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STUDY OF GROWTH OF CERTAIN SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the solutions of second-order linear differential equations by considering various conditions on the coefficients of the homogeneous linear differential equation and non-homogeneous linear differential equation.

Keywords: entire function, order of growth, homogeneous linear differential equation, non-homogeneous linear differential equation.

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1. Introduction

Consider the homogeneous linear complex differential equation:

$$f'' + A(z)f' + B(z)f = 0, \tag{1}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. All solutions of equation (1) are of finite order if and only if the coefficients $A(z)$ and $B(z)$ are polynomials (see [23]). A natural question arises: what happens when at least one of the coefficients is a transcendental entire function? M. Frei [3] addressed this question, proving that, in such a case, all non-trivial solutions of equation (1) are of infinite order.

The main aim of this work is to identify conditions on the entire coefficients $A(z)$ and $B(z)$ under which all non-trivial solutions of equation (1) are of infinite order. Many researchers have studied this problem previously. Gundersen [6] proved that if $\rho(A) < \rho(B)$, then all non-trivial solutions of equation (1) are of infinite order. It is clear that if the coefficient $A(z)$ is a polynomial and $B(z)$ is a transcendental entire function, then all non-trivial solutions are of infinite order. However, the case where $\rho(A) \geq \rho(B)$ remained

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unexplored until the work of Ozawa [19]. Following Ozawa's paper, other researchers have explored this case partially.

The following result is a collection of such findings.

Theorem A. *All non-trivial solutions of equation* (1) *are of infinite order if the coefficients $A(z)$ and $B(z)$ satisfy any of the following conditions:*

(a) [6] $\rho(A) < \rho(B)$;
(b) [8] $\rho(B) < \rho(A) \leq \frac{1}{2}$;
(c) [6] *$A(z)$ is a transcendental entire function with $\rho(A) = 0$ and $B(z)$ is a polynomial;*
(d) [6] *$A(z)$ is a polynomial and $B(z)$ is a transcendental entire function.*

Example 1. *(i)* $f'' - e^z f' + (e^z - 1)f = 0$ *has the solution* $f(z) = e^z$.
(ii) $f'' + (\sin^2 z - 2\tan z)f' - \tan z f = 0$ *has the solution* $f(z) = \tan z$.

It can be observed from the above examples that differential equations may have finite order solutions when $\rho(A) = \rho(B)$ or $\rho(A) > \rho(B)$ and $\rho(A) > 1/2$. This raises the question: under what conditions does equation (1) possess only non-trivial solutions of infinite order? In the next section, we partially address this question.

The corresponding non-homogeneous second-order linear differential equation is:

$$f'' + A(z)f' + B(z)f = H(z), \tag{2}$$

where $A(z)$, $B(z)$, and $H(z)$ are entire functions. A non-homogeneous linear differential equation can always be reduced to a homogeneous one, so the basic results are similar. If all the coefficients and $H(z)$ are entire functions, then all solutions of equation (2) are also entire functions (see [21]). If all the coefficients are polynomials and $H(z) \not\equiv 0$ has finite order of growth, then all solutions of equation (2) are of finite order (see [4, Lemma 2]). Therefore, if at least one coefficient is a transcendental entire function, then almost all solutions are of infinite order. Let ρ be the minimal order of solutions of equation (1); it is well-known that there may exist at most one solution of order less than ρ for equation (2) (see [13]). Thus, even if all non-trivial solutions of equation (1) are of infinite order, a finite order solution may exist for equation (2). This is illustrated by the following examples.

Example 2. *The equation*

$$f'' + zf' + e^z f = e^{-z}(1 - z) + 1$$

has a finite order solution, $f(z) = e^{-z}$, whereas, using Theorem A(d), we can conclude that the associated homogeneous equation has all non-trivial solutions of infinite order.

Example 3. *Let $b(z)$ be a finite order entire function with a multiply connected Fatou component. Then, the equation*

$$f'' - e^z f' + b(z)f = 0$$

has all non-trivial solutions of infinite order (see [18, Theorem B]*). However, the associated non-homogeneous equation*

$$f'' - e^z f' + b(z)f = e^{-z}(1 + b(z)) + 1$$

has a finite order solution, $f(z) = e^{-z}$.

2. Results

2.1. **Second Order Homogenous Linear Differential Eqaution.** G. Zhang [24] investigated the solutions of the equation (1), where $A(z) = e^{P(z)}$, with $P(z)$ and $B(z)$ are polynomials of degrees m and n respectively.

Theorem B. [24] *Consider the equation*

$$f'' + e^{P(z)}f(z) + Q(z) = 0,$$

where $P(z)$ and $Q(z)$ are polynomials of degrees $n \geq 2$ and $m(\neq 0)$, respectively. All non-trivial solutions are of infinite order if $m + 2 > 2n$ and $n \nmid (m + 2)$.

In our first main result, we replace $e^{P(z)}$ by $h(z)e^{P(z)}$ in Theorem B, where $\rho(h) < n$.

Theorem 1. *Consider the equation*

$$f'' + h(z)e^{P(z)}f' + Q(z)f = 0, \tag{3}$$

where $P(z)$ and $Q(z)$ are polynomials of degrees $n \geq 2$ and $m(\neq 0)$, respectively. All non-trivial solutions are of infinite order if $m + 2 > 2n$ and $n \nmid (m + 2)$.

Example 4. *The equation*

$$f'' + \sin z e^{z^2} f' + z^3 f = 0$$

satisfies the conditions of Theorem 1 and thus all non-trivial solutions are of infinite order.

