TWMS J. App. and Eng. Math. V.15, N.7, 2025, pp. 1796-1809

EQUITABLE COLORINGS OF CARTESIAN PRODUCTS OF SQUARE OF PATHS AND CYCLES WITH SQUARE OF PATHS AND CYCLES

ELUMALAI P.[1] AND PARTHIBAN A.[1,*], §

Abstract. Let $[p] = \{1, 2, 3, \ldots, p\}$ and G be an undirected simple graph. Graph coloring is a special case of labeling, and G is said to admit a proper coloring if no two neighbouring vertices of it are given an identical color. The vertices of an identical color constitute a color class. G is p - colorable if it admits proper p - coloring. The chromatic number, $\chi(G) = \min\ \{p : G$ is proper p - colorable$\}$ and G is equitably p - colorable if it admits proper p - coloring and the absolute difference in size between any distinct pairwise color class is at most 1. The equitable chromatic number, $\chi_=(G) = \min\ \{p : G$ is equitably p - colorable$\}$. The equitable chromatic threshold, $\chi^*_=(G) = \min\ \{p' : G$ is equitably p - colorable $\forall\ p \geq p'\}$. In this paper, we obtain exact values or bounds of $\chi^*_=(G_1 \square G_2)$ and $\chi_=(G_1 \square G_2)$, where $G_1 = P_m^2$ or C_m^2 and $G_2 = P_n^2$ or C_n^2.

Keywords: Square of a path and cycle graph, Cartesian product, Equitable coloring, Equitable chromatic number, Equitable chromatic threshold.

AMS Subject Classification: 05C15, 05C40, 05C76, 05C38

1. Introduction

Consider $G = (V, E)$, or simply G to be a finite, connected, undirected, and simple graph with $V(G)$ and $E(G)$, respectively, denote the vertex set and edge set. For standard graph theory notations and terminologies here, we refer to [1, 2, 7, 14, 19, 23, 18]. A (proper) p - coloring of G is a mapping $h : V(G) \to [p]$ such that $h(a) \neq h(b)$ for $ab \in E(G)$, i.e., neighbouring vertices receive the different colors and the set of all vertices of an identical color constitute a color class or independent set. The chromatic number, $\chi(G) = \min$ $\{p : G$ admits proper p - coloring$\}$. A graph G is said to have an equitable p - coloring if it admits a proper vertex coloring with p colors such that the size difference between any two distinct color classes is at most one. The equitable chromatic number of G, denoted by $\chi_=(G)$, is the minimum number of colors required for an equitable coloring of G. The equitable chromatic threshold, denoted by $\chi^*_=(G)$, is the smallest integer p' such that G is an equitable p - coloring for all $p \geq p'$. The maximum degree of a vertex in G, denoted by

[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632 014, Tamil Nadu, India.
e-mail: elumalai.p@vit.ac.in; https://orcid.org/0009-0002-9063-5730.
e-mail: parthiban.a@vit.ac.in; https://orcid.org/0000-0002-8375-4072.
[*] Corresponding author.
§ Manuscript received: June 13, 2024; accepted: November 14, 2024.

$\triangle(G)$, and $\lceil a \rceil$ and $\lfloor a \rfloor$ stand for, respectively, the least number that is not lower than a and the greatest number that is not larger than a. Meyer [15] proposed first the notation of equitable coloring. Tucker [17] was the source of his inspiration.The equitable graph coloring progress is discussed in [11]. According to Erdős [9], every G, with maximal degree $\triangle(G) < p$ permits an equitable $(p+1)$ - coloring. It is crucial to notice that if $\chi_{=}(G) = p$, then $\chi_{=}(G) = p+1$ is not possible. In contrast, $K_{3,3}$, in which two colors are permitted to be colored equitably but not three. One can observe that (i) $\chi_{=}(P_n) = 2$, (ii) $\chi_{=}(K_{n,n}) = 2$, (iii) $\chi_{=}(K_n) = n$, and hence (iv)

$$\chi_{=}(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

It is straightforward to check that $\chi(G) \leq \chi_{=}(G) \leq \chi^*_{=}(G)$ [3, 4, 20]. In general, strict inequality is valid. For instance,

$$\chi(K_{1,4}) = 2 < \chi_{=}(K_{1,4}) = \chi^*_{=}(K_{1,4}) = 3,$$
$$\chi(K_{3,3}) = \chi_{=}(K_{3,3}) = 2 < \chi^*_{=}(K_{3,3}) = 4,$$
$$\chi(K_{5,8}) = 2 < \chi_{=}(K_{5,8}) = 3 < \chi^*_{=}(K_{5,8}) = 5.$$

In graph G, the shortest distance between vertices a and b, denoted by $d(a,b)$, is equal to the number of edges in the path with the fewest edges that connects them. A square of G is called G^2, with $V(G) = V(G^2)$, and two vertices a and b are neighboring vertices in G^2 if and only if $d(a,b) \leq 2$ in G [8]. Consider the two graphs, G_1 and G_2, the Cartesian (square) product of G_1 and G_2 is represented as $G_1 \Box G_2$, with $V(G_1 \Box G_2)$ and $E(G_1 \Box G_2)$ in such a way

$$\{(a_1, b_1) : a_1 \in V(G), b_1 \in V(G_2))\}, and$$

$$\{(a_1, b_1)(a_2, b_2) : a_1 = a_2 \ and \ b_1 b_2 \in E(G_2) \ or \ b_1 = b_2 \ and \ a_1 a_2 \in E(G_1)\}.$$

Conjecture 1.1. [15]. ***Equitable Coloring Conjecture (ECC).*** *Let G be any graph* $G \neq \{K_n, C_{2n+1}\}$. $\chi_{=}(G) \leq \Delta(G)$.

