *A PFS ψ of U is $\psi = \{(u, \mu(u), \nu(u))|u \in U\}$, $\mu(u) \in [0,1]$ and $\nu(u) \in [0,1]$ are the membership and non membership degrees of $u \in U$ respectively, where $0 \leq \mu^2(u) + \nu^2(u) \leq 1$, $\forall u \in U$.*

Definition 2.6. [10] *Let ψ be a PFS of U. Then ψ is a PFSG of U if*
(i) $\mu^2(u_1 \cdot u_2) \geq \mu^2(u_1) \wedge \mu^2(u_2)$ and $\nu^2(u_1 \cdot u_2) \leq \nu^2(u_1) \vee \nu^2(u_2)$, $\forall u_1, u_2 \in U$,
(ii) $\mu^2(u^{-1}) \geq \mu^2(u)$ and $\nu^2(u^{-1}) \leq \nu^2(u)$, $\forall u \in U$.
Here, $\mu^2(u) = \{\mu(u)\}^2$ and $\nu^2(u) = \{\nu(u)\}^2$ for all $u \in U$.

Proposition 2.1. [17] *The count of maximal chains of $\mathbb{Z}_\lambda \times \mathbb{Z}_\lambda$ is $\lambda + 1$, where λ is a prime.*

3. Pythagorean fuzzy divisible subgroups

This section will introduce the concept of Pythagorean fuzzy divisible subgroups. Through out this section, U stands for an abelian group. The collection of all PFSs of U is denoted by $A_U(\psi)$ and all PFSGs of U is denoted by $AS_U(\psi)$.

Definition 3.1. *Let ψ be a PFS of U. For $n \in \mathbb{N}$, define $n\psi = (n\mu, n\nu)$ as follows:*

$$(n\mu)^2(x) = \begin{cases} \bigvee_{x=ny} \mu^2(y), & \text{when } x \in nU \\ 0, & \text{elsewhere} \end{cases}$$

and

$$(n\nu)^2(x) = \begin{cases} \bigwedge_{x=ny} \nu^2(y), & \text{when } x \in nU \\ 1, & \text{elsewhere} \end{cases}$$

where $nU = \{nu|u \in U\}$.

Example 3.1. *Consider $U = (\mathbb{Z}_3, +)$.*

Assign a PFS $\psi = (\mu, \nu)$ on U by $\mu(\bar{0}) = 0.7$, $\mu(\bar{1}) = 0.8$, $\mu(\bar{2}) = 0.6$, $\nu(\bar{0}) = 0.4$, $\nu(\bar{1}) = 0.3$ and $\nu(\bar{2}) = 0.5$.

Then $\mu^2(\bar{0}) = 0.49$, $\mu^2(\bar{1}) = 0.64$, $\mu^2(\bar{2}) = 0.36$, $\nu^2(\bar{0}) = 0.16$, $\nu^2(\bar{1}) = 0.09$ and $\nu^2(\bar{2}) = 0.25$.

Take $n = 2$. Therefore $2\psi = (2\mu, 2\nu)$ is given by

$(2\mu)^2(\bar{0}) = \bigvee_{\bar{0}=2\bar{y}} \mu^2(\bar{y}) = \mu^2(\bar{0}) = 0.49$, $(2\mu)^2(\bar{1}) = \bigvee_{\bar{1}=2\bar{y}} \mu^2(\bar{y}) = \mu^2(\bar{2}) = 0.36$, $(2\mu)^2(\bar{2}) = \bigvee_{\bar{2}=2\bar{y}} \mu^2(\bar{y}) = \mu^2(\bar{1}) = 0.64$, $(2\nu)^2(\bar{0}) = \bigwedge_{\bar{0}=2\bar{y}} \nu^2(\bar{y}) = \nu^2(\bar{0}) = 0.16$, $(2\nu)^2(\bar{1}) = \bigwedge_{\bar{1}=2\bar{y}} \nu^2(\bar{y}) = \nu^2(\bar{2}) = 0.25$, and $(2\nu)^2(\bar{2}) = \bigwedge_{\bar{2}=2\bar{y}} \nu^2(\bar{y}) = \nu^2(\bar{1}) = 0.09$ where $\bar{y} \in 2\mathbb{Z}_3$.

Definition 3.2. *Let $\psi \in AS_U(\psi)$. Then ψ is a Pythagorean fuzzy divisible subgroup of U if $n\psi = \psi$ for all $n \in \mathbb{N}$.*

Example 3.2. *Consider $U = (\mathbb{Q}, +)$.*

Assign a PFS ψ on U as follows:

$$\mu(u) = \begin{cases} 0.9, & \text{when } u \in \{0\} \\ 0.7, & \text{elsewhere} \end{cases}$$

and

$$\nu(u) = \begin{cases} 0.4, & \text{when } u \in \{0\} \\ 0.5, & \text{elsewhere.} \end{cases}$$

Clearly $\psi \in AS_{\mathbb{Q}}(\psi)$.