Definition 1. [16] *Consider a polynomial $P(z) = a_n z^n + \ldots + a_0$, where $a_n = \alpha + \iota\beta \neq 0$. A ray $\arg z = \theta$ is known as a critical ray of $e^{P(z)}$, if $\delta(P, \theta) = 0$, where*

$$\delta(P, \theta) = \Re(a_n e^{\iota n\theta}) = \alpha \cos(n\theta) - \beta \sin(n\theta).$$

The concept of critical rays was first introduced by Long et al.[16]. According to their findings, there are a total of $2n$ critical rays for the function $e^{P(z)}$. These critical rays divide the entire complex plane into $2n$ sectors, each with an equal angular length of $\frac{\pi}{n}$. Furthermore, these sectors exhibit a particular property. In n distinct sectors, $\delta(P, \theta) \geq 0$, and in the remaining n sectors, $\delta(P, \theta) \leq 0$ (see [17]).

Lemma 1. [2] *Let $P(z)$ be a polynomial of degree n and $h(z)$ an entire function of order less than n. Consider $A(z) = h(z)e^{P(z)}$. There exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for every $\epsilon > 0$, the following hold:*

(i) For $\theta \in [0, 2\pi) \setminus E$ with $\delta(P, \theta) > 0$, there exists $R > 1$ such that

$$\exp((1 - \epsilon)\delta(P, \theta)r^n) \leq |A(re^{\iota\theta})|,$$

for $r > R$, and

(ii) For $\theta \in [0, 2\pi) \setminus E$ with $\delta(P, \theta) < 0$, there exists $R > 1$ such that

$$|A(re^{\iota\theta})| \leq \exp((1 - \epsilon)\delta(P, \theta)r^n),$$

for $r > R$.

The following Lemma is given by Langley [15].

Lemma 2. [15] *Assume S is the strip*

$$z = x + \iota y, \quad x \geq x_0, \quad |y| \leq 4.$$

Assume that in S,

$$Q(z) = a_n z^n + O(|z|^{n-2}),$$

where n is a natural number and $a_n > 0$. Then, there exists a path Γ tending to ∞ in S such that all the solutions of

$$w'' + Q(z)w = 0$$

tend to zero on Γ.

The following Lemma gives the logarithmic estimate of a meromorphic function outside an R-set. It is Proposition 5.12 from [14].

Lemma 3. [14] *Assume f is a meromorphic function of finite order. Then, there exists $N = N(f) > 0$ such that*

$$\left|\frac{f'(z)}{f(z)}\right| = O(r^N),$$

holds outside an R-set.

The growth estimate in the following Lemma is deduced in [2] from Herold's Comparison Theorem [9].

Lemma 4. [15] *Suppose $A(z)$ is analytic in a sector containing the ray $\Gamma : re^{\iota\theta}$ and that as $r \to \infty$, $A(re^{\iota\theta}) = O(r^n)$ for some $n \geq 0$. Then, all solutions of*

$$w'' + A(z)w = 0$$

satisfy

$$\log^+ |w(re^{\iota\theta})| = O(r^{\frac{n+2}{2}})$$

on Γ.

Remark 1. *Let $f(re^{\iota\theta_1}) \to a$ and $f(re^{\iota\theta_2}) \to b$ as $r \to \infty$, and assume $f(z)$ is analytic and bounded between the angles θ_1 and θ_2. Then, $a = b$, and $f(z) \to a$ uniformly in the angle. The straight lines $z = re^{\iota\theta_1}$ and $z = re^{\iota\theta_2}$ may be replaced by curves approaching ∞.*

Remark 2. *In Theorem* 1, if $m = 1$ or 2, the possible values of n are 0 and 1 satisfying $m + 2 = 2n$. However, since $n \geq 2$, the result is proven for $m > 2$.

Proof of Theorem 1. Let us suppose that the equation (3) has a solution f of finite order and assume

$$f = w \exp\left\{-\frac{1}{2}\int_0^z h(z)e^{P(z)}dz\right\}. \tag{4}$$

Substituting (4), the equation (3) is transformed into

$$w'' + \left(Q(z) - \frac{1}{4}(he^{P(z)})^2 - \frac{1}{2}h'(z)e^{P(z)} - \frac{1}{2}h(z)P'(z)e^{P(z)}\right)w = 0. \tag{5}$$

By a translation, we can assume that

$$Q(z) = a_m z^m + a_{m-2}z^{m-2} + \cdots, \quad m > 2.$$

We define the critical ray for the polynomial $Q(z)$ as the ray $re^{\iota\theta}$, for which

$$\theta_j = \frac{-\arg a_m + 2j\pi}{m+2}, \tag{6}$$

where $j = 0, 1, 2, \ldots, m+1$. Substituting $z = xe^{\iota\theta_j}$, where $\theta_j, j = 0, 1, \ldots, n+1$ is a critical ray of the polynomial $Q(z)$, transforms equation (5) into

$$\frac{d^2w}{dx^2} + (Q_1(x) + P_1(x))w = 0, \tag{7}$$

where

$$Q_1(x) = \alpha_1 x^m + O(x^{m-2}), \quad \alpha_1 > 0$$

and

$$P_1(x) = -\frac{1}{4}(he^{P(xe^{\iota\theta_j})})^2 - \frac{1}{2}h'(xe^{\iota\theta_j})e^{P(xe^{\iota\theta_j})} - \frac{1}{2}h(xe^{\iota\theta_j})P'(xe^{\iota\theta_j})e^{P(xe^{\iota\theta_j})}.$$

