

MULTI-DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS AND COMPLETE GRAPHS INTO BANNERS AND STARS OF SIZE FIVE

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ABSTRACT. A decomposition of a graph G is a set of edge disjoint subgraphs $H_1, H_2, \dots, H_r$ of G such that every edge of G belongs to exactly one H_i . If all the subgraphs in the decomposition of G are isomorphic to a graph H , then we say that G is H -decomposable. The graph G has an (H_1, H_2) -multi-decomposition if α copies of H_1 and β copies of H_2 decompose G , where α and β are non-negative integers. In this paper, we have obtained the multi-decomposition of complete bipartite graphs and complete graphs into banners and stars of size 5.

Keywords: Banners, Stars, Complete graphs, Complete Bipartite graphs, Multi-decomposition.

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1. INTRODUCTION

All graphs considered here are finite, simple and undirected. Let K_n denotes the complete graph on n vertices and $K_{m,n}$ denotes the complete bipartite graph with vertex partite sets of cardinality m and n . A *banner* is a graph in which one edge is attached to any vertex of the square C_4 and it is denoted by $C_4 + e$ and $S_5 (= K_{1,5})$ denotes a *star* on 5 edges. For a graph G , if $E(G)$ can be partitioned into subsets $E_1, \dots, E_k$ such that the subgraph of G induced by E_i is H_i for each $1 \leq i \leq k$, we write $G = H_1 \oplus \dots \oplus H_k$, where $\oplus$ denotes the edge disjoint sum of the subgraph H_i . For $1 \leq i \leq k$, if $H_i \cong H$, we say that G has an H -decomposition. If G can be decomposed into α copies of H_1 and β copies of H_2 , then we say that G has an $\{H_1^\alpha, H_2^\beta\}$ -decomposition or an (H_1, H_2) -multi-decomposition. If the necessary conditions for the existence of an $\{H_1^\alpha, H_2^\beta\}$ -decomposition of G are satisfied by the pairs (α, β) , $\alpha, \beta \geq 0$, then we say that (α, β) is an admissible pair.

The study of $\{H_1^\alpha, H_2^\beta\}$ - decomposition has been introduced by Abueida and Daven [1]. In [2], the existence of multi-decomposition of $K_m(\lambda)$ into stars and cycles of same

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size has been studied. The multi-decomposition of complete graphs into kites, cycles, stars and paths has been studied in [7, 10, 11]. In [4], the kite and star multi-decomposition of complete graphs has been obtained. The banner decomposition of graphs such as complete graphs, complete equi-partite graphs and some regular graphs has been studied in [3, 5, 8]. The multi-decomposition of complete bipartite graphs into cycles, stars and paths has been studied in [6, 9]. In this paper, we investigate the multi-decomposition of $K_{m,n}$ and K_n into banners and stars of size 5, for the given positive integers $m \geq 2$ and $n \geq 5$.

Notations:

- Let $V(G) = \{v_i\}$, $i = 1, 2, \dots, n$. Then a banner $C_4 + e$ in a graph G is denoted by $(v_a, v_b, v_c, v_d; v_a v_f)$, where v_a, v_b, v_c, v_d is cycle C_4 and $v_a v_f$ is the pendant edge e attached to the vertex v_a of C_4 .
- A star S_5 in a graph G is of the form $(v_a; v_b, v_c, v_d, v_e, v_f)$, where v_a is the center vertex (of degree 5) and v_b, v_c, v_d, v_e, v_f are the end vertices (pendant vertices). The $n + 1$ stars with the same end vertices v_1, v_2, v_3, v_4, v_5 and different center vertices $a_0, a_1, \dots, a_n$ are denoted by $(a_0, a_1, \dots, a_n; v_1, v_2, \dots, v_5)$. This notation is same as for two or more twins having same end vertices.
- Let $V(K_{p,q}) = (M, N)$, where $M = \{m_1, m_2, \dots, m_p\}$ and $N = \{n_1, n_2, \dots, n_q\}$.

Remark 1.1. If two graphs G_1 and G_2 have an $\{H_1^\alpha, H_2^\beta\}$ -decomposition, then $G_1 \oplus G_2$ also has an $\{H_1^\alpha, H_2^\beta\}$ -decomposition.

Remark 1.2. For a graph G and a subgraph H of G , if sH denotes s edge disjoint copies of H in G , then sH has an $\{H_1^\alpha, H_2^\beta\}$ -decomposition whenever H has such a decomposition.

Observation 1.3. If there exist two stars S_5 having the same end vertices $v_1, v_2, \dots, v_5$ and centers at a_0 & a_1 are denoted by $(a_0, a_1; v_1, v_2, v_3, v_4, v_5)$. Further they can be decomposed into two banner graphs given by $(a_0, v_1, a_1, v_2; a_0 v_3)$, $(a_0, v_4, a_1, v_5; a_1 v_3)$. Hence, we call this structure as Twins denoted by $\mathcal{T}$.

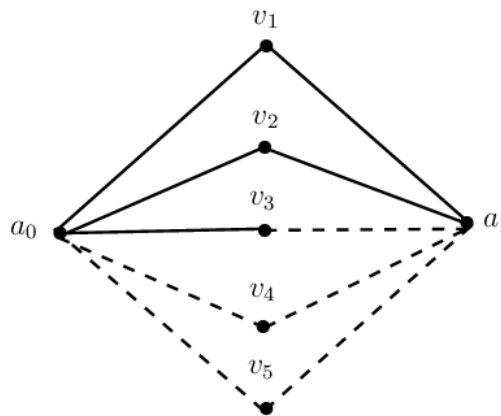


FIGURE 1. $\mathcal{T}$ (Twins)

2. $\{(C_4 + e)^\alpha, S_5^\beta\}$ - DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS

In this section, we prove the sufficient condition for the existence of a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of $K_{m,n}$ for $m \geq 2$ and $n \geq 5$. The following lemmas are useful in proving our main result.

