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A STUDY ON UPPER DEG-CENTRIC GRAPHS

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Abstract. The upper deg-centric graph of a simple, connected graph G, denoted by G_{ud}, is a graph constructed from G such that $V(G_{ud}) = V(G)$ and $E(G_{ud}) = \{v_iv_j : d_G(v_i, v_j) \geq deg_G(v_i)\}$. This paper introduces and discusses the concepts of upper deg-centric graphs and iterated upper deg-centrication of a graph.

Keywords: Distance, eccentricity, deg-centric graphs, upper deg-centric graphs, upper deg-centrication process.

AMS Subject Classification: 83-02, 99A00

1. Introduction

For a basic terminology of graph theory, we refer to [3, 1]. For further topics on graph classes, [11, 7, 10]. All graphs discussed in this paper are finite, simple, connected, and undirected. Without loss of generality, the vertex set of a graph G of order n will be $V(G) = \{v_i : 1 \leq i \leq n\}$. The order and size of G are denoted by $|G|$ and $\varepsilon(G)$, respectively. Recall that the distance between two distinct vertices v_i and v_j of G, denoted by $d_G(v_i, v_j)$, is the length of the shortest path joining them. The eccentricity of a vertex $v_i \in V(G)$, denoted by $e(v_i)$, is the furthest distance from v_i to some vertex of G. Vertices at a distance $e(v_i)$ from v_i are called the eccentric vertices of v_i. An *eccentric graph* of a graph G denoted by G_e, is obtained from the same set of vertices as G with two vertices v_i and v_j being adjacent in G_e if and only if v_j is an eccentric vertex of v_i or v_i is an eccentric vertex of v_j (see[1, 2]). The *iterated eccentric graph* of G, denoted by G_{e^k}, is defined in [8], as the derived graph obtained by taking the eccentric graph successively k-times; that is, $G_{e^k} = ((G_e)_e \ldots)_e$, ($k$-times).

The *degree centric graph* or *deg-centric graph* of G is the graph G_d with $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \leq deg_G(v_i)\}$ (see[4]). Let G be a graph and G_d be the deg-centric graph of G. Then, the *successive iteration deg-centric graph* of G, denoted by G_{d^k}, is defined as the derived graph obtained by taking the deg-centric graph successively k times; that is $G_{d^k}= ((G_d)_d...)_d$, (k-times). This process is known as *deg-centrication*

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process (see[4]). The *exact degree centric graph* or *exact deg-centric graph* of a graph G and denoted by G_{ed}, is the graph with $V(G_{ed}) = V(G)$ and $E(G_{ed}) = \{v_iv_j : d_G(v_i, v_j) = deg_G(v_i)\}$. This graph transformation is called exact deg-centrication (see[5]). Let G be a graph and G_{ed} be the exact deg-centric graph of G. Then, the iterated *exact deg-centric graph* of G, denoted by G_{ed^k}, is defined as the graph obtained by applying *exact deg-centrication* successively k-times; That is, $G_{ed^k} = ((G_{ed})_{ed} \ldots)_{ed}$, ($k$-times) (see[5]). The *coarse degree centric graph* or *coarse deg-centric graph* of a graph G, denoted by G_{cd}, is the graph with $V(G_{cd}) = V(G)$ and $E(G_{cd}) = \{v_iv_j : d_G(v_i, v_j) > deg_G(v_i)\}$. Then the iterated *coarse deg-centric graph* of G, denoted by G_{cd^k}, is defined as the graph obtained by applying *coarse deg-centrication* successively k-times; That is, $G_{cd^k} = ((G_{cd})_{cd} \cdots)_{cd}$, ($k$-times) (see[6].

Motivated by the studies mentioned above, in this paper, we introduce a new class of transformed graphs, called the upper deg-centric graphs, and investigate the properties and structural characteristics of this type of transformed graph.