We denote the sectors as follows:

$$S^+ = \{re^{\iota\theta} : 0 < r < +\infty, \frac{2i\pi}{n} < \theta < \frac{(2i+1)\pi}{n}\},$$

$$S^- = \{re^{\iota\theta} : 0 < r < +\infty, \frac{(2i+1)\pi}{n} < \theta < \frac{2(i+1)\pi}{n}\},$$

where $i = 0, 1, \ldots, n-1$, and where $\delta(P,\theta) > 0$ on S^+ and $\delta(P,\theta) < 0$ on S^-. By using Lemma 1, we obtain

$$\begin{aligned} |P_1(x)| \leq & |(h(xe^{\iota\theta_j})e^{P(xe^{\iota\theta_j})})^2| + |h'(xe^{\iota\theta_j})e^{P(xe^{\iota\theta_j})}| + |h(xe^{\iota\theta_j})e^{P(xe^{\iota\theta_j})}P'(xe^{\iota\theta_j})| \\ & \leq \exp\{\delta(P,\theta)x^n\} + \exp\{\frac{1}{2}\delta(P,\theta)x^n\} + \exp\{\frac{1}{2}\delta(P,\theta)x^n\}O(x^{n-1}) \to 0 \qquad (8) \end{aligned}$$

for $xe^{\iota\theta_j} \in S^-$ as $x \to \infty$. Using (7), (8), and Lemma 2 for any critical line $\arg z = \theta_j \in S^-$, there exists a path Γ_{θ_j} tending to ∞ such that $\arg z \to \theta_j$ on Γ_{θ_j} and

$$y(z) \to 0. \qquad (9)$$

Moreover, we have

$$\begin{aligned} |\exp\{-\frac{1}{2}\int_0^z h(z)e^{P(z)}dz\}| &\leq \exp\left\{\frac{1}{2}\left|\int_0^z h(z)e^{P(z)}dz\right|\right\} \\ &\leq \exp\left\{\frac{1}{2}r\exp\{\delta(P,\theta)r^n\}\right\} \to 1 \qquad (10) \end{aligned}$$

for $z \in S^-$ as $r \to \infty$. Using (4), (9), and (10), we have $f(z) \to 0$ along Γ_{θ_j} tending to ∞. Substituting $V = \frac{f'}{f}$, equation (3) transforms into

$$V' + V^2 + h(z)e^{P(z)}V + Q(z) = 0.$$

By Lemma 3, we have

$$|V'| + |V|^2 = O(|z|^N)$$

outside an R-set U, where $N > 0$ is a positive integer. If $z = re^{\iota\phi} \in S^+$ is a ray such that $\arg z = \phi$ meets only finitely many discs of U, we observe that $V = o(|z|^{-2})$ as $|z| \to \infty$. Thus, f tends to a finite nonzero limit.

Applying this reasoning to a set of ϕ outside a set of measure zero, we deduce, by the Phragmén-Lindelöf principle, that without loss of generality, for any sufficiently small positive ϵ,

$$f(re^{\iota\theta}) \to 1, \qquad (11)$$

as $r \to \infty$ such that

$$z = re^{\iota\theta} \in S^+_\epsilon = \{z = re^{\iota\theta} : 0 < r < \infty, \frac{2i\pi}{n} + \epsilon < \theta < \frac{(2i+1)\pi}{n} - \epsilon\}.$$

By Lemma 1, for any $z = re^{\iota\theta} \in S^-$, we have

$$\begin{aligned}|Q(z) - \frac{1}{4}(he^{P(z)})^2 - \frac{1}{2}h'(z)e^{P(z)} - \frac{1}{2}h(z)P'(z)e^{P(z)}| &\leq |Q(z)| + |(he^{P(z)})^2| \\ &\quad + |h'(z)e^{P(z)}| + |h(z)P'(z)e^{P(z)}| \\ &\leq O(r^m) + \exp\{\delta(P,\theta)r^n\} \\ &\quad + \exp\{\frac{1}{2}\delta(P,\theta)r^n\} \\ &\quad + \exp\{\frac{1}{2}\delta(P,\theta)r^n\}O(r^{n-1}) \\ &\leq O(r^m) \end{aligned} \tag{12}$$

for sufficiently large r. Using Lemma 4 along with equations (5) and (12), we have

$$\log^+ |w(re^{\iota\theta})| = O(r^{\frac{m+2}{2}})$$

as $r \to \infty$ for any $z = re^{\iota\theta} \in S^-$. By equations (5) and (10), we have

$$\log^+ |f(re^{\iota\theta})| = O(r^{\frac{m+2}{2}}) \tag{13}$$

as $r \to \infty$ for any $z = re^{\iota\theta} \in S^-$.

Consider the ray $\arg z = \theta_k$ such that $\theta(P, \theta_k) = 0$. On the ray $\arg z = \theta_k$, we have $|e^{P(z)}| = |e^{P_{n-1}(z)}|$. Now, there arise three cases: either $\delta(P_{n-1}, \theta_k) > 0$ or $\delta(P_{n-1}, \theta_k) < 0$ or $\delta(P_{n-1}, \theta_k) = 0$. For the cases $\delta(P_{n-1}, \theta_k) > 0$ or $\delta(P_{n-1}, \theta_k) < 0$, by the same method as above, we get $f(z) \to 1$ or $\log^+ |f(z)| = O(r^{\frac{m+2}{2}})$, respectively, on the ray $\arg z = \theta_k$. If $\delta(P_{n-1}, \theta_k) = 0$, repeating the same steps as above, we finally deduce that either $f(z) \to 1$ or $\log^+ |f(z)| = O(r^{\frac{m+2}{2}})$ on the rays $\arg z = \theta_k$, where $k = 0, 1, \ldots, 2n - 1$.

By equations (4), (11), (13), and the Phragmén-Lindelöf principle, we have

$$\rho(f) \leq \frac{m+2}{2}. \tag{14}$$

We assert that $\frac{(2i+1)\pi}{n}$, where $i = 0, \ldots, n-1$, are critical rays for the function $Q(z)$. If this is not the case, it implies the existence of a critical angle θ_j for $Q(z)$ within the range:

$$\frac{(2i+1)\pi}{n} < \theta_j < \frac{2(i+1)\pi}{n} + \frac{2\pi}{m+2} \quad (i = 0, 1, \ldots, n-1),$$

because $m + 2 > 2n$. Consequently, this implies the existence of an unbounded domain of angular measure of at most $\frac{2\pi}{m+2} + \epsilon$, bounded by a path on which $f(z) \to 0$ and a ray on which $f(z) \to 1$. By Remark 1, this implies $\rho(f) > \frac{m+2}{2}$. However, this contradicts the inequality (14).