Lemma 2.1. *For $2 \leq n \leq 9$, there exists a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of $K_{5,n}$.*

Proof. The following table shows the existence of a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of $K_{5,n}$, where $2 \leq n \leq 9$ for the given admissible pairs (α, β) .

n	$K_{5,n}$	Admissible pairs (α, β)
2	$1\mathcal{T}$	$(0, 2), (2, 0)$
3	$1\mathcal{T} \oplus 1S_5$	$(0, 3), (2, 1)$
4	$2\mathcal{T}$	$(0, 4), (2, 2), (4, 0)$
5	$\mathcal{D}_1$	$(5, 0)$
	$2\mathcal{T} \oplus 1S_5$	$(0, 5), (2, 3), (4, 1)$
6	$3\mathcal{T}$	$(6, 0), (0, 6), (2, 4), (4, 2)$
	$\mathcal{D}_2$	$(3, 3)$
	$K_{5,5} \oplus 1S_5$	$(5, 1)$
7	$K_{5,5} \oplus \mathcal{D}_3$	$(7, 0)$
	$K_{5,6} \oplus 1S_5$	$(0, 7), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$
8	$K_{5,6} \oplus \mathcal{D}_4$	$(8, 0)$
	$K_{5,7} \oplus 1S_5$	$(0, 8), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1)$
9	$K_{5,7} \oplus \mathcal{D}_5$	$(9, 0)$
	$K_{5,8} \oplus 1S_5$	$(0, 9), (2, 8), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1)$

TABLE 1. Multi-decomposition of $K_{5,n}$ into banners and stars

where $\mathcal{D}_1 = \{(m_1, n_1, m_3, n_2; m_1 n_4), (m_1, n_3, m_5, n_5; n_3 m_3), (n_1, m_2, n_3, m_4; n_1 m_5), (m_2, n_2, m_5, n_4; n_2 m_4), (m_3, n_4, m_4, n_5; n_5 m_2)\}$,

$\mathcal{D}_2 = \{(m_1, n_4, m_4, n_1; n_1 m_2), (m_1, n_5, m_4, n_2; n_2 m_3), (m_1, n_6, m_4, n_3; n_3 m_5), (m_2, n_2, n_3, n_4, n_5, n_6), (m_3, n_1, n_3, n_4, n_5, n_6), (m_5, n_1, n_2, n_4, n_5, n_6)\}$,

$\mathcal{D}_3 = \{(n_6, m_1, n_7, m_2; n_6 m_3), (n_6, m_4, n_7, m_5; n_7 m_3)\}$,

$\mathcal{D}_4 = \{(n_7, m_1, n_8, m_2; n_7 m_3), (n_7, m_4, n_8, m_5; n_8 m_3)\}$ and

$\mathcal{D}_5 = \{(n_8, m_1, n_9, m_2; n_8 m_3), (n_8, m_4, n_9, m_5; n_9 m_3)\}$. □

Lemma 2.2. *There exists a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of $K_{5,10}$ for all admissible pairs (α, β) with $\alpha \neq 1$ such that $\alpha + \beta = 10$.*

Proof. The decomposition of $K_{5,10}$ into α copies of $C_4 + e$ and β copies of S_5 for all admissible pairs (α, β) such that $\alpha + \beta = 10$ is discussed in the following cases:

Case 1: $(\alpha, \beta) \in \{(0, 10), (2, 8), (4, 6), (6, 4), (8, 2), (10, 0)\}$.

We can write, $K_{5,10} = 10S_5 = 5\mathcal{T}$. We get the required decomposition for the even pairs $(\alpha, \beta) \in \{(0, 10), (2, 8), (4, 6), (6, 4), (8, 2), (10, 0)\}$ by Observation 1.3.

Case 2: $(\alpha, \beta) \in \{(3, 7), (5, 5), (7, 3)\}$.

Here, we have $K_{5,10} = 3(C_4 + e) \oplus 3S_5 \oplus 2\mathcal{T}$ as given below:

$\{(n_1, m_2, n_6, m_3; n_1 m_4), (m_2, n_3, m_3, n_8; n_3 m_5), (m_2, n_2, m_3, n_7; n_2 m_1)\} \oplus \{(m_1; n_1, n_3, n_6, n_7, n_8), (m_4; n_2, n_3, n_6, n_7, n_8), (m_5; n_1, n_2, n_6, n_7, n_8)\} \oplus \{(n_4, n_9, n_5, n_{10}; m_1, m_2, m_3, m_4, m_5)\}$.

Then the required decomposition for the odd pairs $(\alpha, \beta) \in \{(3, 7), (5, 5), (7, 3)\}$ follows from Observation 1.3.

Case 3: $(\alpha, \beta) = (9, 1)$.

We have, $K_{5,10} = 9(C_4 + e) \oplus 1S_5$, as given below:

$\{(n_1, m_2, n_2, m_1; m_2 n_9), (m_1, n_7, m_2, n_6; m_2 n_{10}), (n_1, m_4, n_2, m_5; n_1 m_3), (m_2, n_5, m_3, n_4; n_4 m_1), (m_1, n_9, m_3, n_{10}; m_1 n_5), (n_4, m_5, n_5, m_4; m_4 n_6), (m_4, n_{10}, m_5, n_9; m_5 n_7), (m_2, n_8, m_4, n_3; m_4 n_7), (n_3, m_1, n_8, m_5; m_5 n_6)\} \oplus \{(m_3; n_2, n_3, n_6, n_7, n_8)\}$.