2. Upper Deg-centric Graphs

Definition 2.1. The *upper degree centric graph* or *upper deg-centric graph* of a graph G, denoted by G_{ud}, is the graph with $V(G_{ud}) = V(G)$ and $E(G_{ud}) = \{v_iv_j : d_G(v_i, v_j) \geq deg_G(v_i)\}$. This graph transformation is called *upper deg-centrication.* Note that this process is independent of the choice of v_i or v_j in the above sets.

The upper deg-centric graph of cycle C_7 is given in Figure 1b for illustration.

The upper deg-centric graph G_{ud} of a graph G need not be a connected graph (For illustration, see Figure1(c)).

Observation 2.1. The upper deg-centric graph of a complete graph K_n of order $n \neq 2$ is an empty graph $\overline{K}_n$.

Observation 2.2. If there exists a vertex $v_i \in V(G)$ with $deg_G(v_i) > e_G(v_i)$, then v_i cannot initiate an edge in G_{ud}.

Lemma 2.1. *The upper deg-centric graph of a graph G is an empty graph if and only if $\delta(G) > diam(G)$.*

Proof. Assume that $\delta(G) > diam(G)$. If any vertex, say v_i, initiates at least one edge, say v_iv_j, it implies that $deg_G(v_i) \geq d_G(v_i, v_j)$. Subsequently, the edge v_iv_k, where $e(v_i) = d_G(v_i, v_k)$, must be formed as well. The aforesaid implies that either $deg_G(v_i) < \delta(G)$ or $e(v_i) > diam(G)$. In both cases, we have a contradiction.

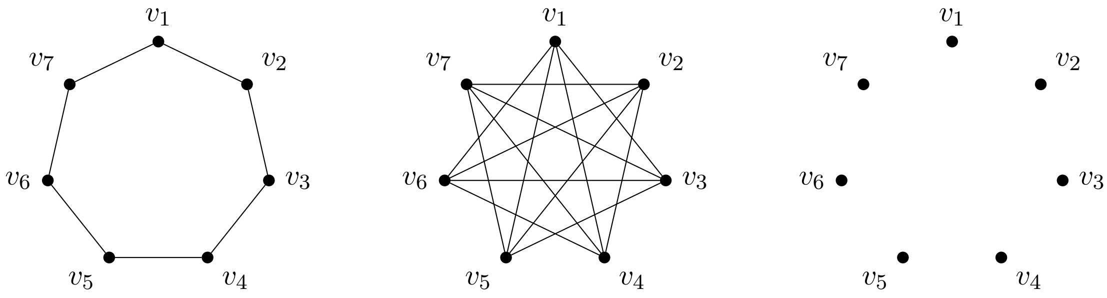
Conversely, assume that $G_{ud} = \overline{K}_n$. In turn, it implies that for each vertex v_i, $deg_G(v_i) > e_G(v_i)$. Therefore, any vertex v_j with $d_G(v_j) = \delta(G)$ has $e_G(v_j) \geq \delta(G)$. This settles the result. □

Definition 2.2. Let G be a graph and G_{ud} be the upper deg-centric graph of G. Then, the *iterated upper deg-centric graph* of G, denoted by G_{ud^k}, is defined as the graph obtained by applying *upper deg-centrication* successively k-times. That is, $G_{ud^k} = ((G_{ud})_{ud} \cdots)_{ud}$, ($k$-times).

The upper deg-centrication process of the cycle C_7 is given in Figure 1.

Lemma 2.2. *For a graph G of order n, which has at least one pendant vertex. Then, any pendant vertex of G will be a universal vertex of G_{ud}.*

Proof. The result is a direct consequence of Definition 2.1. □



(A) C_7 (B) $(C_7)_{ud}$ (C) $(C_7)_{ud^2}$

FIGURE 1. The upper deg-centrication process of C_7.