Thus, there exists a positive integer k satisfying $\frac{2\pi}{n} = k\frac{2\pi}{m+2}$, that is, $m+2 = kn$, which contradicts $n \nmid m + 2$. Hence, we have completed the proof. □

Theorem 2 is inspired by Theorem C as presented by Kumar and Saini [12]. They considered the case where $A(z)$ has Fabry gaps and $\rho(B) < \rho(A)$. We changed the conditions on $A(z)$ to allow for a multiply connected Fatou component.

Theorem C. [12] *Let $A(z)$ and $B(z)$ be entire functions such that $\rho(B) < \rho(A)$ and $A(z)$ has Fabry gaps. Then, $\rho(f) = \infty$ and $\rho_2(f) = \rho(A)$, where f is a non-trivial solution of equation* (1).

Theorem 2. *Let $A(z)$ be a transcendental entire function with a multiply connected Fatou component, and let $B(z)$ be an entire function satisfying $\rho(B) < \rho(A)$. Then, every non-trivial solution of equation (1) is of infinite order. Moreover,*

$$\rho_2(f) = \rho(A).$$

Example 5. *Consider the equation*

$$f'' + Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right) f' + p(z)f = 0,$$

where $p(z)$ is a non-constant polynomial. Here, we have considered

$$A(z) = Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right),$$

where a_n satisfies $1 < a_1 < a_2 < \dots$ and grows so rapidly that $a_{n+1} < A(a_n) < 2a_{n+1}$. This was constructed by Baker [1], and it has a multiply connected Fatou component. As it satisfies the conditions of Theorem 2, we deduce that all non-trivial solutions are of infinite order.

We show in Example 6 that the conditions of Theorem 2 are necessary. If we skip the conditions, we can obtain a solution of finite order.

Example 6. *Consider the equation*

$$f'' + Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right) f' - Cz \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right) f = 0.$$

Here, $A(z)$ satisfies the conditions described in Example 5 and thus has a multiply connected Fatou component. However, it does not satisfy the conditions of Theorem 2, as $\rho(A) = \rho(B)$, and has a solution $f(z) = z$ of finite order of growth.

Lemma 5 is given by Gundersen [5]. He generalized the estimates of logarithmic derivatives of transcendental meromorphic functions of finite order.

Lemma 5. [5] *Suppose f is a transcendental meromorphic function with finite order, and let $\Gamma = \{(k_i, j_i); i = 1, 2, \dots, m\}$ be a finite set of distinct integers satisfying $k_i > j_i \geq 0$. Consider a given constant $\epsilon > 0$. The following statements hold:*

(a) There exists a set $E_1 \subset [0, 2\pi]$ with linear measure zero such that for $\theta \in [0, 2\pi) \setminus E_1$, there exists $R(\theta) > 0$ satisfying the inequality:

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\epsilon)},$$

for all $(k, j) \in \Gamma$, $|z| > R(\theta)$, and $\arg z = \theta$.

(b) There exists a set $E_2 \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| \notin E_2 \cup [0, 1]$, the inequality in statement (a) holds for all $(k, j) \in \Gamma$ and $|z| \geq R(\theta)$.

(c) There exists a set $E_3 \subset [0, \infty)$ with finite linear measure such that for all $|z| \notin E_3$, the inequality

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)+\epsilon)},$$

holds for all $(k, j) \in \Gamma$.

These statements provide conditions on the behavior of the function f with respect to its derivatives and growth in different regions of the complex plane. The sets E_1, E_2, and E_3 account for exceptional cases or specific regions where the given inequalities may not hold.

Recently, Pant and Saini [20] proved the following result for an entire function.

Lemma 6. [20] *Suppose f is a transcendental entire function. Then, there exists a set $F \subset (0,\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \in F$ and $|f(z)| = M(r, f)$, we have*

$$\left|\frac{f(z)}{f^{(m)}(z)}\right| \leq 2r^m,$$

for all $m \in \mathbb{N}$.

Lemma 7. [25] *Let f be a transcendental meromorphic function with at most finitely many poles, and assume that the Julia set $J(f)$ of f consists only of bounded components. Then, for any complex number, there exist a constant $0 < \beta < 1$ and two sequences of positive numbers $\{r_n\}$ and $\{R_n\}$ such that $r_n \to \infty$ and $R_n/r_n \to \infty$ as $n \to \infty$. Furthermore, for all $r \in H = \bigcup_{n=1}^{\infty}\{r : r_n < r < R_n\}$, the following inequality holds:*

$$M(r, f)^\beta \leq L(r, f) \quad \text{for } r \in H,$$

where $M(r, f)$ denotes the maximum modulus of f on the circle $|z| = r$, and $L(r, f)$ denotes the minimum modulus of f on the circle $|z| = r$.

Lemma 8 is presented in [5], and Lemma 5 follows as a corollary of Lemma 8 as demonstrated in the same paper by [5].

Lemma 8. [5] *Let $f(z)$ be a transcendental meromorphic function, and let $\Gamma = \{(k_i, j_i); i = 1, 2, \ldots, m\}$ represent a finite set of distinct pairs of integers such that $k_i > j_i \geq 0$ for $i = 1, 2, \ldots, m$. Suppose $\alpha > 1$ and $\epsilon > 0$ are given real constants. Then, there exists a set $E \subset (1,\infty)$ with finite logarithmic measure, $m_l(E)$, and there exists a constant $c > 0$ depending only on α and Γ, such that the following inequality holds:*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c\left(\frac{T(\alpha r, f)}{r}\log^\alpha r \log T(\alpha r, f)\right)^{k-j}$$

for all z where $|z| = r$, with $r \notin E \cup [0, 1]$, and for all $(k, j) \in \Gamma$.