Hence, we get the desired decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 10$, $\alpha \neq 1$. When $\alpha = 1$, by removing one copy of $C_4 + e$ in $K_{5,10}$, the resulting graph does not have sufficient vertex degree for the decomposition of $9S_5$. $\square$

Theorem 2.1. *Let $m \geq 2$ and $n \geq 5$ be the given integers. Then the graph $K_{m,n}$ admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) with $\alpha \neq 1$, if and only if $mn \equiv 0 \pmod{5}$.*

Proof. The necessary condition is obvious. The sufficiency can be proved in two cases.

Case 1. $m = 5p$ and $n = 5q$.

When both p and q are odd, we can write, $K_{5p,5q} = \frac{p(q-1)}{2}K_{5,10} \oplus \frac{p-1}{2}K_{5,10} \oplus K_{5,5}$. From Lemma 2.2, the graph $K_{5,10}$ has a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition and from Lemma 2.1, the graph $K_{5,5}$ has a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition. Hence, $K_{m,n}$ has a desired decomposition in this case, as given in Remarks 1.1 and 1.2. For other values of p and q , we can write, $K_{5p,5q} = \frac{pq}{2}K_{5,10}$. Then the graph $K_{m,n}$ has a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition, by Lemma 2.2 and Remark 1.2.

Case 2. $m = 5p$ and $n = 5q + r$, where $r \in \{1, 2, 3, 4\}$.

When both p and q are odd, we can write, $K_{5p,5q+r} = pK_{5,5+r} \oplus \frac{p(q-2)}{2}K_{5,10} \oplus \frac{p-1}{2}K_{5,10} \oplus K_{5,5}$. For other values of p and q , we can write, $K_{5p,5q+r} = pK_{5,5+r} \oplus \frac{p(q-1)}{2}K_{5,10}$. Then the proof follows from Lemmas 2.1, 2.2 and Remarks 1.1, 1.2. $\square$

3. $\{(C_4 + e)^\alpha, S_5^\beta\}$ - DECOMPOSITION OF COMPLETE GRAPHS

In this section, the sufficient condition for the existence of a $\{(C_4 + e)^\alpha, S_5^\beta\}$ - decomposition of K_n for $n \geq 10$ is obtained. The following lemmas are useful in proving our main result.

Lemma 3.1. *There exists a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of K_{15} for all admissible pairs (α, β) such that $\alpha + \beta = 21$.*

Proof. We can write, $K_{15} = K_{10} \oplus K_5 \oplus K_{5,10}$. From Appendix 5.1, the graph K_{10} admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 9$. From Appendix 5.3, the graph $K_5 \oplus K_{5,10}$ admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 12$. Hence the proof follows from Remark 1.1. $\square$

Lemma 3.2. *There exists a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of K_{16} for all admissible pairs (α, β) such that $\alpha + \beta = 24$.*

Proof. We can write, $K_{16} = K_{10} \oplus K_6 \oplus K_{6,10}$. From Appendix 5.1, the graph K_{10} admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 9$. From Appendix 5.4, the graph $K_6 \oplus K_{6,10}$ admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 15$. Hence the proof follows from Remark 1.1. $\square$

Observation 3.1. For $n \geq 20$, $p, q \geq 10$, let

$$K_n = K_p \oplus K_q \oplus K_{p,q} \quad (1)$$

and assume that the graphs K_p and K_q admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition. Let the graph $K_{p,q}$ admits such a decomposition for all admissible pairs (α, β) such that $\alpha + \beta = \frac{mn}{5}$ except when $\alpha = 1$. But this does not mean that the graph K_n would not have a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) . Using a different admissible pairs (α, β) for the subgraphs K_p , K_q and $K_{p,q}$, we can get the required decomposition in K_n for all admissible pairs (α, β) .

Theorem 3.1. Let $n > 6$ be the given integer. Then the graph K_n admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) , if and only if $n \equiv 0$ or $1 \pmod{5}$.

Proof. Necessity: We know that the necessary condition for the existence of a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition is $5(\alpha + \beta) = \binom{n}{2}$. Then either $n \equiv 0 \pmod{5}$ or $n \equiv 1 \pmod{5}$.

Sufficiency: Assume that $n \equiv 0$ or $1 \pmod{5}$. Then n can be written as $n = 10s + t$, where $s \geq 1$ and $t \in \{1, 2, 5, 6\}$. Then the sufficiency follows from the four cases as given below:

Case 1: For $t = 0$, we can write,

$$K_{10s} = sK_{10} \oplus \frac{s(s-1)}{2}K_{10,10} \quad (2)$$

Case 2: For $t = 1$, we can write,

$$K_{10s+1} = K_{11} \oplus (s-1)K_{10} \oplus (s-1)K_{11,10} \oplus \frac{(s-1)(s-2)}{2}K_{10,10}. \quad (3)$$

Case 3: For $t = 5$, we can write,

$$K_{10s+5} = K_{15} \oplus (s-1)K_{10} \oplus (s-1)K_{15,10} \oplus \frac{(s-1)(s-2)}{2}K_{10,10} \quad (4)$$

Case 4: For $t = 6$, we can write,

$$K_{10s+6} = K_{16} \oplus (s-1)K_{10} \oplus (s-1)K_{16,10} \oplus \frac{(s-1)(s-2)}{2}K_{10,10} \quad (5)$$

From Appendix 5.1, the graph K_{10} admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 9$. From Appendix 5.2, the graph K_{11} admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 11$. From Lemma 3.1, the graph K_{15} admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 21$. From Lemma 3.2, the graph K_{16} admits a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 24$. From Theorem 2.1, the graphs $K_{10,10}$, $K_{10,11}$, $K_{15,10}$ and $K_{16,10}$ admit a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs except $(\alpha, \beta) \in (1, \beta)$. Hence, the graph K_{10s+t} where $t \in \{1, 2, 5, 6\}$ admits the required decomposition for all admissible pairs by Remarks 1.1, 1.2 and Observation 3.1. $\square$

4. CONCLUSIONS

In this paper, it is proved that there exists a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition in $K_{m,n}$ for all admissible pairs (α, β) with $\alpha \neq 1$, whenever $mn \equiv 0 \pmod{5}$, for $m \geq 2$ and $n \geq 5$. Also, it is proved that there exists a $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition in K_n for all admissible pairs (α, β) , whenever $n \equiv 0$ or $1 \pmod{5}$, for $n > 6$.

