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NEUTROSOPHIC OVER SOFT GENERALIZED CONTINUOUS FUNCTIONS: A PARADIGM SHIFT IN BEST INVENTION COMPETITION MACHINE SELECTION

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ABSTRACT. In today's complex and uncertain world, the emergence of neutrosophic environments is becoming increasingly essential. These frameworks excel at navigating ambiguity, providing valuable tools for understanding and managing uncertainty. A significant advancement in this field is the introduction of Neutrosophic Over Soft Generalized Closed Sets and Continuous Functions. These concepts offer refined methods for grappling with nuanced uncertainties, providing a deeper understanding of complex situations. To illustrate their effectiveness, let's consider a practical example involving the selection of machines for the prestigious Best Invention Competition. By employing tangent similarity measures, we can identify optimal candidates with precision. This numerical demonstration vividly showcases the tangible utility of these concepts in decision-making within intricate and uncertain landscapes. Furthermore, this example hints at the transformative potential of neutrosophic frameworks across various domains. These concepts promise to enhance problem-solving capabilities in contexts where uncertainty is prevalent, enabling the emergence of more informed and resilient decisions.

Keywords: neutrosophic over soft generalized closed set, neutrosophic over soft generalized open set, neutrosophic over soft generalized interior, neutrosophic over soft generalized closure, neutrosophic over soft generalized continuose function.

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1. INTRODUCTION

It all started in 1965 with Zadeh's seminal paper introducing the concept of fuzzy sets [24], which provided a mathematical framework for dealing with uncertainty and imprecision. This laid the foundation for many future works in decision-making and uncertainty modeling. Zadeh further expanded his theory in 1978 by establishing fuzzy sets as a basis for the theory of possibility [25], broadening the applications of fuzzy sets. In 1970, Bellman and Zadeh [2] applied these fuzzy concepts to decision-making,

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demonstrating how fuzzy environments can aid in addressing real-world problems where precise data is scarce.

The late 1990s witnessed a significant development with Molodtsov's introduction of soft set theory in 1999 [14], providing a generalized mathematical tool for handling uncertainty without the limitations of traditional set theory. Concurrently, Smarandache's work in 1999 introduced Neutrosophic Logic [21], which extended fuzzy logic by incorporating the notion of indeterminacy, thus offering a more flexible framework. Maji, Biswas, and Roy continued this momentum by advancing soft set theory in 2003 [12], adding to its theoretical foundation.

Atanassov's 2012 book on intuitionistic fuzzy sets theory [1] presented a comprehensive exploration of intuitionistic fuzzy sets, which added another layer of complexity to fuzzy set theory by allowing the degree of non-membership alongside membership. In 2013, Broumi proposed the generalized neutrosophic soft set [3], pushing the boundaries of neutrosophic theory, and in 2015, Broumi and Deli further explored its applications in medical diagnosis [4], illustrating the practical utility of these sets in healthcare decision-making.

The tangent similarity measure was introduced in 2015 by Pramanik and Mondal [15, 17], providing an innovative method for multiple attribute decision-making, which enhanced decision analysis frameworks. Smarandache and Pramanik (2016) provide an extensive overview of neutrosophic theory and its applications, covering key advancements. This work serves as a fundamental resource in understanding neutrosophic concepts [22]. Smarandache (2016) introduces the concepts of neutrosophic overset, underset, and offset, expanding neutrosophic logic, probability, and statistics. This work enhances understanding of handling over- and under-defined data in uncertain environments [23]. Dhavaseelan and Jafari (2017) introduce generalized neutrosophic closed sets, extending traditional closed set concepts into the neutrosophic environment. This extension aids in handling uncertainty and indeterminate information in decision-making. Their work enriches the field of neutrosophic mathematics [9]. Jansi, Mohana, and Smarandache's 2019 study [10] brought forward the concept of Pythagorean neutrosophic sets, where the dependency of truth and falsity components provided a new way to handle uncertainty in complex situations.

In the recent years, particularly the 2020s, there has been a surge in applying these theories across diverse fields. Saqlain et al. (2020) developed the single and multi-valued neutrosophic hypersoft set [20], expanding the adaptability of neutrosophic sets. Radha et al. (2021) [18] focused on improving correlation coefficients for Pythagorean neutrosophic sets, which enhanced decision-making processes with more refined measures.

Devi and Parthiban emerged as prolific contributors in 2023 and 2024, with their works encompassing multiple aspects of neutrosophic set theory in decision-making. Their studies spanned the decision-making process over neutrosophic Pythagorean soft sets [5], explored decision-making using neutrosophic over soft topological spaces [6], enhanced parental decision-making for school selection [7], and developed applications in healthcare [8]. These contributions have significantly enriched the literature on neutrosophic decision-making.Rodrigo and Maheswari (2023) explore neutrosophic open and closed maps within neutrosophic topological spaces, expanding beyond traditional topological constructs to incorporate indeterminacy. This enhancement increases the adaptability and versatility of the framework, contributing to the theoretical advancement of neutrosophic topology [19].

Simultaneously, Kumaravel et al. (2023) used fuzzy cognitive maps and neutrosophic cognitive maps for analyzing dengue fever [11], and Murugesan et al. (2023) conducted a comparative study on COVID-19 variants using similar approaches [16], showcasing the versatility of fuzzy and neutrosophic techniques in handling complex healthcare issues. Lastly, Majumder et al. (2023) [13] utilized a single-valued pentapartitioned neutrosophic weighted hyperbolic tangent similarity measure to address environmental risks during the COVID-19 pandemic, demonstrating the practical application of these advanced mathematical concepts in real-world problems.

The manuscript introduces the Neutrosophic Over Soft Generalized Closed Set and Neutrosophic Over Soft Generalized Continuous Function, along with their basic definitions and propositions. To demonstrate the practical relevance of these concepts, a numerical illustration is constructed for selecting the best machine in the Best Invention Competition using the tangent similarity measure for the Neutrosophic Over Soft Set. Thus, the manuscript seamlessly integrates theoretical advancements, illustrative examples, and practical applications, making a substantial contribution to the field of study.

2. Preliminary

This section presents the fundamental definitions for Neutrosophic Set (NS), Neutrosophic Over Set (NOS), Neutrosophic Over Soft Set ($\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-set), and Neutrosophic Over Soft Topological Space ($\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-topological space).

Definition 2.1. [21] *Let $\mathcal{H}$ be a non-empty set, and let $\mathcal{J}$ be a Neutrosophic Set (NS). Then*

$$\mathcal{J} = \langle \mathsf{h}, \aleph_{\mathcal{J}}(\mathsf{h}), \eth_{\mathcal{J}}(\mathsf{h}), \Upsilon_{\mathcal{J}}(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{H}$$

where $\aleph, \eth, \Upsilon : \mathcal{H} \to [0, 1]$ and $0 \leq \aleph(\mathsf{h}) + \eth(\mathsf{h}) + \Upsilon(\mathsf{h}) \leq 3$. Here, $\aleph(\mathsf{h})$, $\eth(\mathsf{h})$, and $\Upsilon(\mathsf{h})$ represent the degree of truth membership, indeterminacy, and falsity, respectively.