It has been demonstrated that this conjecture holds for only graphs with six or fewer vertices. To demonstrate that ECC is valid for all bipartite graphs, Lih and Wu [12] provided substantial evidence.
Additionaly, we have a powerful conjecture.

Conjecture 1.2. [5]. ***Equitable Δ - coloring conjecture(EΔCC).*** *If G is a connected graph of degree Δ, other than a complete graph, an odd cycle or a complete bipartite graph $K_{2n+1,2n+1}$ for any $n \geq 1$, then G is equitably Δ - colorable.*

EΔCC is valid for certain kinds of graphs, such as bipartite graph [12], outerplanar graphs with $\Delta \geq 3$ [22], and planar graphs with $\Delta \geq 13$ [21].

Conjecture 1.3. [13]. *Let G_1 and G_2, be any two graphs. Then,* $\chi_{=}(G_1 \Box G_2) \leq \chi(G_1)\chi(G_2)$.

In 1957, Sabidussi posed the following theorem.

Theorem 1.1. [16]. *Let G_1 and G_2 be any graphs. Then,* $\chi(G_1 \Box G_2) = max\{\chi(G_1), \chi(G_2)\}$.

Theorem 1.2. [10]. *Let G_1 and G_2 be equitably p - colorable, then $G_1 \Box G_2$ is equitably p - colorable.*

As a result, we obtain an inequality that describes the minimum number of colors required for an equitable chromatic threshold.

Corollary 1.1. $\chi^*_=(G_1 \square G_2) \leq max\ \{\chi^*_=(G_1), \chi^*_=(G_2)\}$.

2. Cartesian products of square of paths and cycles

In this section, we derive the equitable chromatic number and threshold for square of paths with square of paths and cycles.

Theorem 2.1. *If $m \geq 3$ and $n \geq 3$ are two integers, then* $\chi^*_=(P^2_m \square P^2_n) = \chi_=(P^2_m \square P^2_n) = 3$.

Proof. Suppose that $V(P^2_m) = \{a_s : s \in [m]\}$ and $V(P^2_n) = \{b_t : t \in [n]\}$. Array the vertices of Cartesian product of P^2_m and P^2_n in such a way that $\{(a_s, b_t) : s \in [m], t \in [n]\}$. Note that a color set is any collection of vertices of size no more than $\lceil \frac{mn}{3} \rceil$that have the arrangement $s + t \equiv 2\ (mod\ 3)$, $s + t \equiv 0\ (mod\ 3)$, and $s + t \equiv 1\ (mod\ 3)$.
Consider the following color classes for $P^2_m \square P^2_n$.

$$\sigma_1 = \{(a_s, b_t) : s + t \equiv 2\ (mod\ 3), s \in [m], t \in [n]\}$$
$$\sigma_2 = \{(a_s, b_t) : s + t \equiv 0\ (mod\ 3), s \in [m], t \in [n]\}$$
$$\sigma_3 = \{(a_s, b_t) : s + t \equiv 1\ (mod\ 3), s \in [m], t \in [n]\}.$$

Clearly, $\{\sigma_s : s \in [3]\}$ is a color class of size $\lfloor \frac{mn}{3} \rfloor$ or $\lceil \frac{mn}{3} \rceil$. Therefore, the product $P^2_m \square P^2_n$ is equitably 3 - colorable (see Figure 1).
For $p \geq [3]$, let

$$\sigma_k = \left\lfloor \frac{mn + k - 1}{p} \right\rfloor$$

for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{mn + p - 1}{p} \right\rfloor \leq \left\lceil \frac{mn}{p} \right\rceil,$$

one can divide $V(P^2_m \square P^2_n)$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ in the arrangements. Therefore, $V(P^2_m \square P^2_n)$ is equitably p - colorable.
Contrary, it is straightforward to verify that

$$\chi_=(P^2_m \square P^2_n) \geq \chi(P^2_m \square P^2_n) \geq \chi(P^2_m) = 3.$$

This completes the proof. □

Theorem 2.2. *If $m \geq 3$ and $n \geq 3$ are two integers, then*

$$\chi^*_=(P^2_m \square C^2_n) = \chi_=(P^2_m \square C^2_n) = \begin{cases} 3, & for\ n \equiv 0\ (mod\ 3), \\ 4, & for\ n \equiv 1\ (mod\ 3), \\ 5, & for\ n \equiv 2\ (mod\ 3). \end{cases}$$

Proof. Suppose that $V(P^2_m) = \{a_s : s \in [m]\}$ and $V(C^2_n) = \{b_t : t \in [n]\}$. Array the vertices of Cartesian product of P^2_m and C^2_n such as $\{(a_s, b_t) : s \in [m], t \in [n]\}$.
Claim 1: For $n \equiv 0\ (mod\ 3)$
Note that a color class is a collection of vertices of size no more than $\lceil \frac{mn}{3} \rceil$ that have the

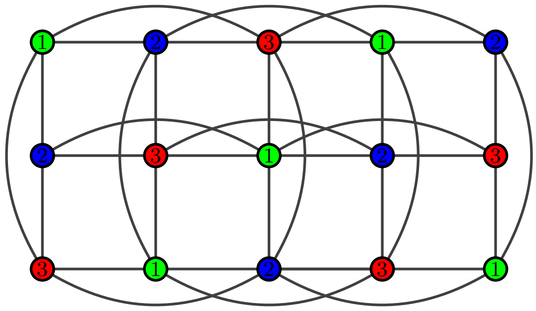

FIGURE 1. Equitable 3 - coloring of $P_3^2 \Box P_5^2$.