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5. APPENDIX

If the desired $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition exists in G , then we write, $G = \alpha(C_4 + e) \oplus \beta S_5$. Here, we discuss the existence of such a decomposition in G for all admissible pairs (α, β) .

5.1. A $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of K_{10} .

Case 1: $(\alpha, \beta) = (9, 0)$

$$K_{10} = 9(C_4 + e) = \{(v_1, v_2, v_4, v_9; v_1 v_8), (v_1, v_3, v_4, v_6; v_3 v_7), (v_1, v_5, v_9, v_7; v_9 v_3), (v_2, v_3, v_5, v_{10}; v_2 v_8), (v_2, v_6, v_8, v_9; v_9 v_{10}), (v_3, v_6, v_7, v_{10}; v_7 v_5), (v_4, v_5, v_6, v_{10}; v_6 v_9), (v_2, v_5, v_8, v_7; v_7 v_4), (v_1, v_4, v_8, v_{10}; v_8 v_3)\}.$$

Case 2: $(\alpha, \beta) = (8, 1)$

$$K_{10} = 8(C_4 + e) \oplus 1S_5 = \{(v_3, v_4, v_9, v_8; v_3 v_{10}), (v_4, v_6, v_5, v_8; v_8 v_2), (v_4, v_5, v_9, v_{10}; v_9 v_2), (v_2, v_3, v_7, v_6; v_7 v_{10}), (v_3, v_5, v_{10}, v_6; v_{10} v_2), (v_2, v_4, v_7, v_5; v_7 v_9), (v_1, v_9, v_6, v_8; v_9 v_3), (v_1, v_7, v_8, v_{10}; v_7 v_2)\} \oplus \{(v_1; v_2, v_3, v_4, v_5, v_6)\}.$$

Case 3: $(\alpha, \beta) = (7, 2)$

$$K_{10} = 7(C_4 + e) \oplus 2S_5 = \{(v_1, v_7, v_8, v_{10}; v_{10} v_2), (v_2, v_8, v_4, v_9; v_4 v_{10}), (v_1, v_3, v_{10}, v_9; v_{10} v_7), (v_5, v_6, v_9, v_7; v_6 v_{10}), (v_3, v_5, v_9, v_8; v_3 v_7), (v_4, v_5, v_8, v_6; v_5 v_{10}), (v_3, v_4, v_7, v_6; v_3 v_9)\} \oplus \{(v_1; v_2, v_4, v_5, v_6, v_8), (v_2; v_3, v_4, v_5, v_6, v_7)\}.$$

Case 4: $(\alpha, \beta) = (6, 3)$

$$K_{10} = 6(C_4 + e) \oplus 3S_5 = \{(v_2, v_9, v_3, v_{10}; v_2 v_8), (v_1, v_9, v_4, v_{10}; v_9 v_7), (v_4, v_5, v_9, v_8; v_8 v_6), (v_1, v_7, v_5, v_8; v_7 v_{10}), (v_6, v_7, v_8, v_{10}; v_7 v_4), (v_5, v_6, v_9, v_{10}; v_6 v_4)\} \oplus \{(v_1; v_2, v_3, v_4, v_5, v_6), (v_2; v_3, v_4, v_5, v_6, v_7), (v_3; v_4, v_5, v_6, v_7, v_8)\}.$$

Case 5: $(\alpha, \beta) = (5, 4)$

$$K_{10} = 5(C_4 + e) \oplus 4S_5 = \{(v_1, v_2, v_5, v_7; v_5 v_{10}), (v_6, v_7, v_9, v_8; v_8 v_{10}), (v_5, v_6, v_{10}, v_9; v_5 v_8), (v_1, v_8, v_7, v_{10}; v_{10} v_4), (v_2, v_9, v_3, v_{10}; v_9 v_6)\} \oplus \{(v_1; v_3, v_4, v_5, v_6, v_9), (v_2; v_3, v_4, v_6, v_7, v_8), (v_3; v_4, v_5, v_6, v_7, v_8), (v_4; v_5, v_6, v_7, v_8, v_9)\}.$$

Case 6: $(\alpha, \beta) = (4, 5)$

$$K_{10} = 4(C_4 + e) \oplus 5S_5 = \{(v_1, v_2, v_8, v_7; v_2 v_9), (v_2, v_3, v_6, v_5; v_3 v_8), (v_3, v_4, v_{10}, v_9; v_9 v_5),$$

$$(v_4, v_5, v_8, v_9; v_5 v_3)\} \oplus \{(v_1; v_3, v_5, v_6, v_8, v_9), (v_4; v_1, v_2, v_6, v_7, v_8), (v_6; v_2, v_7, v_8, v_9, v_{10}), \\ (v_7; v_2, v_3, v_5, v_9, v_{10}), (v_{10}; v_1, v_2, v_3, v_5, v_8)\}.$$