Definition 2.2. [23] *Let $\mathcal{J}$ be an NS in $\mathcal{H}$.If $\mathcal{J}$ is said to be an NOS in an non-empty set $\mathcal{H}$ then it has at-least one neutrosophic component is > 1 and no other component are < 0 is defined as,*

$$\mathcal{J} = \{\langle \mathsf{h}, \aleph_{\mathcal{J}}(\mathsf{h}), \eth_{\mathcal{J}}(\mathsf{h}), \Upsilon_{\mathcal{J}}(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{H}\}$$

Where $\aleph, \eth, \Upsilon : \mathcal{H} \to [0, \Omega]$, $0 \leq \aleph(\mathsf{h}) + \eth(\mathsf{h}) + \Upsilon(\mathsf{h}) \leq 3$ and Ω is said to be over-limit of NOS

Note: $\rho(\mathcal{H})$ is a set of all the $\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$ subset of an non-empty set $\mathcal{H}$

Definition 2.3. [6] *A $\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-set $\odot = e, \langle \mathsf{h}, 0, 0, \Omega\rangle : \mathsf{h} \in \mathcal{H} : e \in \mathcal{E}$ is called a Null $\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-set, and $\circledast = e, \langle \mathsf{h}, \Omega, \Omega, 0\rangle : \mathsf{h} \in \mathcal{H} : e \in \mathcal{E}$ is called a Universal $\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-set.*

Definition 2.4. [6] *Let $\mathcal{H}$ be an non-empty set and $\mathcal{E}$ be a set of parameter on $\mathcal{H}$.Then $\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-set is defined by a set valued function*

$$\lambda_{\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}} : \mathcal{E} \to \rho(\mathcal{H})$$

where $\rho(\mathcal{H})$ is an set of all $\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-set on $\mathcal{H}$.$\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-set is an valued function from the set of parameter $\mathcal{E}$ on $\mathcal{H}$ is defined as

$$\mathcal{J} = (\lambda_{\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}}, \mathcal{E}) = \{(\mathsf{e}, \{\langle \mathsf{h}, \aleph_{\mathcal{J}}(\mathsf{h}), \eth_{\mathcal{J}}(\mathsf{h}), \Upsilon_{\mathcal{J}}(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{H}\}) : \mathsf{e} \in \mathcal{E}\}$$

Definition 2.5. [6] *Let $\mathcal{J} = (\mathcal{J}_{\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}}, \mathcal{E})$ and $\mathcal{W} = (\mathcal{W}_{\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}}, \mathcal{E})$ be a two $\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-set.If $\mathcal{J}$ is said to be a subset of $\mathcal{W}$ i.e.,$\mathcal{J} \subseteq \mathcal{W}$ then*

$$\aleph_{\mathcal{J}}(\mathsf{h}) \leq \aleph_{\mathcal{W}}(\mathsf{h}), \eth_{\mathcal{J}}(\mathsf{h}) \leq \eth_{\mathcal{W}}(\mathsf{h}), \Upsilon_{\mathcal{J}}(\mathsf{h}) \geq \Upsilon_{\mathcal{W}}(\mathsf{h})$$

In other words $\mathcal{W}$ is an super set of $\mathcal{J}$

Note: Let $\mathcal{J} \subset \mathcal{W}$ and $\mathcal{W} \subset \mathcal{J}$ then $\mathcal{J} = \mathcal{W}$

Definition 2.6. [6] *Let $\mathcal{J}$ and $\mathcal{W}$ be any $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-sets, and let $\forall \mathsf{h} \in \mathcal{H}$ and $\mathsf{e} \in \mathcal{E}$. Then the union, intersection, and complement are defined as follows:*

(i) *Union:*

$$\mathcal{J} \mho \mathcal{W} = \{\mathsf{e}, \{\langle \mathsf{h}, \max(\aleph_{\mathcal{J}}(\mathsf{h}), \aleph_{\mathcal{W}}(\mathsf{h})), \max(\eth_{\mathcal{J}}(\mathsf{h}), \eth_{\mathcal{W}}(\mathsf{h})), \min(\Upsilon_{\mathcal{J}}(\mathsf{h}), \Upsilon_{\mathcal{W}}(\mathsf{h}))\rangle\}\}$$

(ii) *Intersection:*

$$\mathcal{J} \Omega \mathcal{W} = \{\mathsf{e}, \{\langle \mathsf{h}, \min(\aleph_{\mathcal{J}}(\mathsf{h}), \aleph_{\mathcal{W}}(\mathsf{h})), \min(\eth_{\mathcal{J}}(\mathsf{h}), \eth_{\mathcal{W}}(\mathsf{h})), \max(\Upsilon_{\mathcal{J}}(\mathsf{h}), \Upsilon_{\mathcal{W}}(\mathsf{h}))\rangle\}\}$$

(iii) *Complement:*

$$\mathcal{J}^{\complement} = \{\mathsf{e}, \{\langle \mathsf{h}, \Upsilon_{\mathcal{J}}(\mathsf{h}), \Omega - \eth_{\mathcal{J}}(\mathsf{h}), \aleph_{\mathcal{J}}(\mathsf{h})\rangle\}\}$$

Definition 2.7. [6] *A neutrosophic over soft topology ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topology) $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ on a non-empty set $\mathcal{H}$ satisfies the following conditions:*

(i) $\odot, \circledast \in \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$.
(ii) *The union of any arbitrary collection of sets in $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ is also in $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$.*
(iii) *The finite intersection of sets in $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ is also in $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$.*

Then, $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is called a neutrosophic over soft topological space ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space). An element of $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ is called a neutrosophic over soft open set ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open set), and the complement of any element in $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ is called a neutrosophic over soft closed set ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-closedset).

Definition 2.8. [6] *For an operator on a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathcal{J} \in \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$, the neutrosophic over soft topological closure and interior, denoted by $cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J})$ and $int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})$, are defined as follows:*

$$cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}) = \Omega \left\{\mathcal{G} : \mathcal{G} \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\text{-closed set in } \mathcal{H} \text{ and } \mathcal{J} \subseteq \mathcal{G}\right\}.$$

$$int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}) = \mho \left\{\mathcal{O} : \mathcal{O} \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\text{-open set in } \mathcal{H} \text{ and } \mathcal{J} \supseteq \mathcal{O}\right\}.$$

Note:

(i) $cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}^{\complement}) = (int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}))^{\complement}$
(ii) $int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}^{\complement}) = (cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}))^{\complement}$

Proposition 2.1. [6] *Let $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space and $\mathcal{J}$ is a subset of $\mathcal{H}$, then*

(i) *$cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{R})$ is the smallest $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} - closedset$ containing $\mathcal{R}$.*
(ii) *$int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J})$ is the largest $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} - openset$ contained in $\mathcal{J}$.*

3. Neutrosophic Over Soft Generalized Continuous Function

Definition 3.1. *Let $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space. A $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathcal{J}$ is said to be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ Generalized Closed Set ($\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set) if $cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}) \subseteq \mathcal{G}$ whenever $\mathcal{J} \subseteq \mathcal{G}$ and $\mathcal{G}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset. The complement of a $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set is called a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ Generalized Open Set ($\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set).*

Definition 3.2. *Let $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space. Then for any $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathcal{J}$, $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ generalized topological interior($int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J})$) and closure($cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J})$)operators are defined as:*
$int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}) = \mho\{\mathcal{O} : \mathcal{O} \text{ is } \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \text{ in } \mathcal{H} \text{ and } \mathcal{J} \supseteq \mathcal{O}\}$ *and*
$cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) = \Omega\{\mathcal{G} : \mathcal{G} \text{ is } \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \text{ in } \mathcal{H} \text{ and } \mathcal{J} \subseteq \mathcal{G}\}$.