arrangement $s+t \equiv 2\ (mod\ 3)$, $s+t \equiv 0\ (mod\ 3)$, and $s+t \equiv 1\ (mod\ 3)$.
Consider the color classes for $P_m^2 \Box C_n^2$.

$$\sigma_1 = \{(a_s, b_t) : s+t \equiv 2\ (mod\ 3), s \in [m], t \in [n]\}$$
$$\sigma_2 = \{(a_s, b_t) : s+t \equiv 0\ (mod\ 3), s \in [m], t \in [n]\}$$
$$\sigma_3 = \{(a_s, b_t) : s+t \equiv 1\ (mod\ 3), s \in [m], t \in [n]\}.$$

Clearly, $\{\sigma_s : s \in [3]\}$ is a color class of size $\lfloor \frac{mn}{3} \rfloor$ or $\lceil \frac{mn}{3} \rceil$. Therefore, $P_m^2 \Box C_n^2$ is equitably 3 - colorable (See Figure 2).

For $p \geq [3]$, let

$$\sigma_k = \left\lfloor \frac{mn+k-1}{p} \right\rfloor$$

and for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{mn+p-1}{p} \right\rfloor \leq \left\lceil \frac{mn}{p} \right\rceil,$$

one can split the vertex set of $P_m^2 \Box C_n^2$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangement. As a result, the product of P_m^2 and C_n^2 is equitably p - colorable.

Contrary, it is uncomplicated to confirm that

$$\chi_=(P_m^2 \Box C_n^2) \geq \chi(P_m^2 \Box C_n^2) \geq \chi(C_n^2) = 3.$$

Claim 2: For $n \equiv 1\ (mod\ 3)$

Note that a color class is any collection of vertices of size no more than $\lceil \frac{mn}{4} \rceil$ that have the arrangement $s+t \equiv 2\ (mod\ 4)$, $s+t \equiv 3\ (mod\ 4)$, $s+t \equiv 0\ (mod\ 4)$ and $s+t \equiv 1\ (mod\ 4)$.
Consider the following color classes for $P_m^2 \Box C_n^2$.

$$\sigma_1 = \{(a_s, b_t) : s+t \equiv 2\ (mod\ 4), s \in [m], t \in [n]\}$$
$$\sigma_2 = \{(a_s, b_t) : s+t \equiv 3\ (mod\ 4), s \in [m], t \in [n]\}$$
$$\sigma_3 = \{(a_s, b_t) : s+t \equiv 0\ (mod\ 4), s \in [m], t \in [n]\}$$
$$\sigma_4 = \{(a_s, b_t) : s+t \equiv 1\ (mod\ 4), s \in [m], t \in [n]\}.$$

Clearly, $\{\sigma_s : s \in [4]\}$ is a color class of size $\lfloor \frac{mn}{4} \rfloor$ or $\lceil \frac{mn}{4} \rceil$. Therefore, the product $P_m^2 \Box C_n^2$ is equitably 4 - colorable (See Figure 3).

For $p \geq [4]$, let

$$\sigma_k = \left\lfloor \frac{mn+k-1}{p} \right\rfloor$$

for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{mn+p-1}{p} \right\rfloor \leq \left\lceil \frac{mn}{p} \right\rceil,$$

one can divide the vertex set of $P_m^2 \Box C_n^2$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangements. Thus, the product graph of P_m^2 and C_n^2 is equitably p - colorable.
Contrary, it is straightforward to confirm that

$$\chi_{=}(P_m^2 \Box C_n^2) \geq \chi(P_m^2 \Box C_n^2) \geq \chi(C_n^2) = 4.$$

Claim 3: For $n \equiv 2\ (mod\ 3)$
Note that a color class is any collection of vertices of size no more than $\lceil \frac{mn}{5} \rceil$ that have the arrangements $s+t \equiv 2\ (mod\ 5)$, $s+t \equiv 3\ (mod\ 5)$, $s+t \equiv 4\ (mod\ 5)$, $s+t \equiv 0\ (mod\ 5)$ and $s+t \equiv 1\ (mod\ 5)$.
Consider the following color classes for $P_m^2 \Box C_n^2$.

$$\begin{aligned}
\sigma_1 &= \{(a_s, b_t) : s+t \equiv 2\ (mod\ 5), s \in [m], t \in [n]\} \\
\sigma_2 &= \{(a_s, b_t) : s+t \equiv 3\ (mod\ 5), s \in [m], t \in [n]\} \\
\sigma_3 &= \{(a_s, b_t) : s+t \equiv 4\ (mod\ 5), s \in [m], t \in [n]\} \\
\sigma_4 &= \{(a_s, b_t) : s+t \equiv 0\ (mod\ 5), s \in [m], t \in [n]\} \\
\sigma_5 &= \{(a_s, b_t) : s+t \equiv 1\ (mod\ 5), s \in [m], t \in [n]\}.
\end{aligned}$$

Clearly, $\{\sigma_s : s \in [5]\}$ is a color class of size $\lfloor \frac{mn}{5} \rfloor$ or $\lceil \frac{mn}{5} \rceil$. Therefore, the product $P_m^2 \Box C_n^2$ is equitably 5 - colorable (see Figure 4).
For $p \geq [5]$, let

$$\sigma_k = \left\lfloor \frac{mn+k-1}{p} \right\rfloor$$

for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{mn+p-1}{p} \right\rfloor \leq \left\lceil \frac{mn}{p} \right\rceil,$$

one can split the vertex set of $P_m^2 \Box C_n^2$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangements. Hence, $P_m^2 \Box C_n^2$ is equitably p - colorable.
Contrary, it is straightforward to verify that

$$\chi_{=}(P_m^2 \Box C_n^2) \geq \chi(P_m^2 \Box C_n^2) \geq \chi(C_m^2) = 5.$$

This completes the theorem. □

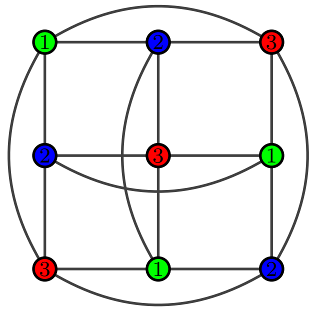

FIGURE 2. Equitable 3 - coloring of $P_3^2 \square C_3^2$.