Case 7: $(\alpha, \beta) = (3, 6)$

$$K_{10} = 3(C_4 + e) \oplus 6S_5 = \{(v_2, v_3, v_9, v_{10}; v_{10} v_4), (v_6, v_9, v_8, v_{10}; v_{10} v_3), (v_1, v_8, v_2, v_9; v_8 v_6)\} \oplus \\ \{(v_1; v_3, v_4, v_5, v_6, v_{10}), (v_2; v_1, v_4, v_5, v_6, v_7), (v_4; v_5, v_6, v_7, v_8, v_9), (v_3; v_4, v_5, v_6, v_7, v_8), \\ (v_5; v_6, v_7, v_8, v_9, v_{10}), (v_7; v_1, v_6, v_8, v_9, v_{10})\}.$$

Case 8: $(\alpha, \beta) = (2, 7)$

$$K_{10} = 2(C_4 + e) \oplus 7S_5 = \{(v_1, v_6, v_7, v_2; v_1 v_{10}), (v_6, v_8, v_7, v_9; v_8 v_3)\} \oplus \{(v_1; v_3, v_4, v_5, v_7, v_8), \\ (v_2; v_4, v_5, v_6, v_8, v_{10}), (v_3; v_2, v_4, v_5, v_6, v_7), (v_4; v_5, v_6, v_7, v_8, v_9), (v_5; v_6, v_7, v_8, v_9, v_{10}), \\ (v_9; v_1, v_2, v_3, v_8, v_{10}), (v_{10}; v_3, v_4, v_6, v_7, v_8)\}.$$

Case 9: $(\alpha, \beta) = (1, 8)$

$$K_{10} = 1(C_4 + e) \oplus 8S_5 = \{(v_1, v_8, v_9, v_2; v_8 v_6)\} \oplus \{(v_1; v_3, v_4, v_5, v_6, v_7), (v_2; v_4, v_5, v_6, v_7, v_8), \\ (v_3; v_2, v_5, v_6, v_7, v_8), (v_4; v_3, v_5, v_6, v_8, v_{10}), (v_5; v_6, v_7, v_8, v_9, v_{10}), (v_7; v_4, v_6, v_8, v_9, v_{10}), \\ (v_9; v_1, v_3, v_4, v_6, v_{10}), (v_{10}; v_1, v_2, v_3, v_6, v_8)\}.$$

Case 10: $(\alpha, \beta) = (0, 9)$

$$K_{10} = 9S_5 = \{(v_1; v_2, v_3, v_4, v_5, v_6), (v_2; v_3, v_4, v_5, v_6, v_7), (v_3; v_4, v_5, v_6, v_7, v_8), \\ (v_4; v_5, v_6, v_7, v_8, v_9), (v_5; v_6, v_7, v_8, v_9, v_{10}), (v_7; v_1, v_6, v_8, v_9, v_{10}), (v_8; v_1, v_2, v_6, v_9, v_{10}), \\ (v_9; v_1, v_2, v_3, v_6, v_{10}), (v_{10}; v_1, v_2, v_3, v_4, v_6)\}.$$

5.2. A $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of K_{11} .

Case 1: $(\alpha, \beta) = (11, 0)$

$$K_{11} = 11(C_4 + e) = \{(v_1, v_2, v_9, v_8; v_1 v_4), (v_2, v_3, v_{10}, v_{11}; v_2 v_8), (v_3, v_4, v_{10}, v_9; v_4 v_7), \\ (v_4, v_5, v_1, v_{11}; v_5 v_2), (v_5, v_6, v_1, v_{10}; v_6 v_2), (v_6, v_7, v_1, v_3; v_6 v_9), (v_7, v_8, v_5, v_3; v_7 v_{10}), \\ (v_2, v_4, v_9, v_7; v_9 v_1), (v_5, v_7, v_{11}, v_9; v_{11} v_8), (v_3, v_8, v_6, v_{11}; v_{11} v_5), (v_4, v_6, v_{10}, v_8; v_{10} v_2)\}.$$

Case 2: $(\alpha, \beta) = (10, 1)$

$$K_{11} = 10(C_4 + e) \oplus 1S_5 = \{(v_1, v_2, v_9, v_8; v_1 v_4), (v_5, v_6, v_1, v_{10}; v_{10} v_{11}), (v_3, v_4, v_{10}, v_9; v_4 v_7), \\ (v_4, v_5, v_1, v_{11}; v_5 v_2), (v_4, v_6, v_{10}, v_8; v_{10} v_3), (v_6, v_7, v_1, v_3; v_6 v_9), (v_7, v_8, v_5, v_3; v_7 v_{10}), \\ (v_2, v_4, v_9, v_7; v_9 v_1), (v_5, v_7, v_{11}, v_9; v_{11} v_8), (v_3, v_8, v_6, v_{11}; v_{11} v_5)\} \oplus \{(v_2; v_3, v_5, v_6, v_8, v_{11})\}.$$

Case 3: $(\alpha, \beta) = (9, 2)$

$$K_{11} = 9(C_4 + e) \oplus 2S_5 = \{(v_1, v_2, v_{10}, v_9; v_9 v_5), (v_2, v_3, v_9, v_8; v_8 v_4), (v_3, v_8, v_{10}, v_4; v_4 v_6), \\ (v_4, v_5, v_{10}, v_{11}; v_{11} v_6), (v_5, v_6, v_1, v_{11}; v_1 v_7), (v_6, v_7, v_{11}, v_9; v_7 v_4), (v_7, v_8, v_1, v_{10}; v_{10} v_6), \\ (v_2, v_4, v_9, v_7; v_4 v_1), (v_2, v_6, v_8, v_{11}; v_2 v_9)\} \oplus \{(v_3; v_1, v_6, v_7, v_{10}, v_{11}), (v_5; v_1, v_2, v_3, v_7, v_8)\}.$$