Theorem 2.1. *Consider a graph G of order $n \neq 2$. If G_{ud} does not have a K_2 component, then the iterated upper deg-centric graph G_{ud^k}, $1 \leq k \leq 2$ is the empty graph $\overline{K}_n$.*

Proof. Note that should one or more trivial graphs K_1 result in G_{ud}, each remains an empty component. On the other hand, a component K_2 remains connected through iterated upper deg-centrication. Therefore, the order to be considered is $n \neq 2$. If $G \equiv K_n$, then $G_{ud} = \overline{K}_n$. Thus $k = 1 < 2$.

Assume that G is not a complete graph. Since $diam(G)$ is finite there exists a pair of vertices say, v_i, v_j such that $d_G(v_i, v_j) = diam(G) = e(v_i)$. Hence, in G_{ud}, the degrees of v_i and v_j have increased and the respective eccentricity decreased. Hence, from Definition 2.1, the vertices v_i, v_j will be isolated vertices in G_{ud^2}. Similar reasoning is valid between all pairs of vertices. Hence, the result is settled by mathematical induction. □

For convenience, a path P_n is depicted on a horizontal line, and the vertices are labelled from left to right as v_1, v_2, v_3,..., v_n.

Proposition 2.1. *Consider a path P_n, $n \geq 4$. If $V = \{v_1, v_2, v_3, \ldots, v_n\}$ is the vertex set of the upper deg-centric graph, then we have*

(a) The vertices v_2, v_{n-1} have a degree of $n-2$.

(b) The vertices $v_3, v_4, \ldots, v_{n-3}, v_{n-2}$ have a degree of $n-3$.

Proof. (a) Consider the vertices v_2 and v_{n-1}. Both $deg(v_2) = deg(v_{n-1}) = 2$, in the path and hence according to Definition 2.1, both v_2 and v_{n-1} will incident an edge to all other vertices except to their respective neighbours. Therefore each of v_2 and v_{n-1} forms exactly $n-3$ edges. However, since v_1 and v_n respectively formed an extra edge, it follows that $deg(v_2) = deg(v_{n-1}) = n-2$ in $(P_n)_{ud}$.

(b) Consider the vertices $v_3, v_4, \ldots, v_{n-3}, v_{n-2}$. By Definition 2.1, each vertex is adjacent to all vertices except their respective neighbours. Hence, the result holds.

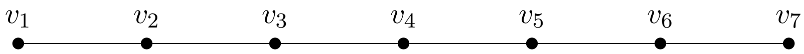
□

An illustration of Proposition 2.1 is given in Figure 2.

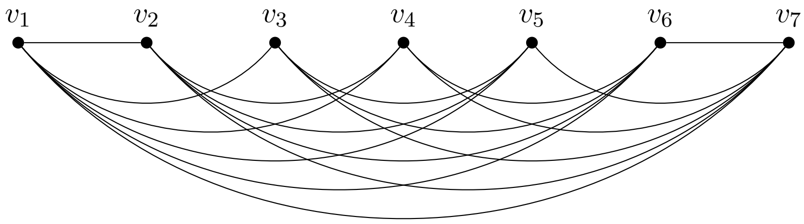
Corollary 2.1. *For a path P_n, $n \geq 4$, $\varepsilon((P_n)_{ud}) = \frac{n^2-3n+6}{2}$.*

Proof. By Lemma 2.2, the degree of each of v_1, v_n is $n-1$. In view of the results above, together with the results of Proposition 2.1, in the well-known formula,

$$\varepsilon(p_n) = \frac{1}{2} \sum_{v_i \in V(p_n)} deg(v_i)$$



(A) P_7



(B) $(P_7)_{ud}$

FIGURE 2. Upper deg-centric graph of P_7.

yields the result. □

A non-trivial *bistar graph*, denoted by $S_{a,b}$, is a graph obtained by joining the centers of two non-trivial star graphs $K_{1,a}$, $a \geq 1$ and $K_{1,b}$, $b \geq 1$ with the edge v_0u_0.

