TWMS J. App. and Eng. Math. V.15, N.7, 2025, pp. 1726-1741

NON-EXPLOSION AND PATHWISE UNIQUENESS OF STRONG SOLUTIONS FOR JUMP-TYPE STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY OPTIONAL SEMIMARTINGALES UNDER NON-LIPSCHITZ CONDITIONS

M. HADDADI[1*], K. AKDIM[2], §

Abstract. This paper is devoted to the question of the pathwise uniqueness and the non-explosion property of strong solutions for a class of jump-type stochastic differential equations (JSDEs) with respect to optional semimartingales under non-Lipschitz conditions. Optional semimartingales have right and left limits (làdlàg) but are not necessarily continuous, therefore, defined on unusual probability spaces. Some models in financial and insurance mathematics which can be described by the jump-type stochastic differential equations (JSDEs) are presented.

Keywords: jump-type stochastic differential equation; optional semimartingale; Non-explosive solution; Non-Lipschitz condition.

AMS Subject Classification: 60H10, 60H30

1. Introduction

Optional semimartingales have been introduced by Gal'čuk (see [14, 16, 17]), Lenglart [22] and Mertens [23]. The first motivation of this paper is to bring attention to the theory of semimartingales in a general framework where the semimartingales are optional. Such processes have been studied in mathematical finance for portfolio optimization when the transaction costs are not neglected, in particular the dual optimizer under transaction costs is in general a làdlàg strong optional supermartingale (see Czichowsky and Schachermayer [13]). Recently, this direction received a new impulse mostly by the works of Abdelghani and Melnikov [5, 1, 2, 4], Abdelghani, Melnikov and Pak [7, 6]. Apparently, this direction attracts substantial attention and many works have appeared during the last couple of years, to mention a few [20, 9, 21, 8].

[1] Science and Technology Research Laboratory (LRST), Department of Science and Technology, The Higher School of Education and Training, Ibn Zohr University, Agadir, Morocco.
e-mail: m.haddadi@uiz.ac.ma; ORCID No. 0000-0002-4780-8790.
[*] Corresponding author.
[2] Department of Mathematics, Faculty of Sciences and Techniques, Cadi Ayyad University, Marrakech, Morocco.
e-mail: k.akdim@uca.ma; ORCID no. 0000-0002-1287-128X.

§ Manuscript received: April 29, 2024; accepted: September 17, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.7;

The second motivation of this paper is to develop the theory of optional processes for stochastic differential equations (SDEs), by studying the problem of existence and uniqueness of SDEs driven by làdlàg optional semimartingales. Specially, the study of Jump-type stochastic differential equations (JSDEs), as natural extensions of SDEs, for their potential applications to mathematical finance and physics. On the financial applications, Shreve [24] and Tankov [25] have listed several financial models that can be modeled using JSDEs. As for the applications in physics, Chudley and Elliott [11] applied JSDEs to describe atomic diffusion, which typically involves jumps between vacant lattice sites. For more applications see [19, 27]. In addition, the important viewpoints on risk assessment and market dynamics are provided by mathematical models ([19, 9, 10]).

This work is a further extension of Gou et al. [19] for the optional framework. As in [19], the first task of this paper is to provide a sufficient super linear growth condition for ensuring the non-explosion of strong optional solutions for JSDEs. The second task is to guarantee the existence and the uniqueness of strong solutions to JSDEs under non-lipschitz condition.

As an example in this paper, we prove under some assumptions that the following JSDE:

$$\begin{aligned} X(t) =& X_0 - \int_0^t |X(s)| \ln |X(s)| ds + \int_0^t \sqrt{|X(s)|} dB_s \\ &+ \int_0^t \int_{|u|\le 1} \sqrt{|X(s)|}\widetilde{N_1^d}(ds,du) + \int_0^t \int_{|u|\le 1} \sqrt{|X(s)|}\widetilde{N_1^g}(ds,du) \\ &+ \int_0^t \int_{|u|>1} \gamma |u| X(s) N_2^d(ds,du) + \int_0^t \int_{|u|>1} \beta |u| X(s) N_2^g(ds,du), \end{aligned}$$

possesses a unique non-explosive strong solution.

The paper is organized as follows. The section 2 presents, in the first, some material on stochastic processes on unusual probability spaces and their stochastic calculus is presented, covering important topics, such as, optional làdlàg martingales, optional increasing and finite variation processes and optional ladlag semimartingales. Also, the definition of stochastic integrals with respect to optional ladlag semimartingales and elements of the stochastic calculus of optional processes are presented. In the second, some necessary preliminaries about JSDEs including assumptions and lemmas. In section 3, we prove the main results which are the non-explosion and the pathwise uniqueness of strong optional solutions for JSDEs under super linear growth and non-Lipschitz conditions. In section 4, some models in financial and insurance mathematics which can be described by the jump-type stochastic differential equations (JSDEs) are presented.

2. Preliminaries

2.1. **Optional semimartingale process.** Let us consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F})_t, P)$, because $\mathcal{F}$ contains all P null sets. The space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F})_t, P)$ is unusual probability space, because the family $\mathbb{F}$ is not assumed to be right or left continuous.

We use the following notation:

- $\mathcal{B}(\mathbb{R})$ is the Borel σ–algebra on $\mathbb{R}$.
- $\mathcal{O}(\mathbb{F})$ is the optional σ–algebras on $(\Omega, \mathbb{R}^+)$.
- $\mathcal{P}(\mathbb{F})$ is the predictable σ–algebras on $(\Omega, \mathbb{R}^+)$.
- $\mathcal{M}_{loc}$ is the set of the optional local martingale.
- $\mathcal{M}_{loc}^r$ is the set of the right continuous local martingale.
- $\mathcal{M}_{loc}^d$ is the set of the discrete right continuous local martingale.

- $\mathcal{M}^g_{loc}$ is the set of the left continuous local martingale.
- $\mathcal{V}$ is the set of the optional finite variation.
- $\mathcal{S}(\mathbb{F}, P)$ is the set of optional semimartingales.
- $\mathcal{S}_p(\mathbb{F}, P)$ the set of special optional semimartingales.

A random process $X = (X_t)$, is said to be optional if it is $\mathcal{O}(\mathbb{F})$–measurable. In general, optional processes have right and left limits but are not necessarily continuous in $\mathbb{F}$. For an optional process we can define the following properties: $X_- = (X_{t-})_{t\geq 0}$ and $X_+ = (X_{t+})_{t\geq 0}$, $\Delta X = (\Delta X_t)_{t\geq 0}$, $\Delta X = X_t - X_{t-}$ and $\Delta^+ X = (\Delta^+ X_t)_{t\geq 0}$, $\Delta^+ X = X_{t+} - X_t$. An Optional semimartingale $X = (X_t)$ can be decomposed to an optional local martingale and an optional finite variation

$$X = X_0 + M + A$$

where $M \in \mathcal{M}_{loc}$, $A \in \mathcal{V}$. A semimartingale X is called special if the above decomposition exists with a predictable process A. If $X \in \mathcal{S}_p(\mathbb{F}, P)$ then the semimartingale decomposition is unique.