Proposition 3.1. *Let $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space. Let $\mathcal{J}$ and $\mathcal{W}$ be any two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$. Then the $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set satisfy the following properties:*

(i) $\mathcal{J} \subseteq cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J})$
(ii) $int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq \mathcal{J}$
(iii) $\mathcal{J} \subseteq \mathcal{W} \implies cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W})$
(iv) $\mathcal{J} \subseteq \mathcal{W} \implies int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{W})$

(v) $cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}\mho\mathcal{W}) = cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J})\mho cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W})$
(vi) $int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}\Omega\mathcal{W}) = int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J})\Omega int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{W})$
(vii) $(cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}))^{\complement} = int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}^{\complement})$
(viii) $(int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}))^{\complement} = cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}^{\complement})$

Proof. (i) $cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) = \Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J} \subseteq \mathcal{G}\}$
Thus $\mathcal{J} \subseteq cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J})$

(ii) $int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}) = \mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J} \supseteq \mathcal{O}\}$
Thus $int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq \mathcal{J}$

(iii) $\mathcal{J} \subseteq \mathcal{W}$

$$\begin{aligned} cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W}) &= \Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{W} \subseteq \mathcal{G}\} \\ &\supseteq \Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J} \subseteq \mathcal{G}\} \\ &\supseteq cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) \end{aligned}$$

Thus $cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W})$

(iv) $\mathcal{J} \subseteq \mathcal{W}$

$$\begin{aligned} int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{W}) &= \mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{W} \supseteq \mathcal{O}\} \\ &\supseteq \mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J} \supseteq \mathcal{O}\} \\ &\supseteq int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}) \end{aligned}$$

Thus $int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{W})$

(v)
$$\begin{aligned} cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}\mho\mathcal{W}) &= \Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad (\mathcal{J}\mho\mathcal{W}) \subseteq \mathcal{G}\} \\ &= (\Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J} \subseteq \mathcal{G}\})\mho \\ &\quad (\Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{W} \subseteq \mathcal{G}\}) \\ &= cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J})\mho cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W}) \end{aligned}$$
$$\therefore cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}\mho\mathcal{W}) = cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J})\mho cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W})$$

(vi)
$$\begin{aligned} int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}\Omega\mathcal{W}) &= \mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad (\mathcal{J}\Omega\mathcal{W}) \supseteq \mathcal{O}\} \\ &= (\mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J} \supseteq \mathcal{O}\})\Omega \\ &\quad (\mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{W} \supseteq \mathcal{O}\}) \\ &= int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J})\Omega int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{W}) \end{aligned}$$
$$\therefore int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}\Omega\mathcal{W}) = int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J})\Omega int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{W})$$

(vii) $cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) = \Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J} \subseteq \mathcal{G}\}$
$$\begin{aligned} (cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}))^{\complement} &= \mho\{\mathcal{G}^{\complement} : \mathcal{G}^{\complement} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J}^{\complement} \supseteq \mathcal{G}^{\complement}\} \\ &= \mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J}^{\complement} \supseteq \mathcal{O}\} \\ &= int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}^{\complement}) \end{aligned}$$
$$\therefore (cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}))^{\complement} = int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}^{\complement})$$

(viii) $int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}) = \mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J} \supseteq \mathcal{O}\}$
$$\begin{aligned} (int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}))^{\complement} &= \Omega\{\mathcal{O}^{\complement} : \mathcal{O}^{\complement} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J}^{\complement} \subseteq \mathcal{O}^{\complement}\} \\ &= \Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}} \quad in \quad \mathcal{H} \quad \text{and} \quad \mathcal{J}^{\complement} \subseteq \mathcal{G}\} \\ &= cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}^{\complement}) \end{aligned}$$
$$\therefore (int_{\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}}(\mathcal{J}))^{\complement} = cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}^{\complement})$$

□

Proposition 3.2. *Let $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space. If $\mathcal{W}$ is a $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ and $\mathcal{W} \subseteq \mathcal{J} \subseteq cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W})$, then $\mathcal{J}$ is a $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$.*

Proof. Let $\mathcal{L}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ such that $\mathcal{J} \subseteq \mathcal{L}$
Since $\mathcal{W} \subseteq \mathcal{J}$
$\mathcal{W} \subseteq \mathcal{L}$

Now, $\mathcal{W}$ is $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set and

$cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W}) \subseteq \mathcal{L}$

But $cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W})$

Since $cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{W}) \subseteq \mathcal{L}$

$\implies cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq \mathcal{L}$

Hence $\mathcal{J}$ is a $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set □

Proposition 3.3. *Let $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space and $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$. Then $\mathcal{J}$ is a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set if and only if $\mathcal{W} \subseteq int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J})$,whenever $\mathcal{W}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-closedset and $\mathcal{W} \subseteq \mathcal{J}$.*

Proof. The proof is obvious. □

Proposition 3.4. *If $int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}) \subseteq \mathcal{W} \subseteq \mathcal{J}$ and if $\mathcal{J}$ is a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set then $\mathcal{W}$ is also a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set.*

Proof. Now, $\mathcal{J}^{\complement} \subseteq \mathcal{W}^{\complement} \subseteq (int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}))^{\complement} = cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}^{\complement})$

Since $\mathcal{J}$ is a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set, then $\mathcal{J}^{\complement}$ is a $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set

By proposition (3.6)

$\mathcal{W}^{\complement}$ is a $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set $\implies$ $\mathcal{W}$ is a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set, □

Definition 3.3. *Let $\mathcal{H}$ and $\mathcal{I}$ be any two nonempty sets, and let $\mathfrak{f} : \mathcal{H} \to \mathcal{I}$ be a function. The notions of image and preimage of a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set are defined as follows:*

(i) *If $\mathcal{K} = \{\langle \mathfrak{i}, \aleph_{\mathcal{K}}(\mathfrak{i}), \eth_{\mathcal{K}}(\mathfrak{i}), \Upsilon_{\mathcal{K}}(\mathfrak{i})\rangle : \mathfrak{i} \in \mathcal{I}\}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $\mathcal{I}$, then the preimage of $\mathcal{K}$ under $\mathfrak{f}$, denoted by $\mathfrak{f}^{\rightarrow}(\mathcal{K})$, is the $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $\mathcal{H}$ defined by*

$$\mathfrak{f}^{\rightarrow}(\mathcal{K}) = \{(\mathsf{e}, \{\langle \mathsf{h}, \mathfrak{f}^{\rightarrow}(\aleph_{\mathcal{K}})(\mathsf{h}), \mathfrak{f}^{\rightarrow}(\eth_{\mathcal{K}})(\mathsf{h}), \mathfrak{f}^{\rightarrow}(\Upsilon_{\mathcal{K}})(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{H}\}) : \mathsf{e} \in \mathcal{E}\}.$$