Proposition 2.2. *For $a, b \geq 1$,*

$$\varepsilon((S_{a,b})_{ud}) = \binom{a+b+2}{2} - 1.$$

Proof. Note that all pendant vertices of $S_{a,b}$ will be adjacent to all other vertices in the upper deg-centric graph,$(S_{a,b})_{ud}$. Also, the central vertices of $S_{a,b}$ cannot be adjacent to each other in the upper deg-centric graph since they are at a distance of one and their degree greater than one. Therefore the upper deg-centric graph of $S_{a,b}$ is isomorphic to $K_{a+b+2} - \{u_0, v_0\}$. Hence,

$$\varepsilon((S_{a,b})_{ud}) = \binom{a+b+2}{2} - 1.$$

□

Proposition 2.3. *For a cycle C_n, $n \geq 5$, the upper deg-centric graph, $(C_n)_{ud}$ is always a $(n-3)$-regular graph.*

Proof. Because $deg_{C_n}(v_i) = 2$, for all $v_i \in V(C_n)$, any vertex v_i in $(C_n)_{ud}$ is adjacent to all vertices in $V(C_n) \backslash N_{C_n}[v_i]$. It immediately follows that $(C_n)_{ud}$ is always a $(n-3)$-regular graph. □

A *wheel graph*, denoted by $W_{1,n}$, $n \geq 3$, is obtained by taking a cycle C_n, $n \geq 3$ (the rim with rim-vertices) and adding the central vertex v_0 with *spokes* namely, edges v_0v_i, $1 \leq i \leq n$. Note that, in view of Lemma 2.1, the upper deg-centric graph of a wheel graph $W_{1,n}$ is the empty graph $\overline{K}_{n+1}$. Since minimum degree, $\delta(W_{1,n}) > diam(W_{1,n})$.

Recall that the sequence of the *second pentagonal numbers* denoted by p_n is generated by $p_n = \frac{n(3n+1)}{2}$, $n = 0, 1, 2, \ldots$. This sequence is: $0, 2, 7, 15, 26, 40, 57, 77, 100, 126, 155, 187, \ldots$

The relation between the size of the upper deg-centricated Helm graphs and the second pentagon numbers follows immediately as a proposition.

A *helm graph*, denoted by $H_{1,n}$, $n \geq 3$, is a graph obtained from a wheel graph $W_{1,n}$ by attaching a pendant vertex u_i to the corresponding rim vertex v_i (see Figure 3 for illustration).

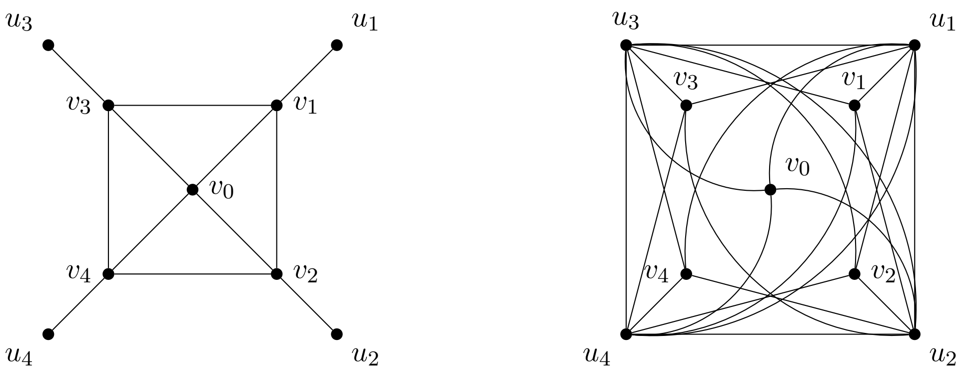
Proposition 2.4. *For $n \geq 3$, $\varepsilon((H_{1,n})_{ud}) = p_n$.*