By optional martingale decomposition and decomposition of predictable processes (see [16] and [18]), we can decompose a semimartingale further to

$$X = X_0 + X^r + X^g,$$

with $X^r = A^r + M^r$, $X^g = A^g + M^g$ and $M^r = M^c + M^d$ where A^r and A^g are finite variation processes right and left continuous, respectively. $M^r \in \mathcal{M}^r_{loc}$ right continuous local martingale, $M^d \in \mathcal{M}^d_{loc}$ discrete right continuous local martingale and $M^g \in \mathcal{M}^g_{loc}$ a left continuous local martingale. This decomposition is useful for defining integration with respect to optional semimartingales. A stochastic integral with respect to optional semimartingale was defined by Gal'čuk [18],

$$\varphi \circ X_t = \int_0^t \varphi_s dX_s = \int_{0+}^t \varphi_{s-} dX^r_s + \int_0^{t-} \varphi_s dX^g_{s+}, \tag{1}$$

where

$$\int_{0+}^t \varphi_{s-} dX^r_s = \int_{0+}^t \varphi_{s-} dM^r_s + \int_{0+}^t \varphi_{s-} dA^r_s$$

and

$$\int_0^{t-} \varphi_s dX^g_{s+} = \int_0^{t-} \varphi_s dM^g_{s+} + \int_0^{t-} \varphi_s dA^g_{s+}.$$

The stochastic integral with respect to the finite variation processes or strongly predictable process A^r and A^g are interpreted as usual, in the Lebesgue sense. The integral $\int_{0+}^t \varphi_{s-} dM^r_s$ is our usual stochastic integral with respect to RCLL local martingale where $\int_0^{t-} \varphi_s dM^g_{s+}$ is Gal'čuk stochastic integral (see [16] and [18]) with respect to left continuous local martingale. In general, the stochastic integral with respect to optional semimartingale X can be defined as a bilinear form $(\varphi, \phi) \circ X_t$ such that

$$\begin{aligned} Y_t &= (\varphi, \phi) \circ X_t = \varphi \cdot X^r_t + \phi \odot X^g_t, \\ \varphi \cdot X^r_t &= \int_{0+}^t \varphi_{s-} dX^r_s, \;\; \phi \odot X^g_t = \int_0^{t-} \phi_s dX^g_{s+}, \end{aligned}$$

where Y is again an optional semimartingale $\varphi_- \in \mathcal{P}(\mathbb{F})$, and $\phi \in \mathcal{O}(\mathbb{F})$. Note that, the stochastic integral over optional semimartingale is defined on a much larger space of integrands, the product space of predictable and optional processes, $\mathcal{P}(\mathbb{F}) \times \mathcal{O}(\mathbb{F})$.

Let us recall the change of variables formula for optional semimartingales which are not necessarily càdlàg. The result can be seen as a generalization of the classical Itô formula.

Lemma 2.1 (Theorem 8.2. [16])**.** *Let $n \in \mathbb{N}$. Let X be n-dimensional optional semimartingale, i.e, $X = (X^1, \dots, X^n)$ is an n-dimensional optional process with decomposition $X^k = X^k_0 + M^k + A^k + B^k$, for all $k \in \{1, \dots, n\}$, where M^k is a (càdlàg) local martingale, A^k is a right–continuous process of finite variation such that $A_0 = 0$, and B^k is a left–continuous process of finite variation which is purely discontinuous and such that $B_0 = 0$. Let $h(x) = h(x^1, \dots, x^n)$ is twice continuously differentiable function on $\mathbb{R}^n$. Then $h(X)$ is a semimartingale, and for all $t \in \mathbb{R}_+$,*

$$\begin{aligned} h(X_t) &= h(X_0) + \sum_{k=1}^{n} \int_{]0,t]} D^k h(X_{s-}) d(A^k + M^k)_s + \frac{1}{2} \sum_{1 \le k,l \le n} \int_{]0,t]} D^k D^l h(X_{s-}) d \left\langle M^{kc}, M^{lc} \right\rangle_s \\ &\quad + \sum_{0<s\le t} \left\{ h(X_s) - h(X_{s-}) - \sum_{k=1}^{n} D^k h(X_{s-}) \Delta X^k_s \right\} + \sum_{k=1}^{n} \int_{[0,t[} D^k h(X_s) d(B^k)_{s+} \\ &\quad + \sum_{0\le s<t} \left\{ h(X_{s+}) - h(X_s) - \sum_{k=1}^{n} D^k h(X_s) \Delta^+ X^k_s \right\}, \end{aligned}$$

where D^k is the differentiation operator with respect to the k–th coordinate, the process M^{kc} denotes the continuous part of M^k.

2.2. **Jump-type stochastic differential equations driven by optional semimartingales.** Consider the Lusin space $(\mathbb{E}, \mathcal{E})$, where $\mathbb{E} = (\mathbb{R} \setminus \{0\}) \cup \{\delta^d_1\} \cup \{\delta^d_2\} \cup \{\delta^g_1\} \cup \{\delta^g_2\}$, δ^d_i and δ^g_i, $i = 1, 2$ are Poisson point processes on U_1 and U_2, $\mathcal{E} = \mathcal{B}(\mathbb{E})$ is the Borel σ–algebra in $\mathbb{E}$. Also, define the spaces

$$\tilde{\Omega} = \Omega \times \mathbb{R}_+ \times \mathbb{E}, \quad \tilde{\mathbb{E}} = \mathbb{R}_+ \times \mathbb{E}, \quad \tilde{\mathcal{E}} = \mathcal{B}(\mathbb{R}_+) \times \mathcal{E},$$
$$\tilde{\mathcal{O}}(\mathbb{F}) = \mathcal{O}(\mathbb{F}) \times \mathcal{E}, \quad \tilde{\mathcal{P}}(\mathbb{F}) = \mathcal{P}(\mathbb{F}) \times \mathcal{E}.$$

It was shown by Gal'čuk in [18] that there exist sequences $(S_n)_{n\ge1}$, $(T_n)_{n\ge1}$ and $(U_n)_{n\ge1}$ of predictable stopping times, totally inaccessible stopping times and totally inaccessible stopping times in the broad sense respectively, absorbing all jumps of the process X such that the graphs of these stopping times do not intersect within each sequence. On $\tilde{\Omega}$, let $N^d_i(\omega, ., .)$ and $N^g_i(\omega, ., .)$, $i = 1, 2$ four Poisson random measures defined on the $\sigma-$algebra $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ that are associated with the sequences of stopping times that are associated with X. On the $\sigma-$algebra $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{E})$ we define the Poisson random measures by the following relations,

$$N^d_1(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(T_n, \beta^d_{T_n}), \qquad N^g_1(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(U_n, \beta^g_{U_n}),$$
$$N^d_2(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(S_n, \beta^d_{S_n}), \qquad N^g_2(B \times \Gamma) = \sum_n \mathbf{1}_{B\times\Gamma}(S_n, \beta^g_{S_n}),$$

where $B \in \mathcal{B}(\mathbb{R}_+)$ and $\Gamma \in \mathcal{B}(\mathbb{E})$, $\beta^d_t = \Delta X_t$ if $\Delta X_t \neq 0$ and $\beta^d_t = \delta^d_i$ if $\Delta X_t = 0$ for $i = 1, 2$, $\beta^g_t = \Delta^+ X_t$ if $\Delta^+ X_t \neq 0$ and $\beta^g_t = \delta^g_i$ if $\Delta^+ X_t = 0$ for $i = 1, 2$.

For the measures $N^d_i(ds, dz)$ and $N^g_i(ds, dz)$, for $i = 1, 2$, there exists unique random measures $\nu^d_i(du)ds$ and $\nu^g_2(du)ds$, for $i = 1, 2$, respectively, which satisfy $\int_{\mathbb{R}\setminus\{0\}}(|x|^2 \wedge 1)\nu^j_i(dx) < \infty$, for $i = 1, 2$ and $j = d, g$, such that for any $\mathcal{P}(\mathbb{F})-$function $f_i : \mathbb{R}\times U_i \to \mathbb{R}$, for $i = 1, 2$ and $\mathcal{O}(\mathbb{F})-$function $g_i : \mathbb{R}\times U_i \to \mathbb{R}$:

- The process $\int_{0+}^{t} \int_{U_i} f_i(X(s-), u)\nu^d_i(du)ds$ and $\int_{0}^{t-} \int_{U_i} g_i(X(s), z)\nu^g_i(dz)ds$ for $i = 1, 2$ is $\mathcal{P}(\mathbb{F})$-measurable and $\mathcal{O}(\mathbb{F})$-measurable respectively.