(ii) *If $\mathcal{J} = \{(\mathsf{e}, \{\langle \mathsf{h}, \aleph_{\mathcal{J}}(\mathsf{h}), \eth_{\mathcal{J}}(\mathsf{h}), \Upsilon_{\mathcal{J}}(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{H}\}) : \mathsf{e} \in \mathcal{E}\}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $\mathcal{H}$, then the image of $\mathcal{J}$ under $\mathfrak{f}$, denoted by $\mathfrak{f}(\mathcal{J})$, is the $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $\mathcal{I}$ defined by*

$$\mathfrak{f}(\mathcal{J}) = \{(\mathsf{e}, \{\langle \mathfrak{i}, \mathfrak{f}(\aleph_{\mathcal{J}})(\mathfrak{i}), \mathfrak{f}(\eth_{\mathcal{J}})(\mathfrak{i}), (1 - \mathfrak{f}(1 - \Upsilon_{\mathcal{J}})(\mathfrak{i}))\rangle : \mathfrak{i} \in \mathcal{I}\}) : \mathsf{e} \in \mathcal{E}\}.$$

where,

$$\mathfrak{f}(\aleph_{\mathcal{J}})(\mathfrak{i}) = \begin{cases} \sup_{\mathsf{h}\in\mathfrak{f}^{\rightarrow}(\mathfrak{i})} \aleph_{\mathcal{J}}(\mathsf{h}), & \textit{if } \mathfrak{f}^{\rightarrow}(\mathfrak{i}) \neq \emptyset \\ 0, & \textit{otherwise} \end{cases}$$

$$\mathfrak{f}(\eth_{\mathcal{J}})(\mathfrak{i}) = \begin{cases} \sup_{\mathsf{h}\in\mathfrak{f}^{\rightarrow}(\mathfrak{i})} \eth_{\mathcal{J}}(\mathsf{h}), & \textit{if } \mathfrak{f}^{\rightarrow}(\mathfrak{i}) \neq \emptyset \\ 0, & \textit{otherwise} \end{cases}$$

$$\mathfrak{f}(\Upsilon_{\mathcal{J}})(\mathfrak{i}) = \begin{cases} \inf_{\mathsf{h}\in\mathfrak{f}^{\rightarrow}(\mathfrak{i})} \Upsilon_{\mathcal{J}}(\mathsf{h}), & \textit{if } \mathfrak{f}^{\rightarrow}(\mathfrak{i}) \neq \emptyset \\ 1, & \textit{otherwise} \end{cases}$$

Corollary 3.1. *Let $\mathcal{J}_{\mathfrak{n}}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $\mathcal{H}(\forall \mathfrak{n} = 1, 2, \dots)$, $\mathcal{K}_{\mathfrak{m}}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $\mathcal{I}(\forall \mathfrak{m} = 1, 2, \dots)$ and $\mathfrak{f} : \mathcal{H} \to \mathcal{I}$ be a function. Then*

(i) $\mathcal{J}_1 \subseteq \mathcal{J}_2 \implies \mathfrak{f}(\mathcal{J}_1) \subseteq \mathfrak{f}(\mathcal{J}_2)$
(ii) $\mathcal{K}_1 \subseteq \mathcal{K}_2 \implies \mathfrak{f}^{\rightarrow}(\mathcal{K}_1) \subseteq \mathfrak{f}^{\rightarrow}(\mathcal{K}_2)$
(iii) $\mathcal{J} \subseteq \mathfrak{f}^{\rightarrow}(\mathfrak{f}(\mathcal{J}))\{$*if* $\mathfrak{f}$ *is injective, then* $\mathcal{J} = \mathfrak{f}^{\rightarrow}(\mathfrak{f}(\mathcal{J}))\}$
(iv) $\mathfrak{f}(\mathfrak{f}^{\rightarrow}(\mathcal{K})) \subseteq \mathcal{K}\{$*if* $\mathfrak{f}$ *is surjective, then* $\mathfrak{f}(\mathfrak{f}^{\rightarrow}(\mathcal{K})) = \mathcal{K}\}$
(v) $\mathfrak{f}^{\rightarrow}(\mho\mathcal{K}_{\mathfrak{m}}) = \mho\mathfrak{f}^{\rightarrow}(\mathcal{K}_{\mathfrak{m}})$
(vi) $\mathfrak{f}^{\rightarrow}(\Omega\mathcal{K}_{\mathfrak{m}}) = \Omega\mathfrak{f}^{\rightarrow}(\mathcal{K}_{\mathfrak{m}})$
(viii) $\mathfrak{f}(\mho\mathcal{J}_{\mathfrak{n}}) = \mho\mathfrak{f}(\mathcal{J}_{\mathfrak{n}})$

(viii) $\mathfrak{f}(\Omega\mathcal{J}_{\mathfrak{n}}) \subseteq \Omega\mathfrak{f}(\mathcal{J}_{\mathfrak{n}})$ {*if* $\mathfrak{f}$ *is injective, then* $\mathfrak{f}(\Omega\mathcal{J}_{\mathfrak{n}}) = \Omega\mathfrak{f}(\mathcal{J}_{\mathfrak{n}})$}
(ix) $\mathfrak{f}^{\rightarrow}(\circledast) = \circledast$
(x) $\mathfrak{f}^{\rightarrow}(\odot) = \odot$
(xi) $\mathfrak{f}(\circledast) = \circledast$, *if* $\mathfrak{f}$ *is surjective*
(xii) $\mathfrak{f}(\odot) = \odot$
(xiii) $(\mathfrak{f}(\mathcal{J}))^{\complement} \subseteq \mathfrak{f}(\mathcal{J}^{\complement})$, *if* $\mathfrak{f}$ *is surjective*
(xiv) $\mathfrak{f}(\mathcal{J}^{\complement}) = (\mathfrak{f}(\mathcal{J}))^{\complement}$

Definition 3.4. *Let* $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ *and* $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *be any two* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-topological spaces. Let* $\mathfrak{f} : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}}) \rightarrow (\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *is a function.*

(i) $\mathfrak{f}$ *is said to be a* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *Generalized Continuous Function* ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$*-function) if the inverse image of every* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-closedset in* $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *is a* $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$*-set in* $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$
Similarly if the inverse image of every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-openset in* $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *is a* $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$*-set in* $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$
(ii) $\mathfrak{f}$ *is said to be a Strongly* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *Continuous Function (strongly-*$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$*-function)if* $\mathfrak{f}^{\rightarrow}(\mathcal{J})$ *is both* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-openset and* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-closedset in* $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ *for each* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-set in* $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$
(iii) $\mathfrak{f}$ *is said to be a Strongly* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *Generalized Continuous Function (strongly-*$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$*-function) if the inverse image of every* $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$*-set in* $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *is a* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-openset in* $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$