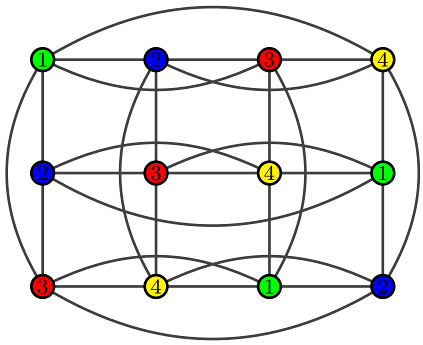

FIGURE 3. Equitable 4 - coloring of $P_3^2 \square C_4^2$.

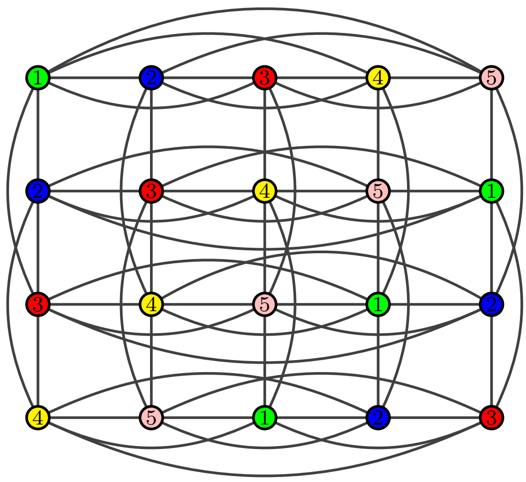

FIGURE 4. Equitable 5 - coloring of $P_4^2 \square C_5^2$.

3. Cartesian products of square of cycles

In this section, the exact values of the equitable chromatic number and threshold of square of cycles with square of cycles are obtained and equal.

Theorem 3.1. [6, 10]. *Let* $m, n \geq 3$. *Then*

$$\chi^*_=(C_m \square C_n) = \chi_=(C_m \square C_n) = \begin{cases} 2, & \text{for } mn \text{ is even;} \\ 3, & \text{otherwise.} \end{cases}$$

Theorem 3.2. *If* $m \geq 1$ *is an integer, then* $\chi^*_=(C^2_{3m} \square C^2_{3m}) = \chi_=(C^2_{3m} \square C^2_{3m}) = 3$.

Proof. Suppose that $V(C^2_{3m}) = \{a_1, a_2, \ldots, a_{3m}\}$ and $V(C^2_{3m}) = \{b_1, b_2, \ldots, b_{3m}\}$. Arrange the vertices of Cartesian products of C^2_{3m} and C^2_{3m} in such a way $\{(a_s, b_t) : s \in [3m], t \in [3m]\}$. Note that a color class is any collection of vertices of size no more than $3m^2$ that have the arrangement $s + t \equiv 2\ (mod\ 3)$, $s + t \equiv 0\ (mod\ 3)$, and $s + t \equiv 1\ (mod\ 3)$. Consider the color classes for $C^2_{3m} \square C^2_{3m}$.

$$\begin{aligned} \sigma_1 &= \{(a_s, b_t) : s + t \equiv 2\ (mod\ 3), s \in [m], t \in [n]\} \\ \sigma_2 &= \{(a_s, b_t) : s + t \equiv 0\ (mod\ 3), s \in [m], t \in [n]\} \\ \sigma_3 &= \{(a_s, b_t) : s + t \equiv 1\ (mod\ 3), s \in [m], t \in [n]\}. \end{aligned}$$

Clearly, $\{\sigma_i : i \in [3]\}$ is a color class of size $3m^2$. Therefore, $P^2_m \square P^2_n$ is equitably 3 - colorable (see Figure 2).

For $p \geq [3]$, let

$$\sigma_k = \left\lfloor \frac{(3m)^2 + k - 1}{p} \right\rfloor$$

for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{(3m)^2 + p - 1}{p} \right\rfloor \leq \left\lceil \frac{(3m)^2}{p} \right\rceil,$$

one can split $V(C^2_{3m} \square C^2_{3m})$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangement. Therefore, the Cartesian product of C^2_{3m} and C^2_{3m} is equitably p - colorable.

Contrary, it is straightforward to show that

$$\chi_=(C^2_{3m} \square C^2_{3m}) \geq \chi(C^2_{3m} \square C^2_{3m}) \geq \chi(C^2_{3m}) = 3.$$

Hence the theorem.

□

Theorem 3.3. *If* $m \geq 1$ *is an integer, then* $\chi^*_=(C^2_{3m} \square C^2_{3m+1}) = \chi_=(C^2_{3m} \square C^2_{3m+1}) = \chi^*_=(C^2_{3m+1} \square C^2_{3m+1}) = \chi_=(C^2_{3m+1} \square C^2_{3m+1}) = 4$.

Proof. Suppose that $V(C^2_{3m+1}) = \{a_1, a_2, \ldots, a_{3m+1}\}$ and $V(C^2_{3m+1}) = \{b_1, b_2, \ldots, b_{3m+1}\}$. Arrange the vertices of Cartesian product of C^2_{3m+1} and C^2_{3m+1} are in such a way that $\{(a_s, b_t) : s \in [3m], t \in [3m + 1]\}$.