- The following equalities
$$\begin{aligned} E\left[\int_0^t \int_{U_i} f_i(X(s-),u)N_i^d(ds,du)\right] &= E\left[\int_0^t \int_{U_i} f_i(X(s-),u)\nu_i^d(du)ds\right], \\ E\left[\int_0^t \int_{U_i} g_i(X(s),u)N_i^g(ds,du)\right] &= E\left[\int_0^T \int_{U_i} g_i(X(s),u)\nu_i^g(du)ds\right] \end{aligned}$$
are valid.

In this paper, we consider the following JSDE:
$$\begin{aligned} X(t) = X_0 &+ \int_0^t \sigma(X(s))dB_s + \int_0^t \int_{U_1} f_1(X(s-),u)\widetilde{N_1^d}(ds,du) \\ &+ \int_0^t \int_{U_1} g_1(X(s),u)\widetilde{N_1^g}(ds,du) + \int_0^t b(X(s))ds \\ &+ \int_0^t \int_{U_2} f_2(X(s-),u)N_2^d(ds,du) + \int_0^t \int_{U_2} g_2(X(s),u)N_2^g(ds,du) \end{aligned} \tag{2}$$
with $\mathbb{E}[|X_0|^2] < \infty$, where
$$\widetilde{N_1^d}(dt,du) = N_1^d(dt,du) - \nu_1^d(du)dt$$
$$\widetilde{N_1^g}(dt,du) = N_1^g(dt,du) - \nu_1^g(du)dt$$
are the compensated Poisson random measures of $N_1^d(dt,du)$ and $N_1^g(dt,du)$, respectively.

Definition 2.1. *A process $X(t)$ is said to be a optionl strong solution of (2) if it is $\mathscr{F}_t$-adapted almost surely for every $t \geq 0$, where $\mathscr{F}_t = \sigma\left(\{B_t\}, \{\delta_1^d(t)\}, \{\delta_2^d(t)\}, \{\delta_1^g(t)\}, \{\delta_2^g(t)\}\right)$ is the augmented natural filtration generated by $\{B_t\}$, $\{\delta_1^d(t)\}$, $\{\delta_2^d(t)\}$, $\{\delta_1^g(t)\}$ and $\{\delta_2^g(t)\}$ which are independent of each other.*

Lemma 2.2. *Let U_3 be a subsets of U_2 respectively, satisfying $\nu_2^d(U_2\backslash U_3) < \infty$, $\nu_2^g(U_2\backslash U_3) < \infty$ and consider the following JSDE:*
$$\begin{aligned} X(t) &= X_0 + \int_0^t \sigma(X(s))dB_s + \int_0^t \int_{U_1} f_1(X(s-),u)\widetilde{N_1^d}(ds,du) \\ &\quad + \int_0^t \int_{U_1} g_1(X(s),u)\widetilde{N_1^g}(ds,du) + \int_0^t b(X(s))ds \\ &\quad + \int_0^t \int_{U_3} f_2(X(s-),u)N_2^d(ds,du) + \int_0^t \int_{U_3} g_2(X(s),u)N_2^g(ds,du). \end{aligned} \tag{3}$$
Then (2) has an strong optional solution if (3) has an strong optional solution. Moreover, the pathwise uniqueness of strong solutions holds for (2) if it holds for (3).

Proof. If $\nu_1^d(U_1 \setminus U_2) = 0$ and $\nu_1^g(U_1 \setminus U_2) = 0$, the lemma is trivially valid. Therfore we suppose that $0 < \nu_1^d(U_1 \setminus U_2) < \infty$ and $0 < \nu_1^g(U_1 \setminus U_2) < \infty$. We assume that (3) has a strong optional solution $\{x_0(t)\}$. Let $\{S_k : k = 1, 2, \cdots\}$ be the set of jump times of the Poisson process
$$t \mapsto \int_0^t \int_{U_2\backslash U_3} N_2^d(ds,du) + \int_0^t \int_{U_2\backslash U_3} N_2^g(ds,du).$$

It's clear that $S_k \to \infty$ as $k \to \infty$. For $0 \le t < S_1$ set $y(t) = x_0(t)$. Suppose that $y(t)$ has been defined for $0 \le t < S_k$ and let

$$X_0 = y(S_k) + \int_{\{S_k\}} \int_{U_2} f_1(y(S_k-), u) N_2^d(ds, du) + \int_{\{S_k\}} \int_{U_2} g_1(y(S_k), u) N_2^g(ds, du). \quad (4)$$

By the assumption there is also a strong solution $\{x_k(t)\}$ to

$$\begin{aligned} x(t) = X_0 &+ \textstyle\int_0^t \sigma(x(s)) dB(S_k + s) + \int_0^t \int_{U_1} f_1(x(s-), u) \tilde{N}_1^d(S_k + ds, du) \\ &+ \textstyle\int_0^t \int_{U_1} g_1(x(s), u) \tilde{N}_1^g(S_k + ds, du) + \int_0^t b(x(s)) ds \\ &+ \textstyle\int_0^t \int_{U_3} f_2(x(s-), u) N_2^d(S_k + ds, du) + \int_0^t \int_{U_3} g_2(x(s), u) N_2^g(S_k + ds, du). \end{aligned} \quad (5)$$

For $S_k \le t < S_{k+1}$ we set $y(t) = x_k(t - S_k)$. By (3) and (5) it is not hard to show that $\{y(t)\}$ is a strong solution to (2). On the other hand, if $\{y(t)\}$ is a solution to (2), it satisfies (3) for $0 \le t < S_1$ and satisfies (5) for $S_k \le t < S_{k+1}$ with X_0 given by (4). Then the pathwise uniqueness for (2) follows from that for (3) and (5). □

As in [19], let us consider the following assumptions:

Assumption 2.1. *Assume that there exists a continuous, non-decreasing and concave function $\rho : [0, \infty) \to [0, \infty)$ such that $\rho(x) > 0$ for $x > 0$ satisfing*

$$\int_{0+} \frac{ds}{\rho(s)} = \infty. \quad (6)$$

It is clear that the following functions satisfy (6):

$$\rho(x) = x (x > 0); \qquad \rho(x) = -x \ln x (0 < x \le \frac{1}{e});$$

$$\rho(x) = x \ln(-\ln x) (0 < x \le \frac{1}{e}); \qquad \rho(x) = 1 - x^x (0 < x \le \frac{1}{e}).$$

Assumption 2.2. *Suppose that there exists a non-decreasing and continuously differentiable function $\Upsilon : [0, \infty) \to [1, \infty)$ satisfying*

(i) $\lim_{x \to \infty} \Upsilon(x) = +\infty$;

(ii) $\int_0^\infty \frac{ds}{s\Upsilon(s) + 1} = +\infty$;

(iii) $2xb(x) + |\sigma(x)|^2 + \int_{U_1} |f_1(x-, u)|^2 \nu_1^d(du) + \int_{U_1} |g_1(x, u)|^2 \nu_1^g(du)$
$+2 \int_{U_3} |f_2(x-, u)|^2 \nu_2^d(du) + 2 \int_{U_3} |g_2(x, u)|^2 \nu_2^g(du) \le k[x^2 \Upsilon(x^2) + 1]$ *for all* $x \in \mathbb{R}$, *where $k \ge 0$ is a fixed constant.*

It's clear that the following functions satisfy Assumption 2.2:

$$\Upsilon(x) = \ln x \ (x \ge e); \ \ \Upsilon(x) = \ln x \ln(\ln x) \ (x \ge e^2).$$

Assumption 2.3. *Assume that there exists a constant $\delta_0 > 0$ such that, for any $x, y \in \mathbb{R}$ with $0 < |x - y| \le \delta_0$,*

(i) $\max\left\{(x - y)(b(x) - b(y)), |\sigma(x) - \sigma(y)|^2\right\} \le |x - y|^{2-\alpha} \rho(|x - y|^\alpha)$;

(ii) $\int_{U_1} \max\{|f_1(x, u) - f_1(y, u)|^\alpha, |x - y|^{\alpha-1} \cdot |f_1(x, u) - f_1(y, u)|\} \nu_1^d(du) \le \rho(|x - y|^\alpha)$;

(iii) $\int_{U_3} \max\{|f_2(x, u) - f_2(y, u)|^\alpha, |x - y|^{\alpha-1} \cdot |f_2(x, u) - f_2(y, u)|\} \nu_2^d(du) \le \rho(|x - y|^\alpha)$.