Proposition 3.5. *Let* $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ *and* $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *be any two* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-topological space. Let* $\mathfrak{f} : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}}) \rightarrow (\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *is said to be a* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$*-function. Then for every* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-set* $\mathcal{J}$ *in* $\mathcal{H}$,$\mathfrak{f}(cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J})) \subseteq cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathfrak{f}(\mathcal{J}))$

Proof. Let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$.
Since $cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathfrak{f}(\mathcal{J}))$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-closedset and $\mathfrak{f}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$-function
$\implies \mathfrak{f}^{\rightarrow}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathfrak{f}(\mathcal{J})))$ is a $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set and $\mathfrak{f}^{\rightarrow}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathfrak{f}(\mathcal{J}))) \supseteq \mathcal{J}$
Now, $cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J}) \subseteq \mathfrak{f}^{\rightarrow}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathfrak{f}(\mathcal{J})))$
$\therefore \mathfrak{f}(cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathcal{J})) \subseteq cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathfrak{f}(\mathcal{J}))$ □

Proposition 3.6. *Let* $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ *and* $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *be any two* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-topological space. Let* $\mathfrak{f} : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}}) \rightarrow (\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *is said to be a* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$*-function. Then for every* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-set* $\mathcal{J}$ *in* $\mathcal{I}$,$cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathfrak{f}^{\rightarrow}(\mathcal{J})) \subseteq \mathfrak{f}^{\rightarrow}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}))$

Proof. Let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$.Let $\mathcal{K} = \mathfrak{f}^{\rightarrow}(\mathcal{J})$ then
$\mathfrak{f}(\mathcal{K}) = \mathfrak{f}(\mathfrak{f}^{\rightarrow}(\mathcal{J})) \subseteq \mathcal{J}$
By the proposition (3.5),
$\mathfrak{f}(cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathfrak{f}^{\rightarrow}(\mathcal{J}))) \subseteq cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathfrak{f}(\mathfrak{f}^{\rightarrow}(\mathcal{J})))$
Thus, $cl_{\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}}(\mathfrak{f}^{\rightarrow}(\mathcal{J})) \subseteq \mathfrak{f}^{\rightarrow}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{J}))$ □

Proposition 3.7. *Let* $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ *and* $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *be any two* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-topological space. Let* $\mathfrak{f} : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}}) \rightarrow (\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ *is said to be a* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *Continuous function(*$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$*-function) then it is a* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$*-function.*

Proof. Let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset in $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$.
Since $\mathfrak{f}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$-function, $\mathfrak{f}^{\rightarrow}(\mathcal{J})$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$.
Every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset is a $\mathcal{N}_{\mathfrak{sc}}^{\mathfrak{og}}$-set.
Now, $\mathfrak{f}^{\rightarrow}(\mathcal{J})$ is a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$.
Hence $\mathfrak{f}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$-function □

Proposition 3.8. *The converse of the proposition is not necessarily true. That is, if $\mathfrak{f} : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}}) \to (\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$-function, it does not imply that $\mathfrak{f}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ Continuous function ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$-function).*

Proof. It is proved by the help of the example (3.1). □

Example 3.1. *Let $\mathcal{H} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ and $\mathcal{E} = \mathsf{e}$. Let $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-sets $\mathcal{J}$ and $\mathcal{K}$ in $\mathcal{H}$ as follows:*

$$\mathcal{J} = \{\langle \mathfrak{a}, 1.4, 0.7, 0.3\rangle, \langle \mathfrak{b}, 0.6, 0.9, 1.1\rangle, \langle \mathfrak{c}, 1.2, 0.9, 0.5\rangle\}$$

$$\mathcal{K} = \{\langle \mathfrak{a}, 1.4, 0.7, 0.2\rangle, \langle \mathfrak{b}, 0.7, 0.9, 1.1\rangle, \langle \mathfrak{c}, 1.3, 0.7, 0.4\rangle\}$$

Then two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topologies $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}} = \{\odot, \circledast, \mathcal{J}\}$ and $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}} = \{\odot, \circledast, \mathcal{K}\}$. Thus $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ and $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ are two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological spaces.
Define $\mathfrak{f}_1 : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}}) \to (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ as $\mathfrak{f}_1(\mathfrak{a}) = \mathfrak{b}$,$\mathfrak{f}(\mathfrak{b}) = \mathfrak{a}$ and $\mathfrak{f}(\mathfrak{c}) = \mathfrak{c}$.

$$\mathfrak{f}_1^{-1}(\mathcal{K}) = \{\mathsf{e}, \{\langle \mathfrak{a}, 0.7, 0.9, 1.1\rangle, \langle \mathfrak{b}, 1.4, 0.7, 0.2\rangle, \langle \mathfrak{c}, 1.3, 0.7, 0.4\rangle\} : \mathsf{e} \in \mathcal{E}\}$$

$$(\mathfrak{f}_1^{-1}(\mathcal{K}))^{\complement} = \{\mathsf{e}, \{\langle \mathfrak{a}, 1.1, 0.6, 0.7\rangle, \langle \mathfrak{b}, 0.2, 0.8, 1.4\rangle, \langle \mathfrak{c}, 0.4, 0.8, 1.3\rangle\} : \mathsf{e} \in \mathcal{E}\}$$

$$(\mathfrak{f}_1^{-1}(\mathcal{K}))^{\complement} \subseteq \mathcal{G}$$

$$\text{where,} \mathcal{G} = \{\circledast, \mathcal{J}\}$$

$$(\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})^{\complement} = \{\odot, \circledast, (\mathcal{J})^{\complement}\}$$

$$(\mathcal{J})^{\complement} = \{\mathsf{e}, \{\langle \mathfrak{a}, 0.3, 0.8, 1.4\rangle, \langle \mathfrak{b}, 1.1, 0.6, 0.6\rangle, \langle \mathfrak{c}, 0.5, 0.6, 1.2\rangle\} : \mathsf{e} \in \mathcal{E}\}$$

$$cl_{\tau^{\mathfrak{o}}_{\mathcal{N}_{\mathfrak{s}}}}(\mathfrak{f}^{\rightarrow}(\mathcal{K})^{\complement}) = \{\circledast\} \subseteq \mathcal{G}$$

$$\implies cl_{\tau^{\mathfrak{o}}_{\mathcal{N}_{\mathfrak{s}}}}(\mathfrak{f}^{\rightarrow}(\mathcal{K})^{\complement}) \subseteq \mathcal{G}$$

Then $\mathfrak{f}_1$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$-function.
$\therefore$ *$\mathfrak{f}_1^{-1}(\mathcal{K})$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset*
But $\mathfrak{f}_1^{-1}(\mathcal{K})$ is not $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ $\forall \mathcal{K} \in \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}}$
$\implies$ *$\mathfrak{f}_1$ is not a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$-function.*

Proposition 3.9. *Let $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ and $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ be any two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space. Let $\mathfrak{f} : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}}) \to (\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ is a Strongly Neutrosophic Over Soft Generalized Continuous function (strongly-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$-function) then $\mathfrak{f}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$-function.*