Lemma 9. [12] *Assume that $A(z)$ and $B(z)$ are entire functions of finite order. Then, the order of any solution f of equation (1) satisfies the inequality*

$$\rho_2(f) \leq \max\{\rho(A), \rho(B)\}.$$

In this paper, we utilize Lemma 8 and Lemma 9 to prove the second part of Theorem 2, specifically that $\rho_2(f) = \rho(A)$.

Proof of Theorem 2. Suppose f is a finite order non-trivial solution of equation (1).

Applying Lemma 5, there is a set $E \subset (1,\infty)$ with finite logarithmic measure such that

$$\left|\frac{f''(z)}{f'(z)}\right| \leq |z|^{2\rho(f)}, \tag{15}$$

holds for all z satisfying $|z| \notin E \cup [0, 1]$.

Suppose that $z_r = re^{\iota\theta_r}$ be the points such that $|f(z_r)| = M(r, f)$. Then, applying Lemma 6, there exists a set $G \subset (0, \infty)$ with $m_l(G) < \infty$ such that

$$\frac{f(re^{\iota\theta_r})}{f^{(m)}(re^{\iota\theta_r})} \leq 2r^m, \tag{16}$$

holds for all sufficiently large $r \notin G$ and for all $m \in \mathbb{N}$.

Applying Lemma 7, we have

$$M(r, A)^\gamma \leq |A(re^{\iota\theta})|, \tag{17}$$

for $0 < \gamma < 1$ and $r \in F_1 = \bigcup_{n=1}^{\infty}\{r : r_n < r < R_n\}$.

Let $\rho(B) < \beta < \rho(A)$. Then the definition of order of growth of $B(z)$ implies that

$$|B(re^{\iota\theta})| \leq \exp(r^\beta), \tag{18}$$

for all sufficiently large r.

From equations (1), (15), (16), (17), and (18), there exists a sequence $z = re^{\iota\theta}$ such that for all $r \in F_1 \setminus (G \cup E \cup [0, 1])$, we have

$$\begin{aligned}
|A(re^{\iota\theta})| &\leq \left|\frac{f''(re^{\iota\theta})}{f'(re^{\iota\theta})}\right| + |B(re^{\iota\theta})| \left|\frac{f(re^{\iota\theta})}{f'(re^{\iota\theta})}\right| \\
M(r, A)^\gamma &\leq r^{2\rho(f)} + 2r\exp(r^\beta) \\
&\leq 2r\exp(r^\beta)(1 + o(1)).
\end{aligned}$$

This gives $\rho(A) \leq \beta$, which is a contradiction. Hence, every non-trivial solution of equation (1) is of infinite order.

Now, let f be a non-trivial solution of (1), which implies that $\rho(f) = \infty$. By applying Lemma 8, for any $\epsilon > 0$, there exists a set $E \subset (1, \infty)$ with $m_l(E) < \infty$, such that for all z with $|z| = r$ and $r \notin E \cup [0, 1]$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c[T(2r, f)]^{2(k-j)}, \tag{19}$$

where $k > j$, with some constant $c > 0$. Using equations (1), (16), (17), (18) and (19), we arrive at the inequality

$$\rho(A) \leq \rho_2(f). \tag{20}$$

Finally, by Lemma 9 and (20), it follows that $\rho_2(f) = \rho(A)$.

□

In 2017, Gundersen [7] asked the question: "Does every non-trivial solution f of equation (1) have infinite order, when $A(z)$ satisfies $\lambda(A) < \rho(A)$ and $B(z)$ is a non-constant polynomial?" Long et al. [16] partially answered this question.

Theorem D. [16] *Suppose $A(z) = h(z)e^{P(z)}$ and $B(z) = b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0$. Let $\lambda(A) < \rho(A)$. All the non-trivial solutions of equation* (1) *are of infinite order if any of the following conditions is satisfied:*

(1) $m + 2 < 2n$,
(2) $m + 2 > 2n$ and $m + 2$ is not a multiple of $2kn$ for any integer k,
(3) $m + 2 = 2n$ and $\frac{a_n^2}{b_m}$ is not a real negative number.

Kumar et al.[11], motivated by their result, considered $\rho(A) > n$ and $B(z)$ to be a polynomial in Theorem E.

Theorem E. [11] *Let us consider a transcendental entire function* $A(z) = h(z)e^{P(z)}$, *where* $P(z)$ *is a non-constant polynomial of degree* n, *and* $\rho(h) > n$. *Here, we assume that* $h(z)$ *is bounded away from zero and exponentially blows up in* E^+ *and* E^-, *respectively. Additionally, let* $B(z)$ *be a polynomial. Then, any non-trivial solution of the given equation* (1) *is of infinite order.*

Motivated by Theorem E, we aim to replace the condition on $B(z)$ with certain conditions provided in Theorem 3.

Theorem 3. *Assume that* $A(z)$ *satisfies the conditions outlined in Theorem E, and consider a transcendental entire function* $B(z)$ *that satisfies the following conditions:*

(1) $\rho(B) < \rho(A)$, *or*
(2) $\mu(B) < \rho(A)$.

Then, all non-trivial solutions of the equation (1) *have infinite order.*

We illustrate with Example 7 the fact that the conditions of Theorem 3 are necessary. We can obtain a solution of finite order if we skip the conditions.