(iv) $\int_{U_1} \max\{|g_1(x, u) - g_1(y, u)|^\alpha, |x - y|^{\alpha-1} \cdot |g_1(x, u) - g_1(y, u)|\} \nu_1^g(du) \le \rho(|x - y|^\alpha)$;

(v) $\int_{U_3} \max\{|g_2(x,u)-g_2(y,u)|^\alpha, |x-y|^{\alpha-1}\cdot|g_2(x,u)-g_2(y,u)|\}\nu_2^g(du) \leq \rho(|x-y|^\alpha)$.
Where $0 \leq \alpha < +\infty$ *is a fixed constant and* ρ *is defined in Assumption 2.1.*

In order to prove our results, we need the following lemmas.

Lemma 2.3. *([15]) Suppose that* $X(t) \in \mathbb{R}$ *is an optional process of the following form:*

$$dX(t) = b(t,\omega)dt + \sigma(t,\omega)dB_t + \int_{\mathbb{R}} \phi^d(t,u,\omega)\overline{N}^d(dt,du) + \int_{\mathbb{R}} \phi^g(t,u,\omega)\overline{N}^g(dt,du),$$

where

$$\overline{N^i}(x) = \begin{cases} N^i(dt,du) - \nu^i(du)dt, & \text{if } |u| < c; \\ N^i(dt,du), & \text{if } |u| \geq c \end{cases} \quad \text{for } i = d,g \tag{7}$$

for some $c \in [0,+\infty)$. *Let* $f \in C^2(\mathbb{R}^2)$ *and define* $Y(t) = h(t,X(t))$. *Then* $Y(t)$ *is again an optional process and*

$$\begin{aligned} dY(t) &= \frac{\partial h}{\partial t}(t,X(t))dt + \frac{\partial h}{\partial x}(t,X(t))\left[b(t,\omega)dt + \sigma(t,\omega)dB_t\right] + \frac{1}{2}\sigma^2(t,X(t))\frac{\partial^2 h}{\partial x^2}(t,X(t-))dt \\ &+ \int_{|u|<c} \left\{ h\left(t,X(t-)+\phi^d(t,u)\right) - h(t,X(t-)) - \frac{\partial h}{\partial x}(t,X(t-))\phi^d(t,u) \right\}\nu^d(du)dt \\ &+ \int_{|u|<c} \left\{ h\left(t,X(t)+\phi^g(t,u)\right) - h(t,X(t)) - \frac{\partial h}{\partial x}(t,X(t))\phi^g(t,u) \right\}\nu^g(du)dt \\ &+ \int_{\mathbb{R}} \left\{ h(t,X(t-)+\phi^d(t,u)) - h(t,X(t-)) \right\}\overline{N}^d(dt,du) \\ &+ \int_{\mathbb{R}} \left\{ h(t,X(t)+\phi^g(t,u)) - h(t,X(t)) \right\}\overline{N}^g(dt,du). \end{aligned}$$

In the next, for any $f \in C^n(\mathbb{R})$, we will replace $\frac{\partial^n}{\partial^n x}f(x)$ by $D^{(n)}f(x)$ for convenience.

Lemma 2.4. *([19]) Let* $u(t)$ *and* $g(t)$ *be non-negative continuous functions, and* $f(t)$ *a non-negative continuously differentiable and non-decreasing function for all* $t \geq 0$. *Furthermore, suppose that* $\rho : [0,+\infty) \to [0,+\infty)$ *is a non-negative and non-decreasing continuous function with*

$$\rho(t) = 0 \Longleftrightarrow t = 0 \text{ and } \int_{0+} \frac{ds}{\rho(s)} = \infty.$$

Then the inequality

$$u(t) \leq f(t) + \int_0^t g(s)\rho(u(s))ds$$

implies the inequality

$$u(t) \leq \Omega^{-1}\left[\Omega(f(t)) + \int_0^t g(s)ds\right],$$

where

$$\Omega(t) = \int_0^t \frac{ds}{\rho(s)}, \quad \forall t > 0.$$

Moreover, if $f(t) = 0$ *and* $|g(t)| < +\infty$, *then* $u(t) = 0$.

3. Main results

Theorem 3.1. *Under Assumption 2.2, the solutions for JSDE (2) have no finite explosion time.*

Proof. Define the follwing function

$$\phi(x) = \exp\left\{\int_0^x \frac{ds}{s\Upsilon(s)+1}\right\}, \quad x \geq 0.$$

We have

$$\phi'(x) = \phi(x)\frac{1}{x\Upsilon(x)+1} \geq 0, \quad \phi''(x) = \phi(x)\frac{1-\Upsilon(x)-x\Upsilon'(x)}{(x\Upsilon(x)+1)^2} \leq 0.$$

Then $\phi(x)$ is a concave function with $\phi(x) \to \infty$ as $x \to \infty$. Moreover,

$$D^{(1)}\phi(x^2) = \phi'(x^2)\cdot 2x, \quad D^{(2)}\phi(x^2) = 2\phi'(x^2) + \phi''(x^2)\cdot 4x^2.$$

Since $\phi''(x) \leq 0$, we know that

$$\phi(y) \leq \phi(x) + (y-x)\phi'(x), \quad \forall x, y \in [0,\infty).$$

Consequently,

$$\begin{aligned}
&\phi\left((X(s-) + f_1\left(X(s-),u\right))^2\right) - \phi(X^2(s-)) - D^{(1)}\phi\left((X^2(s-)\right)\cdot f_1(X(s-),u)\\
&\leq \phi'(X^2(s-))\left[2X(s-)f_1(X(s-)) + f_1^2\left(X(s-),u\right)\right] +\\
&\quad - \phi'(X^2(s-)).2X(s-)f_1\left(X(s-),u\right)\\
&= \phi'(X^2(s-))f_1^2(X(s-),u).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\phi\left((X(s) + g_1\left(X(s),u\right))^2\right) - \phi(X^2(s)) - D^{(1)}\phi\left((X^2(s)\right)\cdot g_1(X(s),u)\\
&\leq \phi'(X^2(s))g^2(X(s),u).
\end{aligned}$$

Let X be a solution to (2) and τ its lifetime. If we define the stopping time τ_R as