Proof. Let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset in $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$. Every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset is a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set
Now, $\mathcal{J}$ is a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set in $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$
Since $\mathfrak{f}$ is strongly-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$, $\mathfrak{f}^{\rightarrow}(\mathcal{J})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$
Hence, $\mathfrak{f}$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$-function. □

Remark 3.1. *Converse of proposition 3.9 need not be true(shown in example 3.2).*

Example 3.2. *Let $\mathcal{H} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ and $\mathcal{E} = \mathsf{e}$. Let $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathcal{K}$ and $\mathcal{L}$ in $\mathcal{H}$ as follows:*

$$\mathcal{K} = \{\mathsf{e}, \{\langle \mathfrak{a}, 1.4, 0.7, 0.2\rangle, \langle \mathfrak{b}, 1.3, 0.7, 0.4\rangle, \langle \mathfrak{c}, 0.7, 0.9, 1.1\rangle\}$$

$$\mathcal{L} = \{\mathsf{e}, \{\langle \mathfrak{a}, 1.4, 0.7, 0.2\rangle, \langle \mathfrak{b}, 0.7, 0.9, 1.1\rangle, \langle \mathfrak{c}, 1.3, 0.7, 0.4\rangle\}$$

Then two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topologies $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}3}} = \{\odot, \circledast, \mathcal{K}\}$ and $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}4}} = \{\odot, \circledast, \mathcal{L}\}$. Thus $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}3}})$ and $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}4}})$ are two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological spaces.
Define $\mathfrak{f}_2 : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}3}}) \to (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}4}})$ as $\mathfrak{f}_2(\mathfrak{a}) = \mathfrak{a}$,$\mathfrak{f}_2(\mathfrak{b}) = \mathfrak{c}$ and $\mathfrak{f}_2(\mathfrak{c}) = \mathfrak{b}$.
Then $\mathfrak{f}_2$ is a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$-function.
Let $\mathcal{M} = \{\mathsf{e}, \{\langle \mathfrak{a}, 1.4, 0.8, 0.1\rangle, \langle \mathfrak{b}, 0.8, 0.9, 1.1\rangle, \langle \mathfrak{c}, 1.3, 0.7, 0.4\rangle\}$ be a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}4}})$.
But $\mathfrak{f}_2^{\rightarrow}(M)$ is not an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}3}})$.
$\implies$ *$\mathfrak{f}_2$ is not a strongly-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$-function.*

Proposition 3.10. *Let $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$ and $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ be two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological spaces. If $\mathfrak{f} : (\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}}) \to (\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$ is a strongly-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$-function, then $\mathfrak{f}$ is a strongly-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$-function.*

Proof. Let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{so}}^{\mathfrak{og}}$-set in $(\mathcal{I}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}2}})$.

Since $\mathfrak{f}$ is a strongly-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}\mathcal{C}}$-function, it follows that $\mathfrak{f}^{\rightarrow}(\mathcal{J})$ is both an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-openset and a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-closedset in $(\mathcal{H}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}1}})$.

Hence, $\mathfrak{f}$ is a strongly-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{og}\mathcal{C}}$-function. □

Definition 3.5. *Let $\mathcal{T}_{\mathsf{i}}$ and $\mathcal{W}_{\mathsf{i}}$ be $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-sets. Where*

$$\mathcal{T}_{\mathsf{i}} = \{\langle \mathsf{h}, \aleph_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}), \eth_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}), \Upsilon_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{H}\}$$

$$\mathcal{W}_{\mathsf{i}} = \{\langle \mathsf{h}, \aleph_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h}), \eth_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h}), \Upsilon_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{H}\}$$

Tangent Similarity Measure :

$$\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = \frac{1}{n}\left(\sum_{\mathsf{i}=1}^{n} 1 - tan\left[\frac{\pi\left[|\aleph_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \aleph_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})| + |\eth_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \eth_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})| + |\Upsilon_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \Upsilon_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})|\right]}{12}\right]\right)$$

Absolute Difference :

$$\mathfrak{d}(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = \left[|\aleph_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \aleph_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})| + |\eth_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \eth_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})| + |\Upsilon_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \Upsilon_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})|\right]$$

Proposition 3.11. *Let $(\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}))$ for any two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-sets $\mathcal{T}_{\mathsf{i}}$ and $\mathcal{W}_{\mathsf{i}}$ over an non-empty set $\mathcal{H}$ .Then it satiesfies the properties*

(i) $0 \leq \rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) \leq 1$
(ii) $\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = 1$ *iff* $\mathcal{T}_{\mathsf{i}} = \mathcal{W}_{\mathsf{i}}$
(iii) $\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = \rho(\mathcal{W}_{\mathsf{i}}, \mathcal{T}_{\mathsf{i}})$
(iv) *For any $\mathcal{T}_{\mathsf{i}} \subseteq \mathcal{W}_{\mathsf{i}} \subseteq \mathcal{V}_{\mathsf{i}}$ then $\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{V}_{\mathsf{i}}) \leq \rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}})$ and $\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{V}_{\mathsf{i}}) \leq \rho(\mathcal{W}_{\mathsf{i}}, \mathcal{V}_{\mathsf{i}})$*

Proof. (i) As the sum of the membership, indeterminacy, and non-membership functions of the $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set lies within the interval [0, 3], and given that the similarity measure based on the tangent function also falls within [0, 1], hence conclude that the value of the tangent function is constrained between 0°to $\frac{\pi}{4}$.

To prove that $\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) \leq 1$

Substitute the lowest range for $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set is 0,

i.e., $\left[|\aleph_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \aleph_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})| + |\eth_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \eth_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})| + |\Upsilon_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \Upsilon_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})|\right] = 0$

then $\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = \frac{1}{n}\left(\sum_{\mathsf{i}=1}^{n} 1 - tan\left[\frac{\pi(0)}{12}\right]\right)$

$$\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = \frac{1}{n}\left(\sum_{\mathsf{i}=1}^{n} 1 - tan(0)\right)$$

$$\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = \frac{1}{n}\left(\sum_{\mathsf{i}=1}^{n} 1 - 0\right)$$

$$\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = 1$$

Substitute the highest range for $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set is 3,

i.e., $\left[|\aleph_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \aleph_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})| + |\eth_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \eth_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})| + |\Upsilon_{\mathcal{T}_{\mathsf{i}}}(\mathsf{h}) - \Upsilon_{\mathcal{W}_{\mathsf{i}}}(\mathsf{h})|\right] = 3$

then $\rho(\mathcal{T}_{\mathsf{i}}, \mathcal{W}_{\mathsf{i}}) = \frac{1}{n}\left(\sum_{\mathsf{i}=1}^{n} 1 - tan\left[\frac{\pi(3)}{12}\right]\right)$

$$\rho(\mathcal{T}_i,\mathcal{W}_i)=\frac{1}{n}\left(\sum_{i=1}^{n}1-tan\left[\frac{\pi}{4}\right]\right)$$
$$\rho(\mathcal{T}_i,\mathcal{W}_i)=\frac{1}{n}\left(\sum_{i=1}^{n}1-tan\left[\frac{\pi}{4}\right]\right)$$
$$\rho(\mathcal{T}_i,\mathcal{W}_i)=0$$