Choose $n = 2$. Then $2\psi = (2\mu, 2\nu)$. Now, $(2\mu)^2(0) = \bigvee_{0=2y} \mu^2(y) = \bigvee_{y=0/2} \mu^2(y) = 0.81$ and $(2\nu)^2(0) = \bigwedge_{0=2y} \nu^2(y) = \bigwedge_{y=0/2} \nu^2(y) = 0.16$. Again, $(2\mu)^2(u) = \bigvee_{u=2y} \mu^2(y) = \bigvee_{y=u/2} \mu^2(y) = 0.49$ and $(2\nu)^2(u) = \bigwedge_{u=2y} \nu^2(y) = \bigwedge_{y=u/2} \nu^2(y) = 0.25$. Therefore $2\psi = \psi$.

Also, choose $n = 3$. Then $3\psi = (3\mu, 3\nu)$. Now, $(3\mu)^2(0) = \bigvee_{0=3y} \mu^2(y) = \bigvee_{y=0/3} \mu^2(y) = 0.81$ and $(3\nu)^2(0) = \bigwedge_{0=3y} \nu^2(y) = \bigwedge_{y=0/3} \nu^2(y) = 0.16$. Again, $(3\mu)^2(u) = \bigvee_{u=3y} \mu^2(y) = \bigvee_{y=u/3} \mu^2(y) = 0.49$ and $(3\nu)^2(u) = \bigwedge_{u=3y} \nu^2(y) = \bigwedge_{y=u/3} \nu^2(y) = 0.25$. Therefore $3\psi = \psi$.

In fact, we can check that, $n\psi = \psi$ for all $n \in \mathbb{N}$.

Therefore, ψ is a Pythagorean fuzzy divisible subgroup of U.

Theorem 3.1. *Let $\psi \in AS_U(\psi)$. Then ψ is a Pythagorean fuzzy divisible subgroup of U if and only if U is a divisible group.*

Proof. Suppose U is not a divisible group.

So, $nU \neq U$ for some $n \in \mathbb{N}$. Assume for $k \in \mathbb{N}$, $kU \neq U$.

So, the domain of $k\psi$ is different from ψ. Thus $k\psi \neq \psi$.

Therefore the contrapositive statement is true. That is if U is a divisible group, ψ is a Pythagorean fuzzy divisible subgroup of U.

Conversely, let ψ be a Pythagorean fuzzy divisible subgroup of U.

Then $n\psi = \psi$ $\forall n \in \mathbb{N}$.

Therefore $(n\mu)^2(t_1) = \bigvee_{t_1=nt_2} \mu^2(t_2) = \mu^2(t_1)$.

So, $\bigvee_{t_1=nt_2} \mu(t_2) = \mu(t_1)$.

Similarly, we have $\bigwedge_{t_1=nt_2} \nu(t_2) = \nu(t_1)$.

Therefore $\forall t_1 \in U$, there exists $t_2 \in U$ such that $t_1 = nt_2$.

Hence U is a divisible group. □

Example 3.3. *Consider $U = (\mathbb{Z}_4, +)$.*

Assign a PFS $\psi = (\mu, \nu)$ on U by $\mu(\bar{0}) = 0.9$, $\mu(\bar{1}) = 0.7 = \mu(\bar{3})$, $\mu(\bar{2}) = 0.8$, $\nu(\bar{0}) = 0.1$, $\nu(\bar{1}) = 0.4 = \nu(\bar{3})$, and $\nu(\bar{2}) = 0.3$. Then $\psi \in AS_{\mathbb{Z}_4}(\psi)$.

Choose $n = 8$. Then $8\psi = (8\mu, 8\nu)$ is described by

$$(8\mu)^2(x) = \begin{cases} \bigvee_{x=8y} \mu^2(y), & \text{when } x \in 8U \\ 0, & \text{elsewhere} \end{cases}$$

and

$$(8\nu)^2(x) = \begin{cases} \bigwedge_{x=8y} \nu^2(y), & \text{when } x \in 8U \\ 1, & \text{elsewhere} \end{cases}$$

where $8U = \{\bar{0}\}$. So, $8\mathbb{Z}_4 \neq \mathbb{Z}_4$.

Therefore $\mathbb{Z}_4$ is not a divisible group.

Now, $(8\mu)^2(\bar{0}) = \bigvee_{\bar{0}=8y} \mu^2(y) = \bigvee_{y=\bar{0}/8} \mu^2(y) = 0.81$ *and* $(8\nu)^2(\bar{0}) = \bigwedge_{\bar{0}=8y} \nu^2(y) = \bigwedge_{y=\bar{0}/8} \nu^2(y) = 0.01$. *Again,* $(8\mu)^2(\bar{1}) = (8\mu)^2(\bar{2}) = (8\mu)^2(\bar{3}) = 0$ *and* $(8\nu)^2(\bar{1}) = (8\nu)^2(\bar{2}) = (8\nu)^2(\bar{3}) = 1$. *Thus* $8\psi \neq \psi$.