Case 4: $(\alpha, \beta) = (8, 3)$

$$K_{11} = 8(C_4 + e) \oplus 3S_5 = \{(v_1, v_2, v_9, v_8; v_2 v_6), (v_2, v_3, v_{10}, v_{11}; v_{10} v_8), (v_3, v_4, v_{10}, v_9; v_3 v_{11}), \\ (v_4, v_5, v_1, v_{11}; v_1 v_9), (v_5, v_6, v_1, v_{10}; v_5 v_3), (v_6, v_7, v_{11}, v_9; v_{11} v_8), (v_6, v_8, v_2, v_{10}; v_6 v_{11}), \\ (v_1, v_3, v_8, v_7; v_3 v_6)\} \oplus \{(v_4; v_1, v_2, v_6, v_8, v_9), (v_5; v_2, v_7, v_8, v_9, v_{11}), (v_7; v_2, v_3, v_4, v_9, v_{10})\}.$$

Case 5: $(\alpha, \beta) = (7, 4)$

$$K_{11} = 7(C_4 + e) \oplus 4S_5 = \{(v_1, v_2, v_9, v_8; v_2 v_5), (v_2, v_3, v_8, v_7; v_3 v_{10}), (v_3, v_4, v_{10}, v_9; v_3 v_{11}), \\ (v_4, v_5, v_{10}, v_{11}; v_5 v_7), (v_5, v_6, v_1, v_{11}; v_{11} v_9), (v_2, v_6, v_7, v_{10}; v_2 v_{11}), (v_3, v_5, v_9, v_7; v_7 v_{11})\} \oplus \\ \{(v_1; v_3, v_5, v_7, v_9, v_{10}), (v_4; v_1, v_2, v_6, v_7, v_9), (v_6; v_3, v_8, v_9, v_{10}, v_{11}), (v_8; v_2, v_4, v_5, v_{10}, v_{11})\}.$$

Case 6: $(\alpha, \beta) = (6, 5)$

$$K_{11} = 6(C_4 + e) \oplus 5S_5 = \{(v_1, v_2, v_9, v_8; v_1 v_4), (v_2, v_3, v_9, v_{10}; v_9 v_{11}), (v_3, v_4, v_8, v_7; v_4 v_9), \\ (v_4, v_5, v_{10}, v_{11}; v_5 v_8), (v_2, v_5, v_7, v_{11}; v_{11} v_8), (v_2, v_4, v_{10}, v_8; v_8 v_6)\} \oplus \{(v_1; v_6, v_7, v_9, v_{10}, v_{11}), \\ (v_3; v_1, v_6, v_8, v_{10}, v_{11}), (v_5; v_1, v_3, v_6, v_9, v_{11}), (v_6; v_2, v_4, v_9, v_{10}, v_{11}), (v_7; v_2, v_4, v_6, v_9, v_{10})\}.$$

Case 7: $(\alpha, \beta) = (5, 6)$

$$K_{11} = 5(C_4 + e) \oplus 6S_5 = \{(v_1, v_6, v_7, v_{11}; v_7 v_{10}), (v_1, v_2, v_8, v_7; v_7 v_9), (v_2, v_3, v_9, v_{10}; v_{10} v_8), \\ (v_4, v_5, v_{10}, v_{11}; v_{11} v_9), (v_3, v_4, v_9, v_8; v_8 v_{11})\} \oplus \{(v_1; v_4, v_5, v_8, v_9, v_{10}), (v_2; v_5, v_6, v_7, v_9, v_{11}), \\ (v_3; v_1, v_6, v_7, v_{10}, v_{11}), (v_4; v_2, v_{10}, v_8, v_7, v_6), (v_5; v_3, v_7, v_8, v_9, v_{11}), (v_6; v_5, v_8, v_9, v_{10}, v_{11})\}.$$

Case 8: $(\alpha, \beta) = (4, 7)$

$$K_{11} = 4(C_4 + e) \oplus 7S_5 = \{(v_1, v_2, v_9, v_8; v_8 v_{10}), (v_2, v_3, v_7, v_8; v_7 v_9), (v_3, v_4, v_9, v_{10}; v_{10} v_7), (v_4, v_5, v_{10}, v_{11}; v_4 v_6)\} \oplus \{(v_1; v_5, v_6, v_7, v_9, v_{10}), (v_2; v_5, v_6, v_7, v_{10}, v_{11}), (v_3; v_1, v_6, v_8, v_9, v_{11}), (v_4; v_1, v_2, v_7, v_8, v_{10}), (v_5; v_3, v_6, v_7, v_8, v_9), (v_6; v_7, v_8, v_9, v_{10}, v_{11}), (v_{11}; v_1, v_5, v_7, v_8, v_9)\}.$$

Case 9: $(\alpha, \beta) = (3, 8)$

$$K_{11} = 3(C_4 + e) \oplus 8S_5 = \{(v_1, v_2, v_8, v_9; v_9 v_{11}), (v_2, v_3, v_9, v_{10}; v_{10} v_{11}), (v_3, v_4, v_7, v_8; v_3 v_1)\} \oplus \{(v_1; v_6, v_7, v_8, v_{10}, v_{11}), (v_2; v_5, v_6, v_7, v_9, v_{11}), (v_3; v_5, v_6, v_7, v_{10}, v_{11}), (v_4; v_1, v_2, v_9, v_{10}, v_{11}), (v_5; v_1, v_4, v_9, v_{10}, v_{11}), (v_6; v_4, v_5, v_9, v_{10}, v_{11}), (v_7; v_5, v_6, v_9, v_{10}, v_{11}), (v_8; v_4, v_5, v_6, v_{10}, v_{11})\}.$$