Proof. Note that the helm graph $H_{1,n}$, $n \geq 3$ is of the order $2n+1$. Let $V(H_{1,n}) = \{v_0, v_1, v_2, \dots, v_n, \underbrace{u_1, u_2, \dots, u_n}_{pendant\ vertices}\}$. Since $deg(v_0) = n > e(v_0) = 2$ no edge forms from v_0 in the upper deg-centric graph,$(H_{1,n})_{ud}$. Also, since each $deg_{H_{1,n}}(v_i) = 4 > e(v_i)$, no edge forms from a v_i in $(H_{1,n})_{ud}$. However, since all u_i are pendant vertices, each u_i forms the edge u_iv_i in the upper deg-centric graph. In view of Lemma 2.2, the n pendant vertices $u_1, u_2, \dots, u_n$ are adjacent to all other $2n$ vertices that is $deg(u_n) = 2n$ in $(H_{1,n})_{ud}$. All other $n+1$ vertices are adjacent with each u_n hence, $deg(v_n) = n$ in $(H_{1,n})_{ud}$. Finally,

$$\varepsilon((H_{1,n})_{ud}) = \frac{\sum\limits_{w_i \in V((H_{1,n})_{ud})} deg(w_i)}{2} = \frac{n(2n) + (n+1)(n)}{2} = \frac{n(3n+1)}{2}.$$

□

An illustration to Proposition 2.4 is given in Figure 3.



(A) $H_{1,4}$ (B) $(H_{1,4})_{ud}$

FIGURE 3. Upper deg-centric graph of $H_{1,4}$.

A *closed helm graph*, denoted by $CH_{1,n}$, $n \geq 3$, is the graph obtained from a helm graph $H_{1,n}$ by joining the pendant vertices, in order, forming a cycle, called the outer rim.

Proposition 2.5. *For $n \geq 3$,*

$$\varepsilon((CH_{1,n})_{ud}) = \begin{cases} n(n-3) & \text{if } n = 3, 4, 5. \\ \frac{1}{2}n(3n-11) & \text{if } n \geq 6. \end{cases}$$

Proof. Note that $diam(CH_{1,3}) = 2$ and $\delta(CH_{1,3}) = 3$ and hence its upper deg-centric graph is empty. Also, $diam(CH_{1,4}) = 3$ and $\delta(CH_{1,4}) = 3$. It is easy to see that, $d_{CH_{1,4}}(u_1, v_3) = 3 = diam(CH_{1,4})$. Hence, by Definition 2.1, each outer-rim vertex forms one edge. Therefore, $\varepsilon((CH_{1,4})_{ud}) = 4$. For $CH_{1,5}$ we have $diam(CH_{1,5}) = 3$ and $\delta(CH_{1,3}) = 3$. It is easy to see that, $d_{CH_{1,5}}(u_1, v_3) = d_{CH_{1,5}}(v_4) = 3 = diam(CH_{1,5})$. By Definition 2.1, each outer-rim vertex forms two edges. Therefore, $\varepsilon((CH_{1,5})_{ud}) = 10$.

Consider $CH_{1,n}$, $n \geq 6$. In view of the reasoning in Part (a), it follows that each outer-rim vertex forms $(n-3)$ edges to vertices on the inner-rim. With regards to the outer-rim, any vertex u_i forms $(n-5)$ edges to outer-rim vertices. Altogether, $2n-8$ such edges will incident to the outer rim vertices in the upper deg-centric graph. Therefore, a total of $\frac{n(n-3)+(n-5)}{2}$ edges are formed to obtain $(CH_{1,n})_{ud}$. In inner rim vertices, $n-3$ edges form in the upper deg-centric graphs. Therefore, a total of $\frac{n(n-3)}{2}$ edges are formed to obtain $(CH_{1,n})_{ud}$. The aforesaid yield the result,

$$\varepsilon((CH_{1,n})_{ud}) = \frac{n[(n-3)+(n-5)]}{2} + \frac{n(n-3)}{2} = \frac{1}{2}n(3n-11).$$

□

If the edge v_1v_3 joins vertices v_1 and v_3, then the *subdivision* of v_1v_3 replaces v_1v_3 by a new vertex v_2 and two new edges v_1v_2 and v_2v_3. A *gear graph*, denoted by G_n, $n \geq 3$, is a graph obtained by applying subdivision to each edge of the rim of a wheel graph $W_{1,n}$.