Therefore ψ *is not a Pythagorean fuzzy divisible subgroup of* $\mathbb{Z}_4$.

Definition 3.3. *Let* $\psi \in A_U(\psi)$. *Then* $\psi^* = \{t_1 | t_1 \in U, \mu^2(t_1) > 0, \nu^2(t_1) < 1\}$ *is called the support of the PFS* ψ.

Example 3.4. *Consider* $U = (\mathbb{Z}_4, +)$.

Assign a PFS $\psi = (\mu, \nu)$ *on* U *by* $\mu(\bar{0}) = 0$, $\mu(\bar{1}) = 0.8$, $\mu(\bar{2}) = 1$, $\mu(\bar{3}) = 0.5$, $\nu(\bar{0}) = 1$, $\nu(\bar{1}) = 0.4$, $\nu(\bar{2}) = 0$, *and* $\nu(\bar{3}) = 0.6$.

Then $\mu^2(\bar{0}) = 0$, $\mu^2(\bar{1}) = 0.64$, $\mu^2(\bar{2}) = 1$, $\mu(\bar{3}) = 0.25$, $\nu^2(\bar{0}) = 1$, $\nu^2(\bar{1}) = 0.16$, $\nu^2(\bar{2}) = 0$, *and* $\nu^2(\bar{3}) = 0.36$.

Therefore the support of the PFS ψ *is* $\psi^* = \{\bar{1}, \bar{2}, \bar{3}\}$.

Proposition 3.1. *Let* $\psi \in AS_U(\psi)$. *If* ψ *is a Pythagorean fuzzy divisible subgroup of* U, *the support* ψ^* *of* ψ *forms a divisible group.*

Proof. Here ψ is a Pythagorean fuzzy divisible subgroup of U.

So, $n\psi = \psi$ for all $n \in \mathbb{N}$.

Therefore $(n\mu)^2(t_1) = \bigvee_{t_1=nt_2} \mu^2(t_2) = \mu^2(t_1)$ and $(n\nu)^2(t_1) = \bigwedge_{t_1=nt_2} \nu^2(t_2) = \nu^2(t_1)$.

The support of ψ is $\psi^* = \{t_1 | t_1 \in U, \mu^2(t_1) > 0, \nu^2(t_1) < 1\}$.

Suppose $t_1 \in \psi^*$ and $n \in \mathbb{N}$.

Then $\mu^2(t_1) = \bigvee_{t_1=nt_2} \mu^2(t_2) > 0$ and $\nu^2(t_1) = \bigwedge_{t_1=nt_2} \nu^2(t_2) < 1$.

So, there exists a $t_2 \in U$ such that $\mu^2(t_2) > 0$ and $\nu^2(t_2) < 1$. Thus $t_2 \in \psi^*$.

Therefore $\forall t_1 \in \psi^*$ and $\forall n \in \mathbb{N}$, there exists a $t_2 \in \psi^*$ such that $t_1 = nt_2$.

Hence ψ^* is a divisible group. □

Example 3.5. *In Example 3.2,* $\psi^* = \mathbb{Q}$ *is a divisible group.*

Theorem 3.2. *Let* $\psi \in AS_U(\psi)$. *If* ψ *is a Pythagorean fuzzy divisible subgroup of* U, *the PFLSG* $\psi_{(\theta,\tau)}$ *is a divisible group.*

Proof. Given ψ is a Pythagorean fuzzy divisible subgroup of U.

Then $n\psi = \psi$ for all $n \in \mathbb{N}$.

Therefore $(n\mu)^2(t_1) = \bigvee_{t_1=nt_2} \mu^2(t_2) = \mu^2(t_1)$ and $(n\nu)^2(t_1) = \bigwedge_{t_1=nt_2} \nu^2(t_2) = \nu^2(t_1)$.

So, $\mu^2(t_1) = \bigvee_{t_1=nt_2} \mu^2(t_2)$ and $\nu^2(t_1) = \bigwedge_{t_1=nt_2} \nu^2(t_2)$.

We have PFLSG $\psi_{(\theta,\tau)} = \{t_1 | t_1 \in U, \mu^2(t_1) \geq \theta, \nu^2(t_1) \leq \tau\}$, where $0 \leq \theta^2 + \tau^2 \leq 1$.

Suppose $t_1 \in \psi_{(\theta,\tau)}$ and $n \in \mathbb{N}$. Then $\mu^2(t_1) \geq \theta, \nu^2(t_1) \leq \tau$.

Thus $\mu^2(t_1) = \bigvee_{t_1=nt_2} \mu^2(t_2) \geq \theta$ and $\nu^2(t_1) = \bigwedge_{t_1=nt_2} \nu^2(t_2) \leq \tau$.