Hence $0\leq\rho(\mathcal{T}_i,\mathcal{W}_i)\leq 1$

(ii) For any two $\mathcal{N}^{\mathfrak{o}}_{\mathfrak{s}}$-sets $\mathcal{T}_i$ and $\mathcal{W}_i$ IF $\mathcal{T}_i=\mathcal{W}_i$ this implies
$\aleph_{\mathcal{T}_i}(h)=\aleph_{\mathcal{W}_i}(h)$,$\eth_{\mathcal{T}_i}(h)=\eth_{\mathcal{W}_i}(h)$,$\Upsilon_{\mathcal{T}_i}(h)=\Upsilon_{\mathcal{W}_i}(h)$
Then $|\aleph_{\mathcal{T}_i}(h)-\aleph_{\mathcal{W}_i}(h)|=0$,$|\eth_{\mathcal{T}_i}(h)-\eth_{\mathcal{W}_i}(h)|=0$,$|\Upsilon_{\mathcal{T}_i}(h)-\Upsilon_{\mathcal{W}_i}(h)|=0$
Thus $\rho(\mathcal{T}_i,\mathcal{W}_i)=1$
Conversely, $\rho(\mathcal{T}_i,\mathcal{W}_i)=1$
then,$|\aleph_{\mathcal{T}_i}(h)-\aleph_{\mathcal{W}_i}(h)|=0$,$|\eth_{\mathcal{T}_i}(h)-\eth_{\mathcal{W}_i}(h)|=0$,$|\Upsilon_{\mathcal{T}_i}(h)-\Upsilon_{\mathcal{W}_i}(h)|=0$
$\Longrightarrow \mathcal{T}_i=\mathcal{W}_i$

(iii) By the definition of (3.5),

$$\rho(\mathcal{T}_i,\mathcal{W}_i)=\frac{1}{n}\left(\sum_{i=1}^{n}1-tan\left[\frac{\pi\left[|\aleph_{\mathcal{T}_i}(h)-\aleph_{\mathcal{W}_i}(h)|+|\eth_{\mathcal{T}_i}(h)-\eth_{\mathcal{W}_i}(h)|+|\Upsilon_{\mathcal{T}_i}(h)-\Upsilon_{\mathcal{W}_i}(h)|\right]}{12}\right]\right)$$
$$=\frac{1}{n}\left(\sum_{i=1}^{n}1-tan\left[\frac{\pi\left[|-(\aleph_{\mathcal{W}_i}(h)-\aleph_{\mathcal{T}_i}(h))|+|-(\eth_{\mathcal{W}_i}(h)-\eth_{\mathcal{T}_i}(h))|+|-(\Upsilon_{\mathcal{W}_i}(h)-\Upsilon_{\mathcal{T}_i}(h))|\right]}{12}\right]\right)$$
$$\Longrightarrow\ \rho(\mathcal{T}_i,\mathcal{W}_i)=\rho(\mathcal{W}_i,\mathcal{T}_i)$$

(iv) By the definition of absolute difference (3.5)

$$\mathfrak{d}(\mathcal{T}_i,\mathcal{W}_i)=\left[|\aleph_{\mathcal{T}_i}(h)-\aleph_{\mathcal{W}_i}(h)|+|\eth_{\mathcal{T}_i}(h)-\eth_{\mathcal{W}_i}(h)|+|\Upsilon_{\mathcal{T}_i}(h)-\Upsilon_{\mathcal{W}_i}(h)|\right]$$

Since $\mathcal{T}_i\subseteq\mathcal{W}_i\subseteq\mathcal{V}_i$

$$\mathfrak{d}(\mathcal{T}_i,\mathcal{W}_i)\leq\mathfrak{d}(\mathcal{T}_i,\mathcal{V}_i)$$

Tangent function is increasing in the interval $[0,\frac{\pi}{4}]$.

$$tan\left[\frac{\pi\mathfrak{d}(\mathcal{T}_i,\mathcal{W}_i)}{12}\right]\leq tan\left[\frac{\pi\mathfrak{d}(\mathcal{T}_i,\mathcal{V}_i)}{12}\right]$$
$$1-tan\left[\frac{\pi\mathfrak{d}(\mathcal{T}_i,\mathcal{V}_i)}{12}\right]\leq 1-tan\left[\frac{\pi\mathfrak{d}(\mathcal{T}_i,\mathcal{W}_i)}{12}\right]$$

Then $\rho(\mathcal{T}_i,\mathcal{V}_i)\leq\rho(\mathcal{T}_i,\mathcal{W}_i)$
Similarly,

$$\mathfrak{d}(\mathcal{W}_i,\mathcal{V}_i)\leq\mathfrak{d}(\mathcal{T}_i,\mathcal{V}_i)$$

Tangent function is increasing in the interval $[0,\frac{\pi}{4}]$.

$$tan\left[\frac{\pi\mathfrak{d}(\mathcal{W}_i,\mathcal{V}_i)}{12}\right]\leq tan\left[\frac{\pi\mathfrak{d}(\mathcal{T}_i,\mathcal{V}_i)}{12}\right]$$
$$1-tan\left[\frac{\pi\mathfrak{d}(\mathcal{T}_i,\mathcal{V}_i)}{12}\right]\leq 1-tan\left[\frac{\pi\mathfrak{d}(\mathcal{W}_i,\mathcal{V}_i)}{12}\right]$$

Then $\rho(\mathcal{T}_i,\mathcal{V}_i)\leq\rho(\mathcal{W}_i,\mathcal{V}_i)$

□

4. Algorithm

Algorithm to solve tangent similarity of two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set as follows:

Step 1:Collection of data

Consider $\mathfrak{m}$ attributes $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_{\mathfrak{m}}$,$\mathfrak{n}$ replacement $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_{\mathfrak{n}}$ and $\mathfrak{h}$ replacement $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_{\mathfrak{h}}$
($\mathfrak{n} \leq \mathfrak{h}$) for multi-attributes decision making problem(MADMP).

$\mathcal{D}_{\mathfrak{h}} \backslash \mathcal{A}_{\mathfrak{i}}$	$\mathcal{A}_1$	$\mathcal{A}_2$	.	.	.	$\mathcal{A}_m$
$\mathcal{D}_1$	$\mathfrak{d}_{11}$	$\mathfrak{d}_{12}$	.	.	.	$\mathfrak{d}_{1m}$
$\mathcal{D}_2$	$\mathfrak{d}_{21}$	$\mathfrak{d}_{22}$	.	.	.	$\mathfrak{d}_{2m}$
$\mathcal{D}_3$	$\mathfrak{d}_{31}$	$\mathfrak{d}_{32}$	.	.	.	$\mathfrak{d}_{3m}$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
$\mathcal{D}_h$	$\mathfrak{d}_{p1}$	$\mathfrak{d}_{p2}$	.	.	.	$\mathfrak{d}_{pm}$