$$\tau_R := \inf\left\{t > 0 : |X(t)| \geq R\right\},$$

it is clear that τ_R tends to τ as $R \to \infty$.
From Lemma 2.3, we have

$$\begin{aligned}
&\mathbb{E}\left[\phi(X^2(t\wedge\tau_R))\right] = \mathbb{E}[\phi(|X_0|^2)]\\
&+\mathbb{E}\left[\int_0^{t\wedge\tau_R} D^{(1)}\phi(X^2(s))b(X(s)) + \frac{1}{2}\sigma^2(X(s))D^{(2)}\phi(X^2(s))ds\right]\\
&+\mathbb{E}\Bigg[\int_0^{t\wedge\tau_R}\int_{U_1}\phi(|X(s-)+f_1(X(s-),u)|^2) - \phi(X^2(s-))\\
&- D^{(1)}\phi(|X^2(s-)|)f_1(X(s-),u)\nu_1^d(du)ds\Bigg]\\
&+\mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3}\phi\left((X(s-)+f_2(X(s-),u))^2\right) - \phi((X^2(s-))\nu_2^d(du)ds\right]\\
&+\mathbb{E}\Bigg[\int_0^{t\wedge\tau_R}\int_{U_1}\phi(|X(s)+g_1(X(s),u)|^2) - \phi(X^2(s))\\
&- D^{(1)}\phi(|X^2(s)|)g_1(X(s),u)\nu_1^g(du)ds\Bigg]\\
&+\mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3}\phi\left((X(s)+g_2(X(s),u))^2\right) - \phi((X^2(s))\nu_2^g(du)ds\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}[\phi(|X_0)|^2] + \mathbb{E}\left[\int_0^{t\wedge\tau_R} \left(\phi'(X^2(s))\left[2X(s)b(X(s)) + \sigma^2(X(s))\right] \right.\right. + \\
&+2(X^2(s))\phi''(X^2(s))\sigma^2(X(s)) + \int_{U_1} \phi'(X^2(s-))|f_1(X(s-),u)|^2\nu_1^d(du) \\
&\left.\left. + \int_{U_1} \phi'(X^2(s))|g_1(X(s),u)|^2\nu_1^g(du)\right) ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3} \phi'(X^2(s-))(2X(s-)f_2(X(s-),u) + f_2^2(X(s-),u))\nu_2^d(du)ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3} \phi'(X^2(s))(2X(s-)g_2(X(s-),u) + g_2^2(X(s-),u))\nu_2^g(du)ds\right] \\
&\leq \mathbb{E}[\phi(|X_0)|^2] + \mathbb{E}\left[\int_0^{t\wedge\tau_R} \left(\phi'(X^2(s))\left[2X(s)b(X(s)) + \sigma^2(X(s))\right] \right.\right. \\
&+2(X^2(s))\phi''(X^2(s))\sigma^2(X(s)) + \int_{U_1} \phi'(X^2(s-))|f_1(X(s-),u)|^2\nu_1^d(du) \\
&\left.\left. + \int_{U_1} \phi'(X^2(s))|g_1(X(s),u)|^2\nu_1^g(du)\right) ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3} \phi'(X^2(s-))(2|f_2(X(s-),u)|^2 + X^2(s-))\nu_2^d(du)ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3} \phi'(X^2(s))(2|g_2(X(s),u)|^2 + X^2(s))\nu_2^g(du)ds\right] \\
&\leq \mathbb{E}[\phi(|X_0)|^2] + \mathbb{E}\left[\int_0^{t\wedge\tau_R} \left(\phi'(X^2(s))\left[2X(s)b(X(s)) + \sigma^2(X(s))\right] \right.\right. \\
&+ \int_{U_1} \phi'(X^2(s-))|f_1(X(s-),u)|^2\nu_1^d(du) \\
&\left.\left. + \int_{U_1} \phi'(X^2(s))|g_1(X(s),u)|^2\nu_1^g(du)\right) ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3} \phi'(X^2(s-))2|f_2(X(s-),u)|^2\nu_2^d(du)ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3} \phi'(X^2(s))2|g_2(X(s),u)|^2\nu_2^g(du)ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3} \phi'(X^2(s-))X^2(s-)\nu_2^d(du)ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge\tau_R}\int_{U_3} \phi'(X^2(s))X^2(s)\nu_2^g(du)ds\right].
\end{aligned}
$$

Since $\phi''(x) \leq 0$, $\int_{U_3^i} 1\nu_2^i(du) \leq \int_{U_2^i} 1\nu_2^i(du) \leq M$ for $i = d, g$, by Assumption 2.2, we have

$$
\begin{aligned}
\mathbb{E}\left[\phi(X^2(t\wedge\tau_R))\right] \leq& \phi\left(\mathbb{E}[|X_0|^2]\right) + \mathbb{E}\left[\int_0^{t\wedge\tau_R} \phi'(X^2(s))(M+1)k[X^2(s)\Upsilon(X^2(s)) + 1]ds\right] \\
=& \phi\left(\mathbb{E}[|X_0|^2]\right) + (M+1)k\int_0^t \mathbb{E}[\phi(X^2(s\wedge\tau_R))]ds,
\end{aligned}
$$

therefore by using Gronwall stochastic lemma ([2]), we get

$$
\mathbb{E}\left[\phi(X^2(t\wedge\tau_R))\right] \leq \phi\left(\mathbb{E}[|X_0|^2]\right) e^{k(M+1)t}.
$$

Letting $R \to \infty$ in above inequality, by Fatou lemma, we get

$$\mathbb{E}\left[\phi(X^2(t \wedge \tau))\right] \leq \phi\left(\mathbb{E}[|X_0|^2]\right) e^{k(M+1)t}.$$

Letting $|X(t \wedge \tau)| \to \infty$, from (ii) of Assumption 2.2, we have $t \to \infty$ as $\mathbb{E}[|X_0|^2] < \infty$. Therefore we have

$$\mathcal{P}(\tau < \infty) = 0.$$

Consequently, the solutions do not explode in finite time. □

Theorem 3.2. *Under Assumptions 2.2 and 2.3, the pathwise uniqueness of strong optional solutions holds for JSDE (3).*

Proof. From the assumptions imposed on ρ, we get a strictly decreasing sequence $\{a_n\} \subset (0, 1]$ such that

(*i*) $a_0 = 1$;
(*ii*) $\lim\limits_{n\to\infty} a_n = 0$;
(*iii*) $\int_{a_n}^{a_{n-1}} \frac{1}{\rho(r)} dr = n$ for every $n \geq 1$.

It's clear that for each $n \geq 1$, there exists a continuous function ρ_n on $\mathbb{R}$ such that

(*i*) $\rho_n(r)$ has a supported set (a_n, a_{n-1});
(*ii*) $0 \leq \rho_n(r) \leq \frac{2}{n\rho(r)}$ for every $r > 0$;
(*iii*) $\int_{a_n}^{a_{n-1}} \rho_n(r) dr = 1$.

We introduce the following sequence of functions:

$$\psi_n(r) = \int_0^{|r|} \int_0^v \rho_n(u) du dv, \ r \in \mathbb{R}, \ n \geq 1.$$

It's clear that ψ_n is even and twice continuously differentiable (except at $r = 0$) with the following characteristics::

(*i*) $|\psi_n'(r)| \leq 1, \quad r \neq 0$;
(*ii*) $\lim\limits_{n\to\infty} \psi_n(r) = |r|, \quad r \neq 0$;
(*iii*) $\psi_n''(r) \leq \frac{2}{n\rho(r)} I_{(a_n, a_{n-1})}(r), \quad r \neq 0$.

Moreover, for each $r > 0$, the sequence $\{\psi_n(r)\}_{n\geq 1}$ is non-decreasing. For each $n \in \mathbb{N}$, ψ_n, ψ_n' and ψ_n'' all vanish on the interval $(-a_n, a_n)$. We get for $0 \neq x \in \mathbb{R}$,

$$D\psi_n(|x|^\alpha) = \frac{d}{dx}\psi_n(|x|^\alpha) = \psi_n'(|x|^\alpha)\alpha x|x|^{\alpha-2}$$

and

$$D^2\psi_n(|x|^\alpha) = \psi_n''(|x|^\alpha)\alpha^2|x|^{2\alpha-2} + \psi_n'(|x|^\alpha)\alpha(\alpha-1)|x|^{\alpha-2}.$$