This shows that, there exists a $t_2 \in U$ such that $\mu^2(t_2) \geq \theta$ and $\nu^2(t_2) \leq \tau$. So, $t_2 \in \psi_{(\theta,\tau)}$.

Therefore $\forall t_1 \in \psi_{(\theta,\tau)}$ and $\forall n \in \mathbb{N}$, there exists a $t_2 \in \psi_{(\theta,\tau)}$ such that $t_1 = nt_2$.

Hence $\psi_{(\theta,\tau)}$ is a divisible group. □

Definition 3.4. *Let* $\psi \in AS_U(\psi)$. *Then* ψ *is a Pythagorean fuzzy* λ*-divisible subgroup of* U *if* $\lambda\psi = \psi$ *for all prime* λ.

Proposition 3.2. *Every Pythagorean fuzzy divisible subgroup of an abelian group is also a Pythagorean fuzzy* λ*-divisible subgroup of the proposed group.*

Proof. Let $\psi \in AS_U(\psi)$ and ψ be a Pythagorean fuzzy divisible subgroup of U.

Then $n\psi = \psi$, $\forall n \in \mathbb{N}$.

Every prime number is also a natural number.
Therefore $\lambda\psi = \psi$ for all prime λ.
Hence ψ is a Pythagorean fuzzy λ-divisible subgroup of G. □

4. Equivalence relation on PFSGs

Here, we define an equivalence condition on PFSs. We explain some properties of this equivalence relation on PFSGs.

Definition 4.1. *Let X be a group and $A_X(\psi)$ be the set of all PFSs of X. Suppose $\psi_1 = (\mu_1, \nu_1), \psi_2 = (\mu_2, \nu_2) \in A_X(\psi)$. Then ψ_1 is equivalent to ψ_2 if the following conditions holds:*

- $\mu_1^2(x) \geq \mu_1^2(y)$, $\nu_1^2(x) \leq \nu_1^2(y) \iff \mu_2^2(x) \geq \mu_2^2(y)$, $\nu_2^2(x) \leq \nu_2^2(y)$ *for all* $x, y \in X$.
- $\psi_1^* = \psi_2^*$.

If ψ_1 is equivalent to ψ_2 then we denote it $\psi_1 \sim \psi_2$ and $[\psi_1] = \{\psi \in A_X(\psi) | \psi_1 \sim \psi\}$.

Example 4.1. *Consider the group $(\mathbb{Z}_2, +_2)$.*
We assign two PFSs $\psi_1 = (\mu_1, \nu_1)$ and $\psi_2 = (\mu_2, \nu_2)$ on $\mathbb{Z}_2$ by $\mu_1(\bar{0}) = 1$, $\mu_1(\bar{1}) = 0.5$, $\mu_2(\bar{0}) = 1$, $\mu_2(\bar{1}) = 0.3$, $\nu_1(\bar{0}) = 0$, $\nu_1(\bar{1}) = 0.3$, $\nu_2(\bar{0}) = 0$ and $\nu_2(\bar{1}) = 0.8$.
Here, $\mu_1(\bar{0}) > \mu_1(\bar{1}) \Leftrightarrow \mu_2(\bar{0}) > \mu_2(\bar{1})$ and $\nu_1(\bar{0}) < \nu_1(\bar{1}) \Leftrightarrow \nu_2(\bar{0}) < \nu_2(\bar{1})$.
Also, $\psi_1^* = \mathbb{Z}_2 = \psi_2^*$.
Thus $\psi_1 \sim \psi_2$ and $[\psi_1] = [\psi_2]$.

Definition 4.2. *Two PFSGs ψ_1 and ψ_2 of a group U, are distinct if and only if $[\psi_1] \neq [\psi_2]$.*

Example 4.2. *Let $\psi_3 = (\mu_3, \nu_3)$ be a PFS on $\mathbb{Z}_2$, in Example 4.1.*
Assign $\mu_3(\bar{0}) = 1$, $\mu_3(\bar{1}) = 0$, $\nu_3(\bar{0}) = 0$ and $\nu_3(\bar{1}) = 1$.
Here, $\mu_1(\bar{0}) > \mu_1(\bar{1}) \Leftrightarrow \mu_3(\bar{0}) > \mu_3(\bar{1})$ and $\nu_1(\bar{0}) < \nu_1(\bar{1}) \Leftrightarrow \nu_3(\bar{0}) < \nu_3(\bar{1})$.
But, $\psi_1^* = \mathbb{Z}_2 \neq \{\bar{0}\} = \psi_3^*$.
Therefore ψ_1 and ψ_3 are two distinct PFSGs of $\mathbb{Z}_2$.