Example 7. *Consider the differential equation*

$$f'' + (e^{z^2} + 1)e^z f' + (e^{z^2+z} + e^z - 1)f = 0.$$

Here, $h(z) = e^{z^2} + 1$ *and* $P(z) = z$ *satisfy the condition* $\rho(h) > \deg P(z) = 1$. *However, it does not satisfy the conditions of Theorem 3 because* $h(z) = e^{z^2} + 1$ *does not exponentially blow up in* E^- *of* e^z. *Furthermore, for both conditions (i) or (ii), we have* $\mu(B) = \rho(B) = \rho(A)$. *As a result, there exists a solution* $f(z) = e^{-z}$ *of finite order.*

Lemma 10 provides a lower bound for the modulus of an entire function in the neighborhood of θ, where $\theta \in [0, 2\pi)$.

Lemma 10. [22] *Let* $f(z)$ *be an entire function of finite order* ρ *and assume that* $M(r, f) = |f(re^{\iota\theta_r})|$ *for every* $r > 0$, *then for* $\zeta > 0$ *and* $0 < C(\rho, \zeta) < 1$, *there exists* $0 < l_0 < \frac{1}{2}$ *and* $S_1 \subset (1, \infty)$ *such that* $\underline{\log dens}(S_1) \geq 1 - \zeta$ *satisfying*

$$e^{-5\pi} M(r, f)^{1-C} \leq |f(re^{\iota\theta})|,$$

for all very large $r \in S_1$ *and for all* θ *satisfying* $|\theta - \theta_r| \leq l_0$.

The following lemma is a proposition in the research paper of Kumar et al.[10].

Lemma 11. [10] *Assume* $f(z)$ *and* $g(z)$ *are entire functions and* $\rho(g) < \rho(f)$. *Given* $0 < \epsilon \leq \min\{\frac{3\rho(f)}{4}, \frac{\rho(f)-\rho(g)}{2}\}$, *there exists a set* S_2 *satisfying* $S_2 \subset (1, \infty)$ *with* $\overline{\log dens}(S_2) = 1$, *then*

$$|g(z)| = o(M(|z|, f)),$$

for sufficiently large $|z| \in S_2$.

Remark 3. *Lemma 11 would also be true if we replace* $\rho(g)$ *with* $\mu(g)$.

In Lemma 1, consider $A(z) = v(z)e^{P(z)}$, where $v(z)$ is an entire function and $P(z)$ is a polynomial of degree n satisfying $\rho(v) < \deg P$. In contrast, in Lemma 12, the authors considered $\rho(v) > \deg P$ and obtained that

$$|A(re^{\iota\theta})| \geq \exp((1 - \epsilon)\delta(P, \theta)r^n),$$

for $\theta \in E^+ \setminus E$ and also for $\theta \in E^- \setminus E$, where E is a set of linear measure 0.

Lemma 12. [11] *Consider an entire function* $A(z) = h(z)e^{P(z)}$*, where* $P(z)$ *is a polynomial of degree* n *and* $h(z)$ *satisfies the conditions of Theorem E. Then, for* $\epsilon > 0$*, there exists a set* $E \subset [0, 2\pi]$ *of linear measure* 0 *such that the following conditions are satisfied:*

(i) For $\theta \in E^+ \setminus E$*, there exists* $R(\theta) > 1$ *satisfying*

$$|A(re^{\iota\theta})| \geq \exp((1-\epsilon)\delta(P,\theta)r^n), \tag{21}$$

for $r > R(\theta)$*.*

(ii) For $\theta \in E^- \setminus E$*, there exists* $R(\theta) > 1$ *satisfying*

$$|A(re^{\iota\theta})| \geq \exp((1-\epsilon)\delta(P,\theta)r^n), \tag{22}$$

for $r > R(\theta)$*.*

Lemma 13 is proved by Gundersen[6], which provides a logarithmic estimate of the analytic function $f(z)$.

Lemma 13. [6] *Consider an analytic function* f *defined on a ray* $\gamma = re^{\iota\theta}$*. Suppose there exists a constant* $\alpha > 1$ *such that the following condition holds as* z *tends to infinity along the ray* $\arg(z) = \theta$*:*

$$\left|\frac{f'(z)}{f(z)}\right| = O(|z|^{-\alpha}).$$

Then, there exists a non-zero constant c *such that* $f(z) \to c$ *as* z *tends to infinity along the ray* $\arg(z) = \theta$*.*

The proof of Theorem 3 draws inspiration from the proof of Theorem E. However, we have made slight modifications to accommodate the specific conditions stated in the theorem.

Proof of Theorem 3. If $\rho(A) = \infty$, it is evident that $\rho(f) = \infty$ for all non-trivial solutions f of equation (1). Therefore, let us assume that $\rho(A) < \infty$ and there exists a non-trivial solution f of equation (1) with $\rho(f) < \infty$.

From Lemma 5, we obtain a set $E_1 \subset [0, 2\pi]$ of linear measure zero and $m > 0$ such that

$$\left|\frac{f''(re^{\iota\theta})}{f(re^{\iota\theta})}\right| \leq r^m, \tag{23}$$

for $\theta \in [0, 2\pi] \setminus E_1$ and $r > R(\theta)$.

Since $A(z)$ is an entire function of finite order, let $M(r, A) = |A(re^{\iota\theta_r})|$ for every r. Then, from Lemma 10, for $0 < \zeta < 1$ and $0 < C < 1$, there exists $0 < l_0 < \frac{1}{2}$ and a set $S_1 \subset (0, \infty)$ with a lower logarithmic density $\underline{\log dens}(S_1) \geq 1 - \zeta$ such that

$$e^{-5\pi} M(r, A)^{1-C} \leq |A(re^{\iota\theta})|, \tag{24}$$

for all sufficiently large $r \in S_1$ and for all θ satisfying $|\theta - \theta_r| \leq l_0$.