Next we assume that X and $\widetilde{X}$ are two solutions for (3) of the following forms:

$$X(t) = X_0 + \int_0^t \sigma(X(s))dB_s + \int_0^t \int_{U_1} f_1(X(s-), u)\widetilde{N_1^d}(ds, du) + \int_0^t \int_{U_1} g_1(X(s), u)\widetilde{N_1^g}(ds, du) +$$
$$+ \int_0^t b(X(s))ds + \int_0^t \int_{U_3} f_2(X(s-), u)N_2^d(ds, du) + \int_0^t \int_{U_3} g_2(X(s), u)N_2^g(ds, du)$$

and

$$\widetilde{X}(t) = X_0 + \int_0^t \sigma(\widetilde{X}(s))dB_s + \int_0^t \int_{U_1} f_1(\widetilde{X}(s-), u)\widetilde{N_1^d}(ds, du) + \int_0^t \int_{U_1} g_1(\widetilde{X}(s), u)\widetilde{N_1^g}(ds, du) +$$
$$+ \int_0^t b(\widetilde{X}(s))ds + \int_0^t \int_{U_3} f_2(\widetilde{X}(s-), u)N_2^d(ds, du) + \int_0^t \int_{U_3} f_2(\widetilde{X}(s), u)N_2^g(ds, du)$$

for all $t \geq 0$, where $x, \widetilde{x} \in \mathbb{R}$. $\Delta_t := \widetilde{X}(t) - X(t)$ for all $t \geq 0$.
Defined

$$S_{\delta_0} = \inf\{t \geq 0 : |\Delta_t| \geq \delta_0\} = \inf\left\{t \geq 0 : |\widetilde{X}(t) - X(t)| \geq \delta_0\right\}.$$

For $R > 0$, let

$$\tau_R := \inf\left\{t \geq 0 : \max\left\{|\widetilde{X}(t)|\ , |X(t)|\right\} \geq R\right\}.$$

Then, by Theorem 3.1, we have $\tau_R \to \infty$ a.s. as $R \to \infty$. Denote $t' = t \wedge \tau_R \wedge S_{\delta_0}$ and

$$\Delta_{f_i} = f_i(\widetilde{X}(s-), u) - f_i(X(s-), u), \quad i = 1, 2.$$

$$\Delta_{g_i} = g_i(\widetilde{X}(s), u) - g_i(X(s), u), \quad i = 1, 2.$$

From Lemma 2.3, we obtain

$$\begin{aligned}
\mathbb{E}\left[\psi_n(|\Delta_{t'}|^\alpha)\right] = \mathbb{E}&\left[\int_0^{t'} I_{\{\Delta_s \neq 0\}}\left\{D\psi_n(|\Delta_s|^\alpha)(b(\widetilde{X}(s)) - b(X(s))) + \right.\right.\\
&\left.\left. + \frac{1}{2}D^2\psi_n(|\Delta_s|^\alpha)|\sigma(\widetilde{X}(s)) - \sigma(X(s))|^2 ds\right\}\right]\\
&+ \mathbb{E}\left[\int_0^{t'}\int_{U_1}\{\psi_n(|\Delta_s + \Delta_{f_1}|^\alpha) - \psi_n(|\Delta_s|^\alpha) - I_{\{\Delta_s \neq 0\}}D\psi_n(|\Delta_s|^\alpha)\Delta_{f_1}\}\nu_1^d(du)ds\right.\\
&\left. + \int_0^{t'}\int_{U_3}\{\psi_n(|\Delta_s + \Delta_{f_2}|^\alpha) - \psi_n(|\Delta_s|^\alpha)\}\nu_2^d(du)ds\right]\\
&+ \mathbb{E}\left[\int_0^{t'}\int_{U_1}\{\psi_n(|\Delta_s + \Delta_{g_1}|^\alpha) - \psi_n(|\Delta_s|^\alpha) - I_{\{\Delta_s \neq 0\}}D\psi_n(|\Delta_s|^\alpha)\Delta_{g_1}\}\nu_1^g(du)ds\right.\\
&\left. + \int_0^{t'}\int_{U_3}\{\psi_n(|\Delta_s + \Delta_{g_2}|^\alpha) - \psi_n(|\Delta_s|^\alpha)\}\nu_2^g(du)ds\right]\\
= J_1 + J_2 + J_3.&
\end{aligned}$$

Since

$$|\psi_n'(r)| \leq 1, \quad \psi_n''(r) \leq \frac{2}{n\rho(r)}I_{(a_n, a_{n-1})}(r),$$

by Assumption 2.3, we get

$$\begin{aligned}
J_1 \leq &\ \mathbb{E}\Bigg[\int_0^{t'} I_{\{\Delta_s \neq 0\}}\Bigg\{|\psi_n'(|\Delta_s|^\alpha)| \cdot |\alpha| \cdot |\Delta_s|^{\alpha-2}(\widetilde{X}(s) - X(s))(b(\widetilde{X}(s)) - b(X(s)))\\
&+ \frac{1}{2}\left\{|\psi_n''(|\Delta_s|^\alpha)|\alpha^2|\Delta_s|^{2\alpha-2} + |\alpha(\alpha-1)| \cdot \psi_n'(|\Delta_s|^\alpha)|\Delta_s|^{\alpha-2}\right\}|\sigma(\widetilde{X}(s)) - \sigma(X(s))|^2 ds\Bigg\}\Bigg]\\
\leq &\ \mathbb{E}\Bigg[\int_0^{t'} I_{\{\Delta_s \neq 0\}}\Bigg[|\alpha| \cdot |\Delta_s|^{\alpha-2}|\Delta_s|^{2-\alpha}\rho(|\Delta_s|^\alpha)\\
&+ \frac{1}{2}\left\{\frac{2}{n\rho(|\Delta_s|^\alpha)}I_{(a_n, a_{n-1})}(|\Delta_s|^\alpha)\alpha^2|\Delta_s|^{2\alpha-2} + |\alpha(\alpha-1)| \cdot |\Delta_s|^{\alpha-2}\right\}|\Delta_s|^{2-\alpha}\rho(|\Delta_s|^\alpha)\Bigg]ds\Bigg]\\
\leq &\ \mathbb{E}\left[\int_0^{t'}(\frac{1}{2}|\alpha(\alpha-1)| + |\alpha|) \cdot \rho(|\Delta_s|^\alpha) + \frac{\alpha^2|\Delta_s|^\alpha}{n}I_{(a_n, a_{n-1})}(|\Delta_s|^\alpha)ds\right]\\
\leq &\left(\frac{1}{2}|\alpha(\alpha-1)| + |\alpha|\right)\mathbb{E}\left[\int_0^{t'}\rho(|\Delta_s|^\alpha)ds\right] + \frac{\alpha^2 a_{n-1}^\alpha}{n}t'.
\end{aligned}$$

For J_2, by Lagrange's mean value theorem and the fact that $|\psi_n'(r)| \leq 1$, we have the following cases:

Case I. For $0 < \alpha \leq 1$, since $(A+B)^\alpha \leq A^\alpha + B^\alpha$ for all $A, B \geq 0$, we know that there exists some $\xi_1 \in [|\Delta_s|^\alpha, (|\Delta_s| + |\Delta_{f_i}|)^\alpha]$ such that

$$\begin{aligned}
\psi_n(|\Delta_s + \Delta_{f_i}|^\alpha) - \psi_n(|\Delta_s|^\alpha) &\leq \psi_n(|\Delta_s|^\alpha + |\Delta_{f_i}|^\alpha) - \psi_n(|\Delta_s|^\alpha) \\
&\leq |\psi_n'(\xi_1)| \cdot ||\Delta_s|^\alpha + |\Delta_{f_i}|^\alpha - |\Delta_s|^\alpha| \\
&\leq |\Delta_{f_i}|^\alpha.
\end{aligned}$$

Case II. For $1 < \alpha < +\infty$, since $(A+B)^{\alpha-1} \leq (2^{\alpha-2}+1)(A^{\alpha-1}+B^{\alpha-1})$ for all $A, B \geq 0$, there exists some $\xi_2 \in [\min\{\Delta_s, \Delta_s + \Delta_{f_i}\}, \max\{\Delta_s, \Delta_s + \Delta_{f_i}\}]$ such that

$$\begin{aligned}
\psi_n(|\Delta_s + \Delta_{f_i}|^\alpha) - \psi_n(|\Delta_s|^\alpha) &\leq \alpha|\psi_n'(|\xi_2|^\alpha)| \cdot |\xi_2|^{\alpha-1} \cdot |\Delta_s + \Delta_{f_i} - \Delta_s| \\
&\leq \alpha(|\Delta_s| + |\Delta_{f_i}|)^{\alpha-1}|\Delta_{f_i}| \\
&\leq \alpha(2^{\alpha-2}+1)(|\Delta_s|^{\alpha-1}|\Delta_{f_i}| + |\Delta_{f_i}|^\alpha),
\end{aligned}$$

where the second inequality follows from

$$0 \leq |\xi_2| \leq \max\{|\Delta_s|, |\Delta_s + \Delta_{f_i}|\} \leq |\Delta_s| + |\Delta_{f_i}|.$$