Remark 4.1. *The Second condition of the Definition 4.1, can't be omitted.*

Theorem 4.1. *Let $\psi_1 = (\mu_1, \nu_1), \psi_2 = (\mu_2, \nu_2) \in AS_U(\psi)$. Suppose for every (θ_1, τ_1) with $0 \leq \theta_1^2 + \tau_1^2 \leq 1$, there exists a (θ_2, τ_2) with $0 \leq \theta_2^2 + \tau_2^2 \leq 1$ such that $\psi_{1(\theta_1,\tau_1)} = \psi_{2(\theta_2,\tau_2)}$. Then $[\psi_1] = [\psi_2]$.*

Proof. Suppose $\psi_1^* = \emptyset$. Then clearly $\psi_2^* = \emptyset$. So, $\psi_1^* = \psi_2^*$.
Let $\psi_1^* \neq \emptyset$. Suppose $x \in \psi_1^*$. So, $\mu_1^2(x) > 0$ and $\nu_1^2(x) < 0$.
Then $x \in \psi_{1(\theta_1,\tau_1)}$ for some (θ_1, τ_1).
Given that for every (θ_1, τ_1) there exists a (θ_2, τ_2) such that $\psi_{1(\theta_1,\tau_1)} = \psi_{2(\theta_2,\tau_2)}$.
So, $x \in \psi_{2(\theta_2,\tau_2)}$.
This implies $\mu_2^2(x) \geq \theta_2 > 0$ and $\nu_2^2(x) \leq \tau_2 < 0$. Therefore $x \in \psi_2^*$.
Thus $\psi_1^* \subset \psi_2^*$.
Similarly, we can show that $\psi_2^* \subset \psi_1^*$. Therefore $\psi_1^* = \psi_2^*$.
Now, let $\theta_1 = \mu_1^2(x) \geq \mu_1^2(y)$ and $\tau_1 = \nu_1^2(x) \leq \nu_1^2(y)$.
So, $x \in \psi_{1(\theta_1,\tau_1)}$ and $y \notin \psi_{1(\theta_1,\tau_1)}$.
Thus $x \in \psi_{2(\theta_2,\tau_2)}$. Therefore $\mu_2^2(x) \geq \theta_2$ and $\nu_2^2(x) \leq \tau_2$.
If possible let, $\mu_2^2(x) < \mu_2^2(y)$ and $\nu_2^2(x) > \nu_2^2(y)$.
Therefore $\theta_2 \leq \mu_2^2(x) < \mu_2^2(y)$ and $\nu_2^2(y) < \nu_2^2(x) \leq \tau_2$.
So, $y \in \psi_{2(\theta_2,\tau_2)} = \psi_{1(\theta_1,\tau_1)}$, a contradiction.
Hence $\mu_2^2(x) > \mu_2^2(y)$ and $\nu_2^2(x) < \nu_2^2(y)$.
Similarly, $\mu_2^2(x) > \mu_2^2(y)$ and $\nu_2^2(x) < \nu_2^2(y)$ implies $\mu_1^2(x) > \mu_1^2(y)$ and $\nu_1^2(x) < \nu_1^2(y)$.

So, $\psi_1 \sim \psi_2$. Hence $[\psi_1] = [\psi_2]$. □

Theorem 4.2. *Suppose $\psi_1 = (\mu_1, \nu_1), \psi_2 = (\mu_2, \nu_2) \in AS_U(\psi)$ such that $[\psi_1] = [\psi_2]$. Then for all (θ_1, τ_1) with $0 \leq \theta_1^2 + \tau_1^2 \leq 1$, there exists a (θ_2, τ_2) with $0 \leq \theta_2^2 + \tau_2^2 \leq 1$ such that $\psi_{1(\theta_1,\tau_1)} = \psi_{2(\theta_2,\tau_2)}$.*

Proof. We have $\psi_{1(\theta_1,\tau_1)} = \{x|\mu_1^2(x) \geq \theta_1, \nu_1^2(x) \leq \tau_1\}$ and
$\psi_{2(\theta_2,\tau_2)} = \{x|\mu_2^2(x) \geq \theta_2, \nu_2^2(x) \leq \tau_2\}$.

Let $\alpha = \vee\{\mu_1^2(x)|x \in U\}$, $\beta = \vee\{\mu_2^2(x)|x \in U\}$, $\lambda = \wedge\{\nu_1^2(x)|x \in U\}$ and
$\delta = \wedge\{\nu_2^2(x)|x \in U\}$.

For, $\theta_1 = \alpha$, $\tau_1 = \lambda$; choose $\theta_2 = \beta$ and $\tau_2 = \delta$.

Then $\psi_{1(\theta_1,\tau_1)} = \{e\} = \psi_{2(\theta_2,\tau_2)}$.

For, $\theta_1 < \alpha$ and $\tau_1 > \lambda$, consider $\theta = \wedge\{\mu_1^2(x)|x \in \psi_{1(\theta_1,\tau_1)}\}$ and $\tau = \vee\{\nu_1^2(x)|x \in \psi_{1(\theta_1,\tau_1)}\}$.