Claim 1 : $C^2_{3m} \square C^2_{3m+1}$ is equitably 4 - colorable.

Note that a color class is any collection of vertices of size no more than $\left\lceil \frac{(3m)(3m+1)}{4} \right\rceil$.

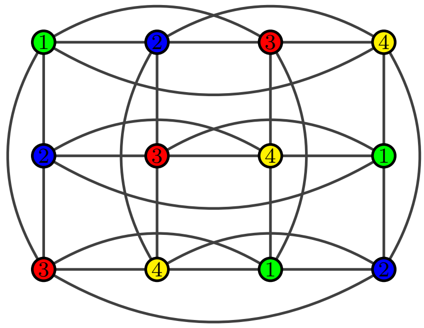

FIGURE 5. Equitable 4- coloring of $C_3^2 \square C_4^2$.

Consider the following color classes for $C_{3m}^2 \square C_{3m+1}^2$.

$$\begin{aligned}
\sigma_1 &= \{(a_s, b_t) : s+t \equiv 2 \ (mod\ 4), s \in [m], t \in [n]\} \\
\sigma_2 &= \{(a_s, b_t) : s+t \equiv 3 \ (mod\ 4), s \in [m], t \in [n]\} \\
\sigma_3 &= \{(a_s, b_t) : s+t \equiv 0 \ (mod\ 4), s \in [m], t \in [n]\} \\
\sigma_4 &= \{(a_s, b_t) : s+t \equiv 1 \ (mod\ 4), s \in [m], t \in [n]\}.
\end{aligned}$$

Clearly, $\{\sigma_s : s \in [4]\}$ is a color class of size $\left\lfloor \frac{3m(3m+1)}{4} \right\rfloor$ or $\left\lceil \frac{3m(3m+1)}{4} \right\rceil$. Hence, $C_{3m}^2 \square C_{3m+1}^2$ is equitably 4 - colorable (see Figure 5).

For $p \geq [4]$, let

$$\sigma_k = \left\lfloor \frac{3m(3m+1)+k-1}{p} \right\rfloor$$

and for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{3m(3m+1)+p-1}{p} \right\rfloor \leq \left\lceil \frac{(3m)(3m+1)}{p} \right\rceil,$$

one can divide the vertex set of $C_{3m}^2 \square C_{3m+1}^2$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangement. Therefore, $C_{3m}^2 \square C_{3m+1}^2$ is equitably p - colorable. Contrary, it is straightforward to verify that

$$\chi_=(C_{3m}^2 \square C_{3m+1}^2) \geq \chi(C_{3m}^2 \square C_{3m+1}^2) \geq \chi(C_{3m+1}^2) = 4.$$

Claim 2 : $C_{3m+1}^2 \square C_{3m+1}^2$ is equitably 4 - colorable.
Note that a color class is any collection of vertices of size no more than $\left\lceil \frac{(3m+1)^2}{4} \right\rceil$.
Consider the following color classes for $C_{3m}^2 \square C_{3m+1}^2$.

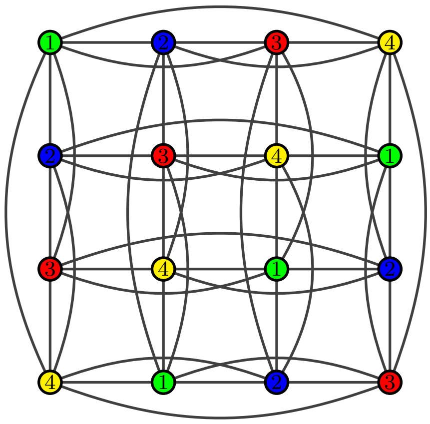

Figure 6. Equitable 4- coloring of $C_4^2 \square C_4^2$.

$$\sigma_1 = \{(a_s, b_t) : s + t \equiv 2\ (mod\ 4), s \in [m], t \in [n]\}$$
$$\sigma_2 = \{(a_s, b_t) : s + t \equiv 3\ (mod\ 4), s \in [m], t \in [n]\}$$
$$\sigma_3 = \{(a_s, b_t) : s + t \equiv 0\ (mod\ 4), s \in [m], t \in [n]\}$$
$$\sigma_4 = \{(a_s, b_t) : s + t \equiv 1\ (mod\ 4), s \in [m], t \in [n]\}.$$

Clearly, $\{\sigma_s : s \in [4]\}$ is a color class of size $\left\lfloor \frac{(3m+1)(3m+1)}{4} \right\rfloor$ or $\left\lceil \frac{(3m+1)(3m+1)}{4} \right\rceil$. Hence, $C_{3m+1}^2 \square C_{3m+1}^2$ is equitably 4 - colorable (see Figure 6).

For $p \geq [4]$, let

$$\sigma_k = \left\lfloor \frac{(3m+1)^2 + k - 1}{p} \right\rfloor$$

for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{(3m+1)^2 + p - 1}{p} \right\rfloor \leq \left\lceil \frac{(3m+1)^2}{p} \right\rceil,$$

one can divide $V(C_{3m+1}^2 \square C_{3m+1}^2)$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangement. Thus, $C_{3m+1}^2 \square C_{3m+1}^2$ is equitably p - colorable.
Contrary, it is uncomplicated to verify that

$$\chi_=(C_{3m+1}^2 \square C_{3m+1}^2) \geq \chi(C_{3m+1}^2 \square C_{3m+1}^2) \geq \chi(C_{3m+1}^2) = 4.$$

Hence the theorem. □

Theorem 3.4. *If $m \geq 1$ is an integer, then* $\chi_=^*(C_{3m}^2 \square C_{3m+2}^2) = \chi_=(C_{3m}^2 \square C_{3m+2}^2) = \chi_=^*(C_{3m+1}^2 \square C_{3m+2}^2) = \chi_=(C_{3m+1}^2 \square C_{3m+2}^2) = \chi_=^*(C_{3m+2}^2 \square C_{3m+2}^2) = \chi_=(C_{3m+2}^2 \square C_{3m+2}^2) = 5$.