Proposition 2.6. *For $n \geq 3$, $\varepsilon((G_n)_{ud}) = \frac{3}{2}n(n-1)$.*

Proof. For a gear graph G_n, $n \geq 3$, is of the order $2n+1$. Let $V(G_n) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $deg_{G_n}(v_0) = n \geq 3 > e(v_0) = 2$, no edges formed from v_0 in $(G_n)_{ud}$. However, since $deg(v_i) = 3$, there are $n-2$ edges incident on any vertex v_i and since $deg(u_i) = 2$, there are $2n-2$ edges incident on any vertex u_i in $(G_n)_{ud}$. Then, $deg_(v_i) = n-2$, $deg(u_i) = 2n-2$ and $deg(v_0) = n$ in $(G_n)_{ud}$. Finally,

$$\varepsilon((G_n)_{ud}) = \frac{\sum\limits_{w_i \in V((G_n)_{ud})} deg(w_i)}{2} = \frac{n(n-2)+n(2n-2)+n}{2} = \frac{3}{2}n(n-1).$$

□

Corollary 2.2. *The gear graph G_n, $n \geq 3$ has the following properties:*

(a) $(G_n)_{ud}$ is a split graph with a clique of order $n+1$.

(b) $(G_n)_{ud}$ is connected.

(c) $diam((G_n)_{ud}) = 3$.

Proof. For a gear graph G_n, $n \geq 3$ is of the order $2n+1$. Let $V(G_n) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$.

(a) Since $deg(u_i) = 2$ in G_n, it follows by Definition 2.1, that each vertex u_i as a consequence of $2n-2$ edges. All vertices u_i together with vertex v_0 are pairwise adjacent. Hence, $K_{n+1} \subset (G_n)_{ud}$. In other words, $(G_n)_{ud}$ contains a complete graph K_{n+1}. Since $deg(v_i) = 3$ in G_n, for all i and $d_{G_n}(v_i, v_j) = 2$, $i \neq j$ it follows that $\{v_i : 1 \leq i \leq n\}$ is a non-empty independent set in $(G_n)_{ud}$. However, each v_i is adjacent to some u_j. Therefore, $(G_n)_{ud}$ is a split graph (a graph that can be partitioned into a clique and an independent set) with a clique of order $n+1$.

(b) From Part (i), it follows that $(G_n)_{ud}$ is connected.

(c) Since $(G_n)_{ud}$ is a connected split graph, $1 \leq diam((G_n)_{ud}) \leq 3$. Further, the non-empty independent set contains all v_i, $1 \leq i \leq n$, $n \geq 3$. Thus, there exists at least one pair of vertices v_i, v_j such that $d_{(G_n)_{ud}}(v_i, v_j) = 3$. The aforesaid implies that $diam((G_n)_{ud}) = 3$.

□

A *web graph*, denoted by $Wb_{1,n}$, $n \geq 3$, is the graph obtained by attaching a pendant vertex w_i to each corresponding vertex u_i of the outer cycle (or rim) of the closed helm graph $CH_{1,n}$.

Proposition 2.7. *The upper deg-centric graph of a web graph $Wb_{1,n}$,$n \geq 3$, contains n vertices with degrees $3n$ and $2n+1$ vertices with degrees n.*