Now, let us consider the following cases:

(i) $\rho(B) < \rho(A)$:

From Lemma 11, for $0 < \epsilon \leq \min\left\{\frac{3\rho(A)}{4}, \frac{\rho(A)-\rho(B)}{2}\right\}$, there exists $S_2 \subset (1, \infty)$ with upper logarithmic density $\overline{\log dens}(S_2) = 1$ such that

$$\frac{|B(z)|}{M(|z|, A)} \to 0, \tag{25}$$

for sufficiently large $|z| \in S_2$.

Using the properties of logarithmic density and the fact that $\overline{\log dens}(S_1 \cup S_2) \leq 1$, we can conclude that

$$\begin{aligned}\overline{\log dens}(S_1 \cap S_2) &\geq \underline{\log dens}(S_1) + \underline{\log dens}(S_2) - \overline{\log dens}(S_1 \cup S_2)\\ &\geq 1 - \zeta + 1 - 1 = 1 - \zeta.\end{aligned}$$

Therefore, we can choose $z_r = re^{\iota\theta_r}$ with $r \to \infty$ such that $r \in S_1 \cap S_2$ and $|A(re^{\iota\theta_r})| = M(r, A)$. We consider the sequence $\{\theta_r\}$, where $r \in S_1 \cap S_2$, such that $\theta_r \to \theta_0$ and $r \in S_1 \cap S_2$.

We now consider the following three subcases:

(a) $\delta(P, \theta_0) > 0$:

From Lemma 12(i), we have

$$|A(re^{\iota\theta_0})| \geq \exp\left(\frac{1}{2}\delta(P, \theta_0)r\right), \tag{26}$$

for sufficiently large r, where $r \in S_1 \cap S_2$ and $\theta_0 \in E^+/E_2$, with E_2 being a set of critical rays of $e^{P(z)}$ of linear measure zero.

From equation (1), we have

$$\left|\frac{f'(re^{\iota\theta_0})}{f(re^{\iota\theta_0})}\right| \leq \left|\frac{f''(re^{\iota\theta_0})}{f(re^{\iota\theta_0})}\right| \frac{1}{|A(re^{\iota\theta_0})|} + \frac{|B(re^{\iota\theta_0})|}{M(r, A)}, \tag{27}$$

for $r \in S_1 \cap S_2$ and $\theta_0 \in E^+/(E_1 \cup E_2)$.

Using equations (23), (24), (25), (26), and (27), we obtain

$$\left|\frac{f'(re^{\iota\theta_0})}{f(re^{\iota\theta_0})}\right| \to 0$$

as $r \to \infty$ and $r \in S_1 \cap S_2$, $\theta_0 \in E^+/(E_1 \cup E_2)$. This implies that

$$\left|\frac{f'(re^{\iota\theta_0})}{f(re^{\iota\theta_0})}\right| = O\left(\frac{1}{r^2}\right), \tag{28}$$

as $r \to \infty$ and $r \in S_1 \cap S_2$. From Lemma 13, we have

$$f(re^{\iota\theta_0}) \to a \tag{29}$$

as $r \to \infty$ and $r \in (S_1 \cap S_2)$, for $\theta_0 \in E^+ \setminus (E_1 \cup E_2)$, where a is a non-zero finite constant.

Since $f(re^{\iota\theta_r}) \to f(re^{\iota\theta_0})$ and using (29), we obtain

$$f(re^{\iota\theta_r}) \to a$$

as $r \to \infty$ and $r \in (S_1 \cap S_2)$.

Thus, the entire function f is bounded over its entire domain. However, since f is a non-constant entire function, $f(re^{\iota\theta})$ is unbounded for all $\theta \in [0, 2\pi]$. This implies that for $\theta_r \in [0, 2\pi]$, the function $f(re^{\iota\theta_r})$ is also unbounded, leading to a contradiction.

(b) $\delta(P, \theta_0) < 0$:

From Lemma 12(ii), we have

$$|A(re^{\iota\theta_0})| \geq \exp\left(\frac{1}{2}\delta(P,\theta_0)r^n\right), \tag{30}$$

for $\theta_0 \in E^-/E_1$ for large r.
Using equations (23), (24), and (30), we have

$$\left|\frac{f'(re^{\iota\theta_0})}{f(re^{\iota\theta_0})}\right| \to 0, \tag{31}$$

as $r \to \infty$ and $\theta_0 \in E^-/(E_1 \cup E_2)$. From Lemma 13, we have

$$f(re^{\iota\theta_0}) \to b, \tag{32}$$

as $r \to \infty$ and $r \in (S_1 \cap S_2)$, for $\theta_0 \in E^- \setminus (E_1 \cup E_2)$, where b is a non-zero finite constant.
Since $f(re^{\iota\theta_r}) \to f(re^{\iota\theta_0})$ and using (32), we obtain

$$f(re^{\iota\theta_r}) \to b,$$

for $r \to \infty$ and $r \in (S_1 \cap S_2)$.
Thus, the entire function f is bounded over its entire domain. However, since f is a non-constant entire function, it must be unbounded for all $\theta \in [0, 2\pi]$. This means that for $\theta_r \in [0, 2\pi]$, the expression $f(re^{\iota\theta_r})$ must also be unbounded. This conclusion leads to a contradiction.

(c) $\delta(P,\theta_0) = 0$:
Let $\theta_0^* \in [0, 2\pi]$ be a neighborhood of θ_0 such that $\delta(P,\theta_0^*) > 0$. Taking the limit as $r \to \infty$, we have $|\theta_0 - \theta_0^*| \leq l_0$. We can choose C and ζ such that $l_0 \to 0$.

$$\left|\frac{f'(re^{\iota\theta_0})}{f(re^{\iota\theta_0})}\right| \sim \left|\frac{f'(re^{\iota\theta_0^*})}{f(re^{\iota\theta_0^*})}\right| \leq \left|\frac{f''(re^{\iota\theta_0^*})}{f(re^{\iota\theta_0^*})}\right| \frac{1}{|A(re^{\iota\theta_0^*})|} + \frac{\left|B(re^{\iota\theta_0^*})\right|}{M(r,A)}, \tag{33}$$

The remaining proof is similar to part (i).