Proof. Suppose that $V(C_{3m+1}^2) = \{a_1, a_2, \ldots, a_{3m+1}\}$ and $V(C_{3m+2}^2) = \{b_1, b_2, \ldots, b_{3m+2}\}$. Arrange the vertices of Cartesian product of C_{3m+1}^2 and C_{3m+2}^2 in such a way that

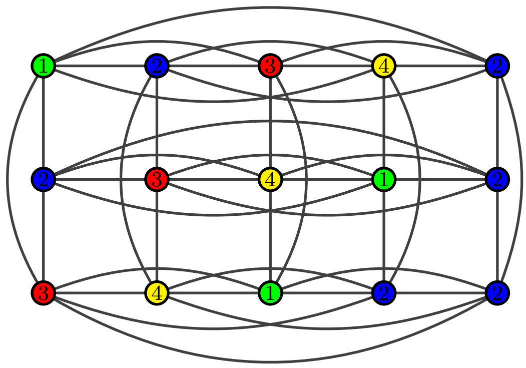

FIGURE 7. Equitable 5 - coloring of $C_3^2 \square C_5^2$.

$\{(a_s, b_t) : s \in [3m+1], t \in [3m+2]\}$.

Claim 1 : $C_{3m}^2 \square C_{3m+2}^2$ is equitably 5 - colorable.
Note that a color class is any collection of vertices of size no more than $\left\lceil \frac{(3m)(3m+2)}{5} \right\rceil$, that have the arrangements $s+t \equiv 2\ (mod\ 5)$, $s+t \equiv 3\ (mod\ 5)$, $s+t \equiv 4\ (mod\ 5)$, $s+t \equiv 0\ (mod\ 5)$ and $s+t \equiv 1\ (mod\ 5)$.
Consider the following color classes for $C_{3m}^2 \square C_{3m+2}^2$.

$$\sigma_1 = \{(a_s, b_t) : s+t \equiv 2\ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_2 = \{(a_s, b_t) : s+t \equiv 3\ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_3 = \{(a_s, b_t) : s+t \equiv 4\ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_4 = \{(a_s, b_t) : s+t \equiv 0\ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_5 = \{(a_s, b_t) : s+t \equiv 1\ (mod\ 5), s \in [m], t \in [n]\}.$$

Clearly, $\{\sigma_s : s \in [5]\}$ is a color class of size $\left\lfloor \frac{3m(3m+2)}{5} \right\rfloor$ or $\left\lceil \frac{3m(3m+2)}{5} \right\rceil$. Hence, $C_{3m}^2 \square C_{3m+2}^2$ is equitably 5 - colorable (see Figure 7).

For $p \geq [5]$, let

$$\sigma_k = \left\lfloor \frac{(3m)(3m+2)+k-1}{p} \right\rfloor$$

for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{(3m)(3m+2)+p-1}{p} \right\rfloor \leq \left\lceil \frac{(3m)(3m+2)}{p} \right\rceil,$$

one can split the vertex set of $C_{3m}^2 \square C_{3m+2}^2$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangement. Hence, $C_{3m}^2 \square C_{3m+2}^2$ is equitably p - colorable.
Claim 2 : $C_{3m+1}^2 \square C_{3m+2}^2$ is equitably 5 - colorable.
Note that a color class is any collection of vertices of size no more than $\left\lceil \frac{(3m+1)(3m+2)}{5} \right\rceil$, that have the arrangement $s+t \equiv 2\ (mod\ 5)$, $s+t \equiv 3\ (mod\ 5)$, $s+t \equiv 4\ (mod\ 5)$, $s+t \equiv 0\ (mod\ 5)$ and $s+t \equiv 1\ (mod\ 5)$.

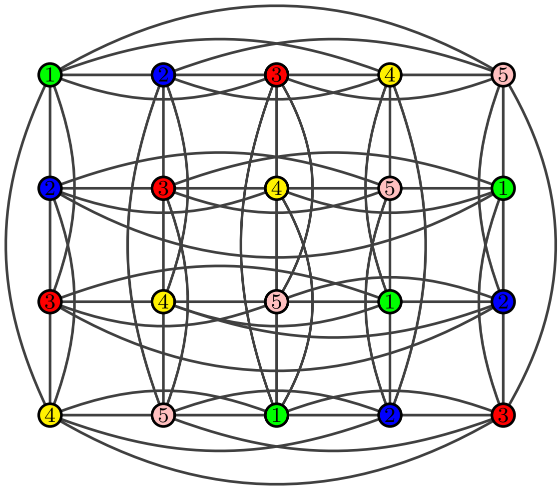

FIGURE 8. Equitable 5 - coloring of $C_4^2 \square C_5^2$.