$\mathcal{A}_{\mathfrak{i}} \backslash \mathcal{C}_{\mathfrak{j}}$	$\mathcal{C}_1$	$\mathcal{C}_2$	.	.	.	$\mathcal{C}_n$
$\mathcal{A}_1$	$\mathfrak{a}_{11}$	$\mathfrak{a}_{12}$	.	.	.	$\mathfrak{a}_{1n}$
$\mathcal{A}_2$	$\mathfrak{a}_{21}$	$\mathfrak{a}_{22}$	.	.	.	$\mathfrak{a}_{2n}$
$\mathcal{A}_3$	$\mathfrak{a}_{31}$	$\mathfrak{a}_{32}$	.	.	.	$\mathfrak{a}_{3n}$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
$\mathcal{A}_m$	$\mathfrak{a}_{m1}$	$\mathfrak{a}_{m2}$	.	.	.	$\mathfrak{a}_{mn}$

Step 2:Calculation

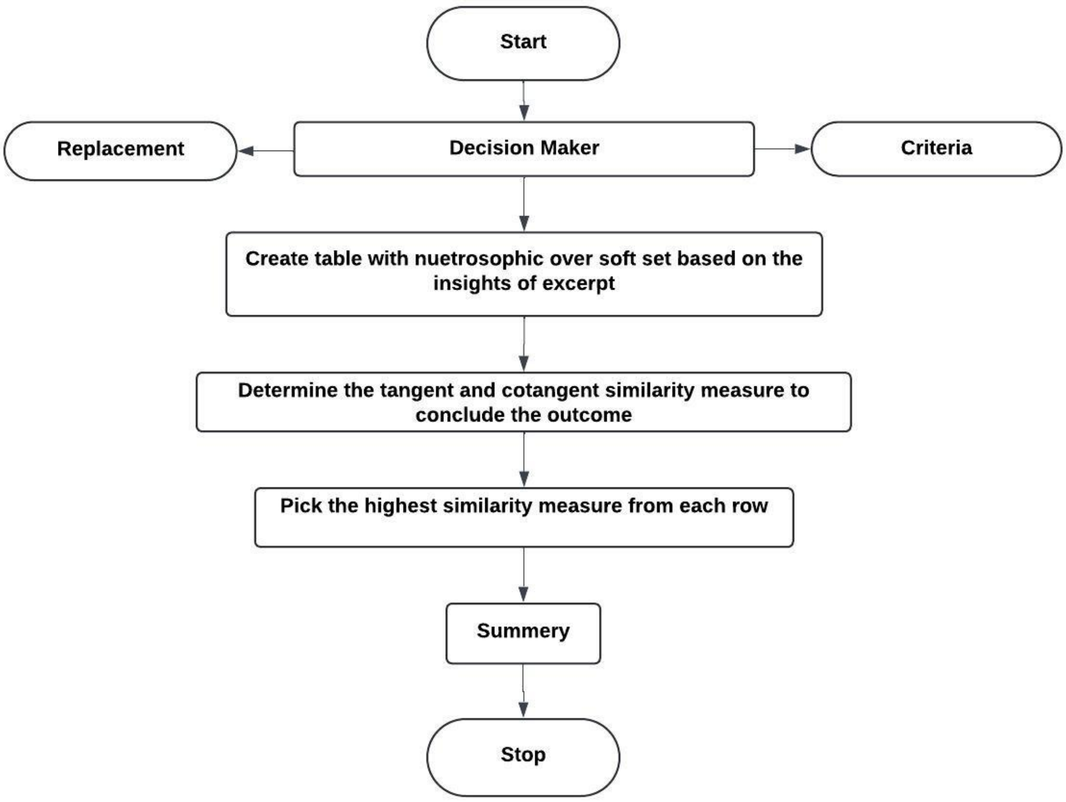
Calculating the tangent similarity measure with collected data.

Step 3:Final Decision

Pick the highest tangent measure from each row.

5. Flowchart

A flowchart visually represents an algorithm's steps in sequential order it simplifies complex processes, making the algorithm easier to understand and follow.



6. Numerical application

Selecting the right machine for competition requires a systematic approach. Firstly, understanding the competition's requirements and setting clear objectives is crucial. This forms the basis for decision-making. Analyzing past competitions provides valuable insights and helps avoid common pitfalls. Assessing team resources such as expertise, budget, and time aids in determining feasible approaches. When evaluating machine types, factors like performance, reliability, and safety are paramount to align with competition goals. Prototyping, testing, and iterating on the design are essential for refining performance. Planning for contingencies and remaining adaptable are key strategies for navigating challenges. Diligently following these steps maximizes the team's chances of success.

Consider similar situation arises in a company to participate in Best Invention Competition. For that they produced three types of machines. To take a good decision regarding the best one among three machines.

Machine=$\{\mathfrak{M}1, \mathfrak{M}2, \mathfrak{M}3\}$,**Criteria**=$\{$Performance, Reliability, and Safety$\}$
Position=$\{$Selected,Non-Selected$\}$

Table 1. Relation between Machine and criteria

$\breve{\mathbf{Y}}$	**Performance**	**Reliability**	**Safety**
$\mathfrak{M}1$	(1.8,0.3,0.1)	(1.5,0.4,0.1)	(1.6,0.4,0.1)
$\mathfrak{M}2$	(1.7,0.3,0.2)	(0.7,1.4,0.0)	(0.7,0.3,1.2)
$\mathfrak{M}3$	(0.8,1.02,1.02)	(0.9,1.1,0.1)	(0.9,0.1,1.2)

Table 2. Relation between criteria and position

$\breve{\mathbf{W}}$	**Non-Selected**	**Selected**
Performance	(1.7,0.1,0.3)	(0.5,0.3,1.3)
Reliability	(1.6,0.3,0.2)	(0.6,0.3,1.2)
Safety	(0.7,1.2,0.1)	(0.7,1.3,0.1)

Table 3. Tangent Similarity Measure

ρ	**Non-Selected**	**Selected**
$\mathfrak{M}1$	**0.9963**	0.9903
$\mathfrak{M}2$	**0.9933**	0.9898
$\mathfrak{M}3$	0.9902	**0.9909**

Table 4. Summary

Machine	**Result**
$\mathfrak{M}1$	Not selected
$\mathfrak{M}2$	Not selected
$\mathfrak{M}3$	Selected

7. Conclusions

This study delves into the application of tangent similarity measures within the framework of Neutrosophic Over Soft Sets , offering a comprehensive exploration of their significance. Additionally, it introduces the concept of Neutrosophic Over Soft Generalized Closed Sets and Neutrosophic Over Soft Generalized Continuous Functions, providing clear definitions and propositions to elucidate their implications. Furthermore, it presents a numerical case study focusing on the selection process for the Best Invention Competition, wherein the analysis demonstrates that the machine denoted as $\mathfrak{M}_3$ emerges as the optimal choice, meeting the specified criteria effectively. By highlighting its applicability beyond theoretical realms, the manuscript underscores the practical significance of employing such measures in real-world scenarios, thereby enhancing decision-making processes and problem-solving methodologies. The manuscript highlights the versatility of the $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set correlation measure across diverse domains, such as medicine, industry, and construction. In medicine, it aids in evaluating patient outcomes, while in industry, it optimizes processes. In construction, it enhances project management by assessing risks and ensuring quality control. This broad applicability underscores its potential as a powerful analytical tool in various environments.

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