Case 10: $(\alpha, \beta) = (2, 9)$

$$K_{11} = 2(C_4 + e) \oplus 9S_5 = \{(v_1, v_2, v_9, v_{10}; v_{10} v_{11}), (v_2, v_3, v_7, v_8; v_8 v_9)\} \oplus \{(v_1; v_6, v_7, v_8, v_9, v_{11}), (v_2; v_5, v_6, v_7, v_{10}, v_{11}), (v_3; v_1, v_8, v_9, v_{10}, v_{11}), (v_4; v_1, v_2, v_3, v_{10}, v_{11}), (v_5; v_1, v_3, v_4, v_{10}, v_{11}), (v_6; v_3, v_4, v_5, v_{10}, v_{11}), (v_7; v_8, v_4, v_5, v_6, v_{10}, v_{11}), (v_9; v_4, v_5, v_6, v_7, v_{11})\}.$$

Case 11: $(\alpha, \beta) = (1, 10)$

$$K_{11} = 1(C_4 + e) \oplus 10S_5 = \{(v_1, v_2, v_9, v_{10}; v_9 v_{11})\} \oplus \{(v_1; v_6, v_7, v_8, v_9, v_{11}), (v_2; v_6, v_7, v_8, v_{10}, v_{11}), (v_3; v_1, v_2, v_9, v_{10}, v_{11}), (v_4; v_1, v_2, v_3, v_{10}, v_{11}), (v_5; v_1, v_2, v_3, v_4, v_{11}), (v_6; v_3, v_4, v_5, v_8, v_{11}), (v_7; v_3, v_4, v_5, v_6, v_{11}), (v_8; v_3, v_4, v_5, v_7, v_{11}), (v_9; v_4, v_5, v_6, v_7, v_8), (v_{10}; v_5, v_6, v_7, v_8, v_{11})\}.$$

Case 12: $(\alpha, \beta) = (0, 11)$

$$K_{11} = 11S_5 = \{(v_1; v_2, v_3, v_4, v_5, v_6), (v_2; v_3, v_4, v_5, v_6, v_7), (v_3; v_4, v_5, v_6, v_7, v_8), (v_4; v_5, v_6, v_7, v_8, v_9), (v_5; v_6, v_7, v_8, v_9, v_{10}), (v_6; v_7, v_8, v_9, v_{10}, v_{11}), (v_7; v_1, v_8, v_9, v_{10}, v_{11}), (v_8; v_1, v_2, v_9, v_{10}, v_{11}), (v_9; v_1, v_2, v_3, v_{10}, v_{11}), (v_{10}; v_1, v_2, v_3, v_4, v_{11}), (v_{11}; v_1, v_2, v_3, v_4, v_5)\}.$$

5.3. A $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of $K_5 \oplus K_{5,10}$.

Case 1: $(\alpha, \beta) \in \{(0, 12), (2, 10), (4, 8), (6, 6)\}$

$$K_5 \oplus K_{5,10} = 6S_5 \oplus 3\mathcal{T} = \{(m_1; m_4, m_5, n_8, n_9, n_{10}), (m_2; m_1, m_5, n_8, n_9, n_{10}), (m_3; m_1, m_2, n_8, n_9, n_{10}), (m_4; m_2, m_3, n_8, n_9, n_{10}), (m_5; m_3, m_4, n_8, n_9, n_{10}), (n_7; m_1, m_2, m_3, m_4, m_5)\} \oplus \{(n_1, n_2, n_3, n_4, n_5, n_6; m_1, m_2, m_3, m_4, m_5)\}.$$

From Observation 1.3, this gives the required decomposition for $(\alpha, \beta) \in \{(2, 10), (4, 8), (6, 6)\}$.

Case 2: $(\alpha, \beta) \in \{(3, 9), (5, 7), (7, 5), (9, 3)\}$

$$K_5 \oplus K_{5,10} = 1(C_4 + e) \oplus 3S_5 \oplus 4\mathcal{T} = \{(m_2, m_3, m_4, m_5; m_2 m_1)\} \oplus \{(m_1; m_3, n_7, n_8, n_9, n_{10}), (m_2; m_4, n_7, n_8, n_9, n_{10}), (m_3; m_5, n_7, n_8, n_9, n_{10})\} \oplus \{(m_4, m_5; m_1, n_7, n_8, n_9, n_{10}), (n_1, n_2, n_3, n_4, n_5, n_6; m_1, m_2, m_3, m_4, m_5)\}.$$

From Observation 1.3, this gives the required decomposition for $(\alpha, \beta) \in \{(3, 9), (5, 7), (7, 5), (9, 3)\}$.

Case 3: $(\alpha, \beta) = (12, 0)$

$$K_5 \oplus K_{5,10} = 12(C_4 + e) = \{(n_2, m_3, n_7, m_4; n_7 m_2), (n_2, m_1, n_7, m_5; n_2 m_2), (n_1, m_3, n_6, m_4; n_6 m_1), (m_1, m_3, m_5, n_1; m_5 m_2), (m_2, m_4, m_5, n_6; m_5 m_1), (n_3, m_1, n_8, m_5; n_3 m_3), (n_3, m_2, n_8, m_4; n_8 m_3), (n_4, m_1, n_9, m_2; n_4 m_4), (n_4, m_3, n_9, m_5; n_9 m_4), (n_5, m_1, n_{10}, m_4; n_{10} m_5), (n_5, m_2, n_{10}, m_3; n_5 m_5), (m_1, m_2, m_3, m_4; m_2 n_1)\}.$$

From this, first two copies of banners can be written as $(n_2, n_7; m_1, m_2, m_3, m_4, m_5)$.

Then, this gives the required decomposition for $(\alpha, \beta) \in (10, 2)$.