Consider the following color classes for $C_{3m+1}^2 \square C_{3m+2}^2$.

$$\sigma_1 = \{(a_s, b_t) : s + t \equiv 2 \ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_2 = \{(a_s, b_t) : s + t \equiv 3 \ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_3 = \{(a_s, b_t) : s + t \equiv 4 \ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_4 = \{(a_s, b_t) : s + t \equiv 0 \ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_5 = \{(a_s, b_t) : s + t \equiv 1 \ (mod\ 5), s \in [m], t \in [n]\}.$$

Clearly, $\{\sigma_s : s \in [5]\}$ is a color class of size $\left\lfloor \frac{(3m+1)(3m+2)}{5} \right\rfloor$ or $\left\lceil \frac{(3m+1)(3m+2)}{5} \right\rceil$. Hence, $C_{3m+1}^2 \square C_{3m+2}^2$ is equitably 5 - colorable (see Figure 8).

For $p \geq [5]$, let

$$\sigma_k = \left\lfloor \frac{(3m+1)(3m+2) + k - 1}{p} \right\rfloor$$

for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{(3m+1)(3m+2) + p - 1}{p} \right\rfloor \leq \left\lceil \frac{(3m+1)(3m+2)}{p} \right\rceil,$$

one can split the vertex set of $C_{3m+1}^2 \square C_{3m+2}^2$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangement. Hence, $C_{3m+1}^2 \square C_{3m+2}^2$ is equitably p - colorable. Contrary, it is uncomplicated to verify that

$$\chi_=(C_{3m+1}^2 \square C_{3m+2}^2) \geq \chi(C_{3m+1}^2 \square C_{3m+2}^2) \geq \chi(C_{3m+2}^2) = 5.$$

Claim 3 : $C_{3m+2}^2 \square C_{3m+2}^2$ is equitably 5 - colorable.

Note that a color class is any collection of vertices of size no more than $\left\lceil \frac{(3m+2)^2}{5} \right\rceil$ that have

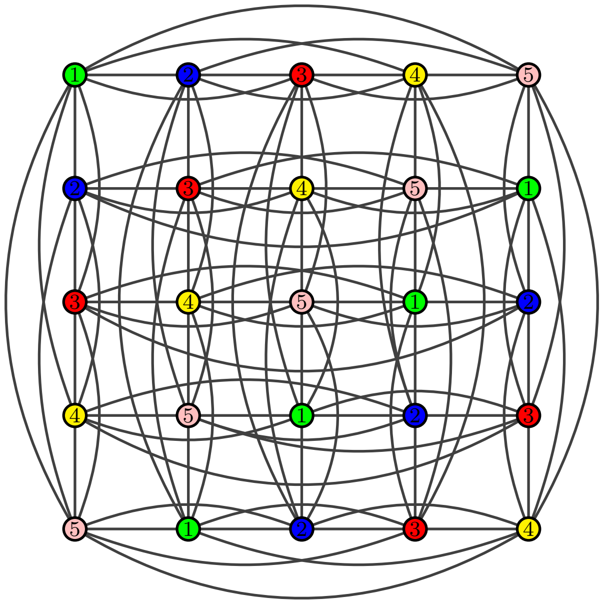

FIGURE 9. Equitable 5- coloring of $C_5^2 \square C_5^2$.

the arrangement $s+t \equiv 2\ (mod\ 5)$, $s+t \equiv 3\ (mod\ 5)$, $s+t \equiv 4\ (mod\ 5)$, $s+t \equiv 0\ (mod\ 5)$ and $s+t \equiv 1\ (mod\ 5)$.
Consider the color classes for $C_{3m+2}^2 \square C_{3m+2}^2$.

$$\sigma_1 = \{(a_s, b_t) : s+t \equiv 2\ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_2 = \{(a_s, b_t) : s+t \equiv 3\ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_3 = \{(a_s, b_t) : s+t \equiv 4\ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_4 = \{(a_s, b_t) : s+t \equiv 0\ (mod\ 5), s \in [m], t \in [n]\}$$
$$\sigma_5 = \{(a_s, b_t) : s+t \equiv 1\ (mod\ 5), s \in [m], t \in [n]\}.$$

Clearly, $\{\sigma_s : s \in [5]\}$ is a color class of size $\left\lfloor \frac{(3m+2)(3m+2)}{5} \right\rfloor$ or $\left\lceil \frac{(3m+2)(3m+2)}{5} \right\rceil$. Therefore, $C_{3m+2}^2 \square C_{3m+2}^2$ is equitably 5 - colorable (see Figure 9).

For $p \geq [5]$, let

$$\sigma_k = \left\lfloor \frac{(3m+2)^2 + k - 1}{p} \right\rfloor$$

for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{(3m+2)^2 + p - 1}{p} \right\rfloor \leq \left\lceil \frac{(3m+2)^2}{p} \right\rceil,$$

one can divide $V(C_{3m+2}^2 \square C_{3m+2}^2)$ into p color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$ subsequently in the arrangement. Thus, $C_{3m+2}^2 \square C_{3m+2}^2$, is equitably p - colorable.

Contrary, it is straightforward to prove that

$$\chi_{=}(C^2_{3m+2}\square C^2_{3m+2}) \geq \chi(C^2_{3m+2}\square C^2_{3m+2}) \geq \chi(C^2_{3m+2}) = 5.$$

Hence the theorem.