□

2.2. **Second Order Non-Homogenous Linear Differential Equation.** Kumar and Saini[12] gave several results for equation (2). In one of their results, they considered $A(z)$ to have Fabry gaps, satisfying $\max(\rho(H), \rho(B)) < \rho(A)$, and proved the following result. We change the condition on $A(z)$ and consider $A(z)$ to be a transcendental entire function having a multiply-connected Fatou component, and we prove Theorem 4.

Theorem F. [12] *If the coefficients and $H(z)$ of equation (2) are entire functions satisfying $\max(\rho(H), \rho(B)) < \rho(A)$, and if $A(z)$ has Fabry gaps, then every non-trivial solution of equation (2) has infinite order.*

Theorem 4. *If the coefficients and $H(z)$ of equation (2) are entire functions satisfying $\max(\rho(H), \rho(B)) < \rho(A)$, and if $A(z)$ has a multiply-connected Fatou component, then every non-trivial solution of equation (2) has infinite order.*

Example 8. *Consider*

$$f''(z) + Cz^2 \prod_{n=1}^{\infty}\left(1 + \frac{z}{a_n}\right) f' + p_1(z) f = p_2(z),$$

where $p_1(z)$ and $p_2(z)$ are non-constant polynomials. Here, $A(z)$ has a multiply-connected Fatou component as described in Example 5. This equation satisfies the conditions of Theorem 4 and thus has all non-trivial solutions of infinite order.

As demonstrated in Example 9, the conditions specified in Theorem 4 are essential. Without these conditions, it is possible to obtain a solution of finite order.

Example 9. *Consider*

$$f'' + Cz^2 \prod_{n=1}^{\infty}\left(1+\frac{z}{a_n}\right) f' - Cz \prod_{n=1}^{\infty}\left(1+\frac{z}{a_n}\right) f = 2.$$

Here, $A(z)$ has a multiply-connected Fatou component as described in Example 5, but it skips the condition of Theorem 4 as $\rho(A) = \rho(B)$ and has a solution $f(z) = z^2$ of finite order of growth.

Proof of Theorem 4. Assume f is a solution of equation (2) with finite order. By applying Lemma 5, there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that the inequality

$$\left|\frac{f''(z)}{f'(z)}\right| \leq |z|^{2\rho(f)}, \tag{34}$$

holds for all z satisfying $|z| \notin E \cup [0, 1]$.

Given that $\max(\rho(H), \rho(B)) < \rho(A)$, choose β such that $\max(\rho(H), \rho(B)) < \beta < \rho(A)$. Using the definition of the order of growth for $B(z)$ and $H(z)$, we have

$$|B(re^{\iota\theta})| \leq \exp(r^{\beta}) \qquad \text{and} \qquad |H(re^{\iota\theta})| \leq \exp(r^{\beta}), \tag{35}$$

for sufficiently large r.

Let $z_r = re^{\iota\theta_r}$ be the points at which $|f(z_r)| = M(r, f)$. Applying Lemma 6, there exists a set $F \subset (0, \infty)$ with $m_l(F) < \infty$ such that the inequality

$$\frac{f(re^{\iota\theta_r})}{f^m(re^{\iota\theta_r})} \leq 2r^m, \tag{36}$$

holds for all sufficiently large $r \notin F$ and all $m \in \mathbb{N}$.

Using Lemma 7, we obtain the inequality

$$M(r, A)^{\gamma} \leq |A(re^{\iota\theta})|, \tag{37}$$

for $0 < \gamma < 1$ and $r \in F_1 = \cup_{n=1}^{\infty}\{r : r_n < r < R_n\}$.

Combining equations (2), (34), (35), (36), and (37), we find a sequence $z = re^{\iota\theta}$ such that for all $r \in F_1 \setminus (E \cup F \cup [0, 1])$, we have

$$\begin{aligned}
|A(re^{\iota\theta})| &\leq \left|\frac{f''(re^{\iota\theta})}{f'(re^{\iota\theta})}\right| + |B(re^{\iota\theta})| \left|\frac{f(re^{\iota\theta})}{f'(re^{\iota\theta})}\right| + \left|\frac{H(re^{\iota\theta})}{f(re^{\iota\theta})}\right| \left|\frac{f(re^{\iota\theta})}{f'(re^{\iota\theta})}\right| \\
M(r, A)^{\gamma} &\leq r^{2\rho(f)} + 2r\exp(r^{\beta}) + 2r\left|\frac{H(re^{\iota\theta})}{M(r, f)}\right| \\
&\leq r^{2\rho(f)} + 4r\exp(r^{\beta}) \\
&\leq 4r\exp(r^{\beta})(1 + o(1)).
\end{aligned}$$

This leads to the contradiction $\rho(A) \leq \beta$. Therefore, every non-trivial solution of equation (2) is of infinite order. □

3. Conclusion

In this paper, we explored the growth of non-trivial solutions for various homogeneous and non-homogeneous linear differential equations that involve entire functions. We established important results regarding the order of these solutions in relation to the properties of the coefficients and the corresponding entire functions. However, the results achieved are only partial and indicate several potential directions for future research.

To make our results more useful, we suggest looking into ways to relax the conditions we currently have. This would allow us to study a broader range of differential equations and their solutions. We should also look into how our findings can apply to non-homogeneous equations and equations of higher order. Exploring these areas could help us learn more about how different solutions behave and improve our understanding of the relationships between the coefficients and the growth rates of these solutions.

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Garima Pant for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.14, N.3.

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