Case 4: $(\alpha, \beta) = (11, 1)$

$$K_5 \oplus K_{5,10} = 11(C_4 + e) \oplus 1S_5 = \{(m_2, m_3, m_4, m_5; m_2 m_1), (n_1, m_2, n_6, m_3; m_3 m_5), (n_1, m_4, n_6, m_5; m_4 m_2), (n_2, m_1, n_7, m_3; n_2 m_2), (n_2, m_4, n_7, m_5; n_7 m_2), (n_3, m_1, n_8, m_2; n_3 m_3), (n_3, m_4, n_8, m_5; n_8 m_3), (n_4, m_1, n_9, m_2; n_4 m_4), (n_4, m_3, n_9, m_5; n_9 m_4), (n_5, m_1, n_{10}, m_2; n_5 m_5), (n_5, m_3, n_{10}, m_4; n_{10} m_5)\} \oplus \{(m_1; n_1, n_6, m_3, m_4, m_5)\}.$$

Case 5: $(\alpha, \beta) = \{(10, 2), (8, 4)\}$

$$K_5 \oplus K_{5,10} = 8(C_4 + e) \oplus 2S_5 \oplus 1\mathcal{T} = \{(m_1, m_3, m_5, n_6; m_5 m_2), \\ (m_1, m_2, m_3, m_4; m_2 n_6), (m_2, m_4, m_5, n_7; m_2 n_8), (m_2, n_9, m_5, n_{10}; m_5 n_8), \\ (n_1, m_1, n_2, m_2; n_1 m_3), (n_1, m_4, n_2, m_5; n_2 m_3), (n_3, m_1, n_4, m_2; n_3 m_3), \\ (n_3, m_4, n_4, m_5; n_4 m_3)\} \oplus \{(m_1; m_5, n_7, n_8, n_9, n_{10}), (n_5; m_1, m_2, m_3, m_4, m_5)\} \\ \oplus \{(m_3, m_4; n_6, n_7, n_8, n_9, n_{10})\}.$$

5.4. **A** $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition of $K_6 \oplus K_{6,10}$.

Case 1: $(\alpha, \beta) = (15, 0)$

$$K_6 \oplus K_{6,10} = 15(C_4 + e) = \{(n_1, m_1, n_6, m_2; m_2 m_4), \\ (n_2, m_1, n_7, m_2; m_1 m_3), (n_2, m_3, n_7, m_4; m_3 m_6), (n_2, m_5, n_7, m_6; m_5 m_2), \\ (n_3, m_5, n_8, m_6; m_5 m_4), (n_3, m_2, n_8, m_4; m_4 m_3), (n_3, m_1, n_8, m_3; m_1 m_2), \\ (n_1, m_3, n_6, m_4; m_3 m_2), (n_1, m_5, n_6, m_6; m_5 m_3), (n_4, m_1, n_9, m_4; m_1 m_6), \\ (n_4, m_2, n_9, m_3; m_2 m_6), (n_4, m_5, n_9, m_6; m_6 m_4), (n_5, m_1, n_{10}, m_3; m_1 m_5), \\ (n_5, m_2, n_{10}, m_6; m_6 m_5), (n_5, m_4, n_{10}, m_5; m_4 m_1)\}.$$

Case 2: $(\alpha, \beta) = (14, 1)$

$$K_6 \oplus K_{6,10} = 14(C_4 + e) \oplus 1S_5 = \{(n_1, m_1, n_6, m_2; m_2 m_3), \\ (n_2, m_1, n_7, m_2; m_1 m_4), (n_2, m_3, n_7, m_4; m_4 n_5), (n_2, m_5, n_7, m_6; m_5 m_2), \\ (n_3, m_5, n_8, m_6; m_5 m_4), (n_3, m_2, n_8, m_4; m_4 m_3), (n_3, m_1, n_8, m_3; m_1 m_2), \\ (n_1, m_3, n_6, m_4; m_3 m_1), (n_1, m_5, n_6, m_6; m_5 m_3), (n_4, m_1, n_9, m_4; m_1 m_5), \\ (n_4, m_2, n_9, m_3; m_2 m_4), (n_4, m_5, n_9, m_6; m_5 n_5), (n_5, m_1, n_{10}, m_3; n_{10} m_5), \\ (n_5, m_2, n_{10}, m_6; n_{10} m_4)\} \oplus \{(m_6; m_1, m_2, m_3, m_4, m_5)\}.$$

Case 3: $(\alpha, \beta) = (13, 2)$

$$K_6 \oplus K_{6,10} = 13(C_4 + e) \oplus 2S_5 = \{(n_1, m_1, n_6, m_2; m_2 m_3), \\ (n_2, m_1, n_7, m_2; m_2 n_5), (n_2, m_3, n_7, m_4; m_4 m_1), (n_2, m_5, n_7, m_6; m_5 n_{10}), \\ (n_3, m_5, n_8, m_6; m_6 n_{10}), (n_3, m_2, n_8, m_4; m_4 m_3), (n_3, m_1, n_8, m_3; m_1 m_2), \\ (n_1, m_3, n_6, m_4; m_4 m_2), (n_1, m_5, n_6, m_6; m_6 n_5), (n_4, m_1, n_9, m_4; m_1 m_3), \\ (n_4, m_2, n_9, m_3; m_2 n_{10}), (n_4, m_5, n_9, m_6; n_{10} m_4), (n_5, m_1, n_{10}, m_3; n_5 m_4)\} \\ \oplus \{(m_5; m_1, m_2, m_3, m_4, n_5), (m_6; m_1, m_2, m_3, m_4, m_5)\}.$$

For all other admissible pairs, we can write $K_6 \oplus K_{6,10} = K_5 \oplus K_{5,10} + 3S_5$. From Appendix 5.3, the graph $K_5 \oplus K_{5,10}$ admits $\{(C_4 + e)^\alpha, S_5^\beta\}$ -decomposition for all admissible pairs (α, β) .



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P. Hemalatha for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, Special Issue.