□

4. Conclusions

In this article, we have obtained the following results for the positive integers m and n

(1) $\chi^*_{=}(P^2_m\square P^2_n) = \chi_{=}(P^2_m\square P^2_n) = 3.$

(2)

$$\chi^*_{=}(P^2_m\square C^2_n) = \chi_{=}(P^2_m\square C^2_n) = \begin{cases} 3, & \text{for } n \equiv 0\ (mod\ 3), \\ 4, & \text{for } n \equiv 1\ (mod\ 3), \\ 5, & \text{for } n \equiv 2\ (mod\ 3). \end{cases}$$

(3) $\chi^*_{=}(C^2_{3m}\square C^2_{3m}) = \chi_{=}(C^2_{3m}\square C^2_{3m}) = 3.$

(4) $\chi^*_{=}(C^2_{3m}\square C^2_{3m+1}) = \chi_{=}(C^2_{3m}\square C^2_{3m+1}) = \chi^*_{=}(C^2_{3m+1}\square C^2_{3m+1}) = \chi_{=}(C^2_{3m+1}\square C^2_{3m+1}) = 4.$

(5) $\chi^*_{=}(C^2_{3m}\square C^2_{3m+2}) = \chi_{=}(C^2_{3m}\square C^2_{3m+2}) = \chi^*_{=}(C^2_{3m+1}\square C^2_{3m+2}) = \chi_{=}(C^2_{3m+1}\square C^2_{3m+2}) = \chi^*_{=}(C^2_{3m+2}\square C^2_{3m+2}) = \chi_{=}(C^2_{3m+2}\square C^2_{3m+2}) = 5.$

In future, we intend to extend our work to equitable total chromatic number of some well known graph structures. Furthermore, it's exciting and challenging to work on the following problem:

Problem: Determining equitable chromatic number and equitable chromatic threshold of different graph products, namely Cartesian product, lexicographic product, rooted product, etc.

Acknowledgement. The authors would like to extend their heartfelt gratitude to the anonymous reviewers for their valuable suggestions and comments to improve presentation of the paper.

References

[1] Bodlaender, Hans L. and Fomin, Fedor V., (2005), Equitable colorings of bounded treewidth graphs, Theoretical Computer Science., 349 (1), pp. 22-30.

[2] Bondy, J. A. and Murty U. S. R., (1976), Graph theory with applications., The Macmillan press LTD, New York.

[3] Chen, B. L. and Yen, C. H., (2012), Equitable Δ-coloring of graphs, Discrete Mathematics., 312 (9), pp. 1512-1517.

[4] Chen, B. L. and Lih, K. W., (1994), Equitable coloring of trees, Journal of Combinatorial Theory Series B., 61 (1), pp. 83-87.

[5] Chen, B. L., Lih, K. W. and Wu, P. L., (1994), Equitable coloring and the maximum degree, European Journal of Combinatorics., 15 (5), pp. 443-447.

[6] Chen, B. L., Lih, K. W. and Yan, J. H., (2009), Equitable coloring of interval graphs and products of graphs. arXiv preprint., arXiv:0903.1396.

[7] Clark, J. and Holton, D. A., (1991), A first look at graph theory, World Scientific.

[8] Elumalai, P. and Parthiban, A., (2024), On equitable colorings of windmill graphs, Contemporary Mathematics., 5 (3), pp. 4064-4078.

[9] Erdős, P., (1964), Problem 9, In: Fielder, M., ed, Theory of graphs and its application, Prague: Czech. Acad. Sci. Publ., 159.

[10] Furmańczyk, H., (2006), Equitable coloring of graph products, Opuscula Mathematica., 26 (1), pp. 31-44.

[11] Lih, K. W., (1998), The equitable coloring of graphs, Handbook of Combinatorial Optimization., 3, pp. 2015-2038.

[12] Lih, K. W. and Wu, P. L., (1996), On equitable coloring of bipartite graphs, Discrete Mathematics., 151, pp. 155-160.
[13] Lin, W. H. and Chang, G. J., (2012), Equitable colorings of Cartesian products of graphs, Discrete Applied Mathematics., 160 (3), pp. 239-247.
[14] Ma, S. and Zuo, L., (2016), Equitable colorings of Cartesian products of square of cycles and paths with complete bipartite graphs, Journal of Combinatorial Optimization., 32, pp. 725-740.
[15] Meyer, W., (1973), Equitable coloring, The American Mathematical Monthly., 80 (8), pp. 920-922.
[16] Sabidussi, G., (1957), Graphs with given group and given graph-theoretical properties, Canadian Journal of Mathematics., 9, pp. 515-525.
[17] Tucker, A., (1973), Perfect graphs and an application to optimizing municipal services, Siam Review., 15 (3), pp. 585-590.
[18] West, D. B., (2001), Introduction to graph theory, Vol. 2, Upper Saddle River Prentice Hall.
[19] Yan, Z., Wang, W. and Zhang, X., (2015), Equitable colorings of Cartesian products with balanced complete multipartite graphs, Discrete Applied Mathematics., 180, pp. 200-203.
[20] Yan, Z. and Wang, W., (2014), Equitable coloring of Kronecker products of complete multipartite graphs and complete graphs, Discrete Applied Mathematics., 162, pp. 328-333.
[21] Yap, H. P. and Zhang, Y., (1998), Equitable colorings of planar graphs, J. Combin. Math. Combin. Comput., 27, pp. 97-105.
[22] Zhang, Y., (1997), The equitable Δ - colouring conjecture holds for outerplanar graphs, Bulletin-Institute of mathematics Academia Sinica., 25, pp. 143-150.
[23] Zuo, L., Wu, F. and Zhang, S., (2015), Equitable colorings of Cartesian products of fans with bipartite graphs, Italian Journal of Pure and Applied Mathematics., 35, pp. 91-100.

ELUMALAI P. is a Ph.D. research scholar in the Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore - 632 014, Tamil Nadu, India. He is doing research in the area of graph coloring in graph theory. He has obtained M.Phil. degree in Mathematics from Thiruvalluvar University, Vellore, India in 2013 and the M.Sc., degree in Mathematics from Presidency College (Autonomous), University of Madras, Chennai in 2011. He has served as an Assistant Professor at Sun Arts and Science College, Tiruvannamalai, Tamilnadu, India (2013-2022).

PARTHIBAN A. for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.14, N.2