Proof. Note that the web graph $Wb_{1,n}$, is of the order $3n+1$. Let $V(Wb_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_{n-1}, v_n, u_1, u_2, u_3, \ldots, u_n, \underbrace{w_1, w_2, w_3, \ldots, w_n}_{\text{pendant vertices}}\}$. Since $deg(v_0) = n > e(v_0) = 3$, no edge incident with v_0 in upper deg-centric graph. Also, since each $deg(v_i) = 4 > e(v_i)$ no edge incident from a v_i in $(Wb_{1,n})_{ud}$. Similarly, no edge incident from any vertex u_i in $(Wb_{1,n})_{ud}$. However, since all w_i is an end vertex, each w_i forms the edges w_iu_j, $1 \leq j \leq n$ as well as w_iv_j, $1 \leq j \leq n$ as well as the edge w_iv_0 and finally, in a commutative fashion, the edges w_iw_j, $1 \leq j \leq n$, $i \neq j$. The summation of incident edges yields the result. □

Proposition 2.8. *For $n \geq 3$,*

$$\varepsilon((Wb_{1,n})_{ud}) = \frac{n}{2}(5n+1)$$

.

Proof. By Lemma 2.2, the degree of each of $w_1, w_2, w_3 \ldots, w_n$ is $n-1$. Utilising those above together with the results of Proposition 2.7 in the well-known formula,

$$\varepsilon(Wb_{1,n}) = \frac{1}{2} \sum_{v_i \in V(Wb_{1,n})} deg(v_i)$$

yields the result.

□

A *double wheel* DW_n is obtained by taking two copies of a wheel W_n, $n \geq 3$, and *merging* the two central vertices. Note that, given Lemma 2.1, the upper deg-centric graph of a double wheel DW_n, $n \geq 3$ is the empty graph $\overline{K}_{2n+1}$.

A *flower graph*, $F_{1,n}$, $n \geq 3$ is a graph obtained from a helm graph $H_{1,n,}$ by joining each of its pendant vertices u_i's to its central vertex v_0.

Proposition 2.9. *For $n \geq 3$, $\varepsilon((F_{1,n})_{ud}) = \frac{3}{2}n(n-1)$.*

Proof. Recall that the flower graph $F_{1,n,}$, $n \geq 3$ is of the order $2n+1$. Let $V(F_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $deg(v_0) = n > e(v_0) = 2$ in $F_{1,n}$, no edge forms from v_0. That is, $deg(v_0) = 0$ in $(F_{1,n})_{ud}$. Also, since each $deg(v_i) = 4 > e(v_i)$ in $F_{1,n}$, no edge exists with a v_i as its end vertex, in $(F_{1,n})_{ud}$. However, since $deg(u_i) = 2$ in $F_{1,n}$, each u_i is adjacent to the vertices with at least a distance of 2 from it. In view of Definition 2.1, the n vertices $u_1, u_2, \ldots, u_n$ are adjacent with $2n-2$ vertices that is, $deg(u_n) = 2n-2$ in $(F_{1,n})_{ud}$. Also, center vertex v_0 is non adjacent with all vertices that is, $deg(v_0) = 0$ in $(F_{1,n})_{ud}$. and $v_1, v_2, v_3 \ldots v_n$ are adjacent with $n-1$ vertices and hence $deg(v_n) = n-1$ in $(F_{1,n})_{ud}$. Finally,

$$\varepsilon((F_{1,n})_{ud}) = \frac{1}{2} \sum_{w_i \in V((F_{1,n})_{ud})} deg_(w_i) = \frac{n(2n-2) + n(n-1)}{2} = \frac{3}{2}n(n-1).$$

□

The *sunflower graph*, denoted by $SF_{1,n}$, $n \geq 3$ is obtained from the wheel $W_{1,n}$ by attaching n vertices u_i, $1 \leq i \leq n$ such that each u_i is adjacent to v_i and v_{i+1} and count the suffix is taken modulo n.