Thus, from Assumption 2.2

$$\begin{aligned}
J_2 &\leq \mathbb{E}\left[\int_0^{t'} 2\alpha(2^{\alpha-2}+1)\rho(|\Delta_s|^\alpha) + \alpha\rho(|\Delta_s|^\alpha) + 2\alpha(2^{\alpha-2}+1)\rho(|\Delta_s|^\alpha)\right] ds \\
&\leq \alpha(2^\alpha+5)\mathbb{E}\left[\int_0^{t'} \rho(|\Delta_s|^\alpha) ds\right].
\end{aligned}$$

Similarly for J_3, we get

$$J_3 \leq \alpha(2^\alpha+5)\mathbb{E}\left[\int_0^{t'} \rho(|\Delta_s|^\alpha) ds\right]$$

then

$$\mathbb{E}\left[\psi_n(|\Delta_{t'}|^\alpha)\right] \leq p(\alpha)\mathbb{E}\left[\int_0^{t'} \rho(|\Delta_s|^\alpha) ds\right] + \frac{\alpha^2 a_{n-1}^\alpha}{n} t,$$

where

$$p(\alpha) = \frac{1}{2}|\alpha(\alpha-1)| + |\alpha| + \alpha(2^{\alpha+1}+10).$$

Since $\lim\limits_{n\to\infty} \psi_n(r) = |r|$, letting $n \to \infty$ yields

$$\begin{aligned}
\mathbb{E}\left[|\Delta_{t'}|^\alpha\right] &\leq p(\alpha)\mathbb{E}\Big[\int_0^{t'} \rho(|\Delta_s|^\alpha) ds\Big] \\
&\leq p(\alpha)\mathbb{E}\Big[\int_0^{t\wedge\tau_R} \rho\left(|\Delta_{s\wedge S_{\delta_0}}|^\alpha\right) ds\Big] \\
&\leq p(\alpha)\int_0^t \rho\left(\mathbb{E}(|\Delta_{s\wedge S_{\delta_0}\wedge\tau_R}|^\alpha)\right) ds,
\end{aligned}$$

we get the last inequality from Jensen's inequality. From using Theorem 3.1, Fatou's lemma and the monotone convergence theorem, we obtain

$$\mathbb{E}[|\Delta_{t\wedge S_{\delta_0}}|^\alpha] \leq \lim_{R\to\infty} \mathbb{E}[|\Delta_{t'}|^\alpha] \leq p(\alpha)\int_0^t \rho\left(\mathbb{E}(|\Delta_{s\wedge S_{\delta_0}}|^\alpha)\right) ds.$$

From Lemma 2.4, we get $\mathbb{E}[|\Delta_{t\wedge S_{\delta_0}}|^\alpha] \to 0$ and so $\Delta_{t\wedge S_{\delta_0}} = 0$ a.s.

On the set $\{S_{\delta_0} \leq t\}$, we have $|\Delta_{t'}| \geq \delta_0$. Observing that $0 = \mathbb{E}[|\Delta_{t\wedge S_{\delta_0}}|^{\alpha}] \geq \delta_0^{\alpha}\mathbb{P}\{S_{\delta_0} \leq t\}$, we have $\mathbb{P}\{S_{\delta_0} \leq t\} = 0$ and hence $\Delta_t = 0$ a.s., which is the desired result. $\square$

Theorem 3.3. *Under Assumptions 2.2 and 2.3, JSDE (2) has a unique non-explosive strong optional solution.*

Proof. By using the result of Theorems 3.1, we get the existence of unique non-explosive strong optional solution for (3). Then from Lemma 2.2, we obtain the existence of unique non-explosive strong optional solution for (2). Indeed similar arguments can be found in the proof of Theorem 2.8 of [27]. $\square$

Example 3.1. *As an application of Theorem 3.3, we will study one particular case: As in [19], we consider the following JSDE:*

$$\begin{aligned} X(t) =& X_0 - \int_0^t |X(s)| \ln |X(s)| ds + \int_0^t \sqrt{|X(s)|} dB_s \\ &+ \int_0^t \int_{|u|\leq 1} \sqrt{|X(s)|} \widetilde{N_1^d}(ds, du) + \int_0^t \int_{|u|\leq 1} \sqrt{|X(s)|} \widetilde{N_1^g}(ds, du) \\ &+ \int_0^t \int_{|u|>1} \gamma |u| X(s) N_2^d(ds, du) + \int_0^t \int_{|u|>1} \beta |u| X(s) N_2^g(ds, du). \end{aligned} \tag{8}$$

Here γ and β are a positive constants, such that $\int_{|u|\leq 1} |\gamma u|^2 \nu^d(du) = 1$ and $\int_{|u|\leq 1} |\beta u|^2 \nu^g(du) = 1$. For any $x > 0$, the coefficient $b(x) = -x \ln x$ satisfies Assumptions 2.1 and 2.2, and for any $x \geq 0$, the coefficient $\sigma(x) = \sqrt{x}$ satisfies Assumption 2.2. Thus, $b(x)$ and $\sigma(x)$ are both non-Lipschitzian due to

$$\lim_{x\to 0^+} b'(x) = \lim_{x\to 0^+} \sigma'(x) = +\infty.$$

Furthermore,

$$|b(x) - b(y)| = |\textstyle\int_x^y 1 + \ln t \, dt| \leq \int_0^{|x-y|} |1 + \ln t| \, dt = b(|x - y|)$$

for all $0 < x, y < \frac{1}{e}$, and

$$(\sigma(x) - \sigma(y))^2 \leq |x - y|$$

for all $x, y \geq 0$. Thus, the coefficients of (8) satisfy Assumptions 2.2. By Theorem (3.3), we know that (8) has a unique non-explosive strong solution.

4. Examples of JSDEs in financial and insurance mathematics

In the first, we give an example of financial model which can be described by the jump-type stochastic differential equations (JSDEs) with respect to optional semimartingales (see [3]).
The constant elasticity of variance (CEV) model was proposed by Cox and Ross [12]. It is often used in mathematical finance to capture leverage effects and stochasticity of volatility. It is also widely used by practitioners in the financial industry for modeling equities and commodities. Consider a modified version of the CEV model where the stock price is said to satisfy the following integral equation,

$$\begin{aligned} S_t &= \rho S.A_t + \sigma S^{\alpha}.M_t, \quad S_0 = s, \\ A_t &= t + \int_{0+}^t \int_{|u|>1} u\mu^d(ds, du) + \int_0^{t-} \int_{|u|>1} u\mu^g(ds, du), \\ M_t &= W_t + \int_{0+}^t \int_{|u|\leq 1} u(\mu^d - \nu^d)(ds, du) + \int_0^{t-} \int_{|u|\leq 1} u(\mu^g - \nu^g)(ds, du), \end{aligned}$$

where ρ and σ are constants and the martingale M is a jump-diffusion process with left and right jumps. W_t is the Wiener process, $\mu^d - \nu^d$ is the measure of right jumps and $\mu^g - \nu^g$ is the measure of left jumps. For $B \in \mathcal{B}(\mathbb{R}_+)$ and $\Gamma \in \mathcal{B}(\mathbb{E})$ the jump measures are defined as follows

$$\begin{aligned}
\mu^d(B \times \Gamma) &:= \left\{(t, \Delta L_t^d) \in B \times \Gamma \mid t > 0 \text{ such that } \Delta L_t^d \neq 0\right\}, \\
\mu^g(B \times \Gamma) &:= \left\{(t, \Delta^+ L_t^g) \in B \times \Gamma \mid t > 0 \text{ such that } \Delta^+ L_t^g \neq 0\right\},
\end{aligned}$$

where L_t^d and L_t^g are independent Poisson random measures with constant intensities γ^d and γ^g respectively and compensators $\nu^d = \gamma^d t$ and $\nu^g = \gamma^g t$.