Then $\theta \geq \theta_1$ and $\tau \leq \tau_1$.

Let $\theta = \mu_1^2(a)$ and $\tau = \nu_1^2(a)$ for some $a \in G$.

Then choose $\theta_2 = \mu_2^2(a)$ and $\tau_2 = \nu_2^2(a)$.

Let $x \in \psi_{1(\theta_1,\tau_1)}$. So, $\mu_1^2(x) \geq \theta_1$ and $\nu_1^2(x) \leq \tau_1$.

Therefore $\mu_1^2(x) \geq \theta = \mu_1^2(a)$ and $\nu_1^2(x) \leq \tau = \nu_1^2(a)$.

This implies $\mu_1^2(x) \geq \mu_1^2(a)$ and $\nu_1^2(x) \leq \nu_1^2(a)$.

As $\psi_1 \sim \psi_2$, $\mu_2^2(x) \geq \mu_2^2(a)$ and $\nu_2^2(x) \leq \nu_2^2(a)$.

Thus $\mu_2^2(x) \geq \theta_2$ and $\nu_2^2(x) \leq \tau_2$.

So, $x \in \psi_{2(\theta_2,\tau_2)}$.

Therefore $\psi_{1(\theta_1,\tau_1)} \subseteq \psi_{2(\theta_2,\tau_2)}$.

Similarly, we can show that $\psi_{2(\theta_2,\tau_2)} \subseteq \psi_{1(\theta_1,\tau_1)}$.

Hence $\psi_{1(\theta_1,\tau_1)} = \psi_{2(\theta_2,\tau_2)}$. □

5. Counting of PFSGs of abelian groups

Here, we introduce PFSGs related with their maximal chains. We investigate all possible PFSGs of finite abelian groups.

Let U be a finite abelian group. If $\psi = (\mu, \nu)$ is a PFSG of U, we assume that $\mu(e) = 1$ and $\nu(e) = 0$, where e is the group's identity.

Definition 5.1. *A maximal chain $\mathbb{Z}_{\lambda^n} \supset \mathbb{Z}_{\lambda^{n-1}} \supset \mathbb{Z}_{\lambda^{n-2}} \supset \cdots \supset \mathbb{Z}_\lambda \supset \{0\}$ defined a PFSG $\psi = (\mu, \nu)$ is as follows:*

$$\mu(a) = \begin{cases} s_n, & \text{when } a \in \mathbb{Z}_{\lambda^n} \setminus \mathbb{Z}_{\lambda^{n-1}} \\ s_{n-1}, & \text{when } a \in \mathbb{Z}_{\lambda^{n-1}} \setminus \mathbb{Z}_{\lambda^{n-2}} \\ \vdots & \\ s_1, & \text{when } a \in \mathbb{Z}_\lambda \setminus \{0\} \\ 1, & \text{when } a \in \{0\} \end{cases}$$

and

$$\nu(a) = \begin{cases} t_n, & \text{when } a \in \mathbb{Z}_{\lambda^n} \setminus \mathbb{Z}_{\lambda^{n-1}} \\ t_{n-1}, & \text{when } a \in \mathbb{Z}_{\lambda^{n-1}} \setminus \mathbb{Z}_{\lambda^{n-2}} \\ \vdots & \\ t_1, & \text{when } a \in \mathbb{Z}_\lambda \setminus \{0\} \\ 0, & \text{when } a \in \{0\} \end{cases}$$

with the conditions $0 \leq s_n \leq s_{n-1} \leq \cdots \leq s_1 \leq 1$ and $1 \geq t_n \geq t_{n-1} \geq \cdots \geq t_1 \geq 0$, where $0 \leq s_i^2 + t_i^2 \leq 1$ for $i = 0, 1, \ldots, n$.

This PFSG $\psi = (\mu, \nu)$ is denoted by $\psi = (1s_1s_2 \cdots s_n, 0t_1t_2 \cdots t_n)$.

Example 5.1. *Consider the group $\mathbb{Z}_\lambda$, where λ is a prime. The maximal chain associated with $\mathbb{Z}_\lambda$ is $\mathbb{Z}_\lambda \supset \{0\}$.*

A PFSG $\psi = (\mu, \nu)$ defined by this maximal chain is

$$\mu(u) = \begin{cases} s, & \text{when } u \in \mathbb{Z}_\lambda \setminus \{0\} \\ 1, & \text{when } u \in \{0\} \end{cases}$$

and

$$\nu(u) = \begin{cases} t, & \text{when } u \in \mathbb{Z}_\lambda \setminus \{0\} \\ 0, & \text{when } u \in \{0\} \end{cases}$$

.

This PFSG $\psi = (\mu, \nu)$ is denoted by $\psi = (1s, 0t)$ with $s, t \in (0, 1)$.

Proposition 5.1. *Every group of prime order has exactly three distinct equivalence class of PFSGs.*

Proof. Any group of prime order λ is cyclic. So, it is isomorphic to $\mathbb{Z}_\lambda$.

The maximal chain for $\mathbb{Z}_\lambda$ is $\{0\} \subset \mathbb{Z}_\lambda$. Then the PFSGs generated by this chain are $\psi = (\mu, \nu)$ where

$$\mu(a) = \begin{cases} \alpha, & \text{when } a \in \mathbb{Z}_\lambda \setminus \{0\} \\ 1, & \text{when } a \in \{0\} \end{cases}$$

and

$$\nu(a) = \begin{cases} \beta, & \text{when } a \in \mathbb{Z}_\lambda \setminus \{0\} \\ 0, & \text{when } a \in \{0\} \end{cases}$$

with the conditions $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $0 \le \mu^2(a) + \nu^2(a) \le 1$.

Here $\alpha = 1$ and $\beta = 0$ represents the PFSG $\psi = (11, 00)$ which is actually $\mathbb{Z}_\lambda$.

Again $\alpha = 0$ and $\beta = 1$ represents the PFSG $\psi = (10, 01)$ which is trivial subgroup $\{0\}$ of $\mathbb{Z}_\lambda$.

Now, $0 < \alpha < 1$ and $0 < \beta < 1$ represents the PFSG $\psi = (1\alpha, 0\beta)$.

Any non-trivial PFSGs of $\mathbb{Z}_\lambda$ is equivalent to the PFSG $\psi = (1\alpha, 0\beta)$.

Hence there are exactly three distinct equivalence class of PFSGs of any group of prime order. □

Proposition 5.2. *Every cyclic group of order λ^2 has exactly seven distinct class of PFSGs, where λ is a prime.*

Proof. Cyclic groups of order λ^2 is isomorphic to $\mathbb{Z}_{\lambda^2}$, where λ is prime.

The maximal chain for $\mathbb{Z}_{\lambda^2}$ is $\{0\} \subset \mathbb{Z}_\lambda \subset \mathbb{Z}_{\lambda^2}$.

Then the PFSGs generated by this chain are $\psi = (\mu, \nu)$ where

$$\mu(a) = \begin{cases} \delta, & \text{when } a \in \mathbb{Z}_{\lambda^2} \setminus \mathbb{Z}_\lambda \\ \alpha, & \text{when } a \in \mathbb{Z}_\lambda \setminus \{0\} \\ 1, & \text{when } a \in \{0\} \end{cases}$$

and

$$\nu(a) = \begin{cases} \sigma, & \text{when } a \in \mathbb{Z}_{\lambda^2} \setminus \mathbb{Z}_\lambda \\ \beta, & \text{when } a \in \mathbb{Z}_\lambda \setminus \{0\} \\ 0, & \text{when } a \in \{0\} \end{cases}$$

with the conditions $0 \le \delta \le \alpha \le 1$, $0 \le \beta \le \sigma \le 1$ and $0 \le \mu^2(a) + \nu^2(a) \le 1$.

By the given conditions, the distinct equivalence classes of PFSGs of $\mathbb{Z}_{\lambda^2}$ are $(111, 000)$, $(11\alpha, 00\beta)$, $(110, 001)$, $(1\alpha\alpha, 0\beta\beta)$, $(1\alpha\delta, 0\beta\sigma)$, $(1\alpha 0, 0\beta 1)$ and $(100, 011)$.

Here $(1\alpha\delta, 0\beta\sigma)$ represents the class of PFSGs

$$\mu(a) = \begin{cases} \delta, & \text{when } a \in \mathbb{Z}_{\lambda^2} \setminus \mathbb{Z}_\lambda \\ \alpha, & \text{when } a \in \mathbb{Z}_\lambda \setminus \{0\} \\ 1, & \text{when } a \in \{0\} \end{cases}$$

and

$$\nu(a) = \begin{cases} \sigma, & \text{when } a \in \mathbb{Z}_{\lambda^2} \setminus \mathbb{Z}_\lambda \\ \beta, & \text{when } a \in \mathbb{Z}_\lambda \setminus \{0\} \\ 0, & \text{when } a \in \{0\} \end{cases}$$

.

Any PFSGs of $\mathbb{Z}_{\lambda^2}$ is belongs to one of the above equivalent classes.

Hence a cyclic group of order λ^2 has exactly seven distinct class of PFSGs. □

Remark 5.1. *We have observed in Proposition 5.2 that there are one more distinct PFSGs with support $\mathbb{Z}_{\lambda^2}$ than the PFSGs whose supports are contained in $\mathbb{Z}_{\lambda^2}$.*

The count of distinct PFSGs whose support is $\mathbb{Z}_{\lambda^2}$ is four, whose support is $\mathbb{Z}_\lambda$ is two and whose support is $\{0\}$ is one.

Proposition 5.3. *Every group of order $\lambda_1 \cdot \lambda_2$ has exactly eleven distinct class of PFSGs, where λ_1, λ_2 are distinct primes.*

Proof. Every group of order $\lambda_1 \cdot \lambda_2$ is isomorphic to $\mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2}$, where λ_1, λ_2 are distinct primes.

The maximal chains of $\mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2}$ are $\{0\} \subset \mathbb{Z}_{\lambda_1} \times \{0\} \subset \mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2}$ and $\{0\} \subset \{0\} \times \mathbb{Z}_{\lambda_2} \subset \mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2}$.

These two chains are equivalent to the maximal chain $\{0\} \subset \mathbb{Z}_\lambda \subset \mathbb{Z}_{\lambda^2}$ in Proposition 5.2, where λ is prime.

So, each chain produce seven distinct class of PFSGs.

Among these PFSGs, three class of PFSGs are equivalent in two chain.

They are $(111, 000)$, $(1\alpha\alpha, 0\beta\beta)$ and $(100, 011)$.

So, the total number of distinct PFSGs of $\mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2}$ is $2 \times 7 - 3 = 11$.

Hence every group of order $\lambda_1 \cdot \lambda_2$ has exactly eleven distinct PFSGs. □

Proposition 5.4. *A non-cyclic group of order λ^2 has $4\lambda + 7$ distinct class of PFSGs, where λ is a prime.*

Proof. Let U be a non-cyclic group of order λ^2, where λ is prime.

Then U is isomorphic to $\mathbb{Z}_\lambda \times \mathbb{Z}_\lambda$.

Since the order of U is λ^2, the maximal chains of U are equivalent to the maximal chains of $\mathbb{Z}_{\lambda^2}$.

According to the Proposition 2.1, there are $(\lambda + 1)$ maximal chains of $\mathbb{Z}_\lambda \times \mathbb{Z}_\lambda$.

Therefore by Proposition 5.2, each maximal chain produce seven distinct class of PFSGs.

Among these seven class of PFSGs of each maximal chain three class of PFSGs are equivalent.

Thus the total class of distinct PFSGs of $\mathbb{Z}_\lambda \times \mathbb{Z}_\lambda$ is $4(\lambda + 1) + 3 = 4\lambda + 7$. □

Theorem 5.1. *Every cyclic group of order λ^m has $\sum_{t=0}^{m} 2^t = 2^{m+1} - 1$ distinct equivalence classes of PFSGs, where λ is a prime and $m \in \mathbb{N}$.*

Proof. Every cyclic group of order λ^m is isomorphic to $\mathbb{Z}_{\lambda^m}$.

Proposition 5.1 and Proposition 5.2 shows that this is true for $m = 1$ and $m = 2$.

Assume that this is also true for $m = d$.

That is $\mathbb{Z}_{\lambda^d}$ has $2^{d+1} - 1 = \sum_{t=0}^{d} 2^d = 2.2^d - 1$ distinct equivalent class of PFSGs.

The count of distinct PFSGs of $\mathbb{Z}_{\lambda^d}$, whose support is $\mathbb{Z}_{\lambda^d}$ is 2^d.

This PFSGs increases two PFSG of $\mathbb{Z}_{\lambda^{d+1}}$ whose support is $\mathbb{Z}_{\lambda^{d+1}}$ and one PFSG whose support is $\mathbb{Z}_{\lambda^d}$.

Thus 2^d PFSGs of $\mathbb{Z}_{\lambda^d}$ increases $2.2^d + 1.2^d$ PFSGs of $\mathbb{Z}_{\lambda^{d+1}}$.

The remaining (2^d-1) PFSGs of $\mathbb{Z}_{\lambda^d}$ have support, contained in $\mathbb{Z}_{\lambda^d}$ and thus yeilds (2^d-1) PFSGs of $\mathbb{Z}_{\lambda^{d+1}}$ by attaching zero to membership grades and one to non-membership grades.

Therefore the total distinct PFSGs of $\mathbb{Z}_{\lambda^{d+1}}$ is $(2.2^d+2^d)+(2^d-1) = 2^d(2+1+1)-1 = 2^{(d+1)+1} - 1 = \sum_{t=0}^{d+1} 2^t$.

Thus by principal of induction, this is true for all $m \in \mathbb{N}$.

Hence $\mathbb{Z}_{\lambda^m}$ has $2^{m+1} - 1$ distinct class of PFSGs. $\square$

6. Conclusion

The primary goal of this paper is to research on distinct PFSGs of a commutative group by a proper equivalence condition. We have introduced the concept of Pythagorean fuzzy divisible subgroups of a group. We have proved that a group is a divisible group if and only if every PFSG of this group is a Pythagorean fuzzy divisible subgroup of it. We have defined an equivalence relation on PFSs. We have explained some properties of this equivalence relation on PFSGs. We have introduced Pythagorean fuzzy subgroups related with their maximal chains. We have investigated the count of PFSGs of finite commutative groups.

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