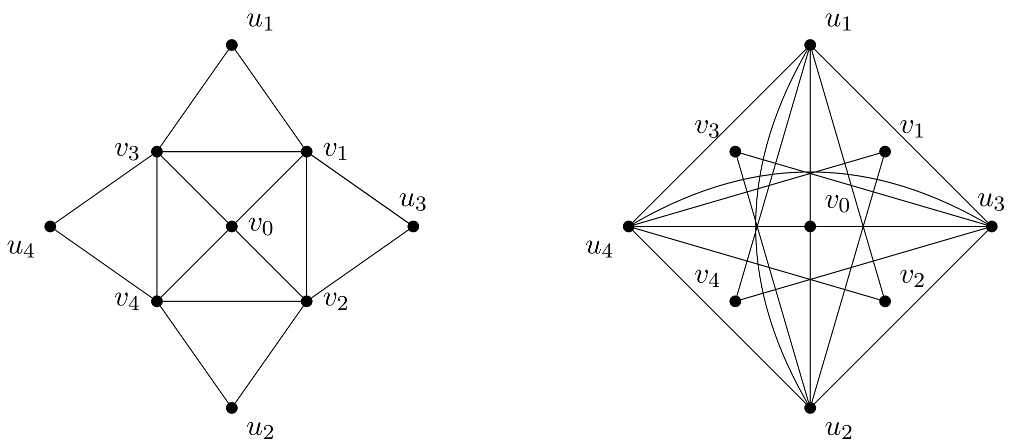
Proposition 2.10. *For $n \geq 3$, $\varepsilon((SF_{1,n})_{ud}) = \frac{3}{2}n(n-1)$.*

Proof. The sunflower graph $SF_{1,n,}$, $n \geq 3$, is of the order $2n+1$. Let $V(SF_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ as mentioned in the definition. Since $deg(v_0) = n > e(v_0) = 2$ in $SF_{1,n}$, no edge forms from v_0 in $(SF_{1,n})_{ud}$. Also, since each $deg(v_i) = 5 > e(v_i)$ in $SF_{1,n}$, no edge forms from a v_i in $(SF_{1,n})_{ud}$. However, since $deg(u_i) = 2$ in $SF_{1,n}$, each u_i forms the edge with at least a distance of two vertices from u_i in $(SF_{1,n})_{ud}$. In view of Definition 2.1, the n vertices $u_1, u_2, \ldots, u_n$ are adjacent with all other $2n-2$ vertices that is $deg(u_n) = 2n-2$ in $(SF_{1,n})_{ud}$. Also center vertex v_0 is adjacent with all u_i that is $deg(v_0) = n$ in $(SF_{1,n})_{ud}$ and $v_1, v_2, v_3 \ldots v_n$ are adjacent with $n-2$ vertices hence, $deg(v_n) = n-2$ in $(SF_{1,n})_{ud}$. Thus, we have

$$\varepsilon((SF_{1,n})_{ud}) = \frac{\sum\limits_{w_i \in V((SF_{1,n})_{ud})} deg(w_i)}{2} = \frac{n(2n-2) + n(n-2) + n}{2} = \frac{3}{2}n(n-1).$$

□

An illustration to Proposition 2.10 is given in Figure 4.



(A) $SF_{1,4}$ (B) $(SF_{1,4})_{ud}$

FIGURE 4. Upper deg-centric graph of $SF_{1,4}$.

A *closed sunflower graph* $CSF_{1,n}$ is obtained by adding the edges $u_iu_{i+1}, 1 \leq i \leq n$, to the sunflower graph.

Proposition 2.11. *The upper deg-centric graph of a closed sunflower graph $CSF_{1,n}$, $n \geq 3$. Then,*

$$\varepsilon((CSF_{1,n})_{ud}) = \begin{cases} 0 & \text{if } 3 \leq n \leq 7. \\ \frac{n^2-7n}{2} & \text{if } n \geq 8. \end{cases}$$

Proof. (a) If $3 \leq n \leq 7$, then the result is a direct consequence of Lemma 2.1.

(b) If $n \geq 8$, for a closed sunflower graph $CSF_{1,n}$. Clearly, the closed sunflower graph is of the order $2n+1$. Let $V(CSF_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since

$deg(v_0) = n > e(v_0) = 2$ in $CSF_{1,