In the second we give an example in mathematical risk theory which is the stochastic model of risk in insurance (see [26]).
Let us consider a risk process whose flow can be summarized by the following equation

$$R_t = u + B_t + N_t + D_t + L_t,$$

where $u > 0$ is the initial capital and $B_0 = N_0 = D_0 = L_0 = 0$.
The process B is a continuous predictable process of finite variation characterizing a stable flow of income payments including premiums and other sources, N is a continuous local martingale representing a random perturbation, D and L are right continuous and left continuous jump processes, respectively. The process L may model some substantial gains or losses in returns on investment. The process D includes a sum of negative jumps representing accumulated claims. In addition, D may also consist of jumps formed by non-anticipated sharp falls or rises in returns on investment.
The risk process can be described by the jump-type stochastic differential equations (JSDEs). Consider an optional semimartingale R which is the risk process with the local characteristics $(a, \langle X^c \rangle, \nu^r, \nu^g)$ and the following representation:

$$\begin{aligned}
R_t &= u + a_t + X^c + \int_{]0,t]} \int_{|x| \leq 1} x(\mu^d - \nu^d)(ds, dx) + \int_{[0,t[} \int_{|x| \leq 1} x(\mu^g - \nu^g)(ds, dx) + \\
&\quad + \int_{]0,t]} \int_{|x| > 1} x\mu^d(ds, dx) + \int_{[0,t[} \int_{|x| > 1} x\mu^g(ds, dx),
\end{aligned}$$

where

$$a_t = B_t + \int_{]0,t]} \int_{|x| \leq 1} x\nu^d(ds, dx) + \int_{[0,t[} \int_{|x| \leq 1} x\nu^g(ds, dx), \quad \langle X^c \rangle_t = \langle N \rangle_t .$$

Acknowledgements

The authors would like to thank the referees for the careful reading of the paper and highly appreciate the comments and suggestions, which significantly contributed to improving the quality of the paper.

References

[1] Abdelghani, M., Melnikov, A., (2016), Financial markets in the context of the general theory of optional processes, Mathematical and Computational Approaches in Advancing Modern Science and Engineering, pp. 519–528.

[2] Abdelghani, M., Melnikov, A., (2017), On linear stochastic equations of optional semimartingales and their applications, Statistics & Probability Letters, 125, pp. 207–214.

[3] Abdelghani, M., Melnikov, A., (2018), A comparison theorem for stochastic equations of optional semimartingales, Stochastics and Dynamics, 4.

[4] Abdelghani, M., Melnikov, A., (2020a), Existence and uniqueness of stochastic equations of optional semimartingales under monotonicity condition, Stochastics, 92, pp. 67–89.

[5] Abdelghani, M., Melnikov, A., (2020b), Optional Processes, Chapman et Hall/CRC, New York, 1st edition edition.
[6] Abdelghani, M., Melnikov, A., Pak, A., (2021), On statistical estimation and inferences in optional regression models, Statistics, 55(2), pp. 445–457.
[7] Abdelghani, M., Melnikov, A., Pak, A., (2022), On comparison theorem for optional sdes via local times and applications, Stochastics, 94(3), pp. 365–385.
[8] Akdim, K., Haddadi, M., (2023), Characterization of optional submartingales of a new class (Σ^l), Boletim da Sociedade Paranaense de Matemática Article in press,
[9] Akdim, K., Haddadi, M., Ouknine, Y., (2020), Strong snell envelopes and rbsdes with regulated trajectories when the barrier is a semimartingale, Stochastics, 92(3), pp. 335–355.
[10] Akdim, K., Ouknine, Y., (2006), Infinite horizon reflected backward sdes with jumps and rcll obstacle, Stochastic analysis and applications, 24(6), pp. 1239–1261.
[11] Chudley, C., Elliott, R., (1961), Neutron scattering from a liquid on a jump diffusion model, Proceedings of the Physical Society, 77(2), pp. 353–361.
[12] Cox, J. C., Ross, S. A., (1976), The valuation of options for alternative stochastic processes, Journal of Financial Economics, 3(1), pp. 145–166.
[13] Czichowsky, C., Schachermayer, W., (2016), Strong supermartingales and limits of nonnegative martingales, The Annals of Probability, 44, pp. 171–205.
[14] Gal'čuk, L., (1977), On the existence of optional modifications for martingales, Theory of Probability and its Applications, 22, pp. 572–573.
[15] Gal'čuk, L., (1979), A change of variables formula, Mathematical Notes 26 (4).
[16] Gal'čuk, L., (1981), Optional martingale, American Mathematical Society, Mathematics of the USSR-Sbornik, 40(4).
[17] Gal'čuk, L., (1982), Decomposition of optional supermartingales, Mathematics of the USSR Sbornik, 43, pp. 145–158.
[18] Gal'čuk, L., (1985), Stochastic integrals with respect to optional semimartingales and random measures, Theory of Probability and its Applications XXIX, pp. 93–108.
[19] Gou, Z., hui Wang, M., jing Huang, N., (2020), Strong solutions for jump-type stochastic differential equations with non-lipschitz coefficients, Stochastics, 92(4), pp. 533–551.
[20] Grigorova, M., Imkeller, P., Offen, E., Ouknine, Y., Quenez, M.-C., (2015), Reflected bsdes when the obstacle is not right-continuous and optimal stopping.
[21] Jarni, I., Ouknine, Y., (2020), On reflection with two-sided jumps, Journal of Theoretical Probability, 34(4).
[22] Lenglart, E., (1980), Tribus de meyer et théorie des processus, Séminaire de probabilités de Strasbourg, 14, pp. 500–546.
[23] Mertens, J., (1972), Théorie des processus stochastiques généraux applications aux surmartingales, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 22, pp. 45–68.
[24] Shreve, S., (2004), Stochastic Calculus for Finance ll: Continuous-Time Models, Springer-Verlag New York.
[25] Tankov, P., (2003), Financial Modelling with Jump Processes, Chapman and Hall/CRC, New York, 1st edition edition.
[26] Valjarević, D., Stojanović, V., Valjarević, A., (2024), Application of the concept of statistical causality in integrable increasing processes and measures, Axioms, 13(2).
[27] Xi, F., Zhu, C., (2019), Jump type stochastic differential equations with non-lipschitz coefficients: Non-confluence, feller and strong feller properties, and exponential ergodicity, Journal of Differential Equations, 266(8), pp. 4668–4711.

M. Haddadi currently working as an assistant professor in the Department of Science and Technology, The Higher School of Education and Training, Ibn Zohr University. Her research interests are Stochastic calculus, Semimartingale theory, Reflected Backward Stochastic Differential Equations (RBSDEs), Applied Probability, Financial Mathematics, Stochastic volatility models and pricing options with mathematical models.

K. Akdim Professor in the Department of Mathematics, Faculty of Sciences and Techniques, Cadi Ayyad University. Her research interests encompass several key areas in the field of mathematics and finance, focusing on advanced stochastic processes and their practical applications. Her primary research involves the study of RBSDEs with Jumps and Their Applications in Finance. She also focuses on identifying and characterizing new classes of submartingales, contributing to a deeper understanding of their properties and applications. Her work extends to developing and analyzing stochastic models that accurately capture occurrences in the real world, especially in the fields of finance, economics, and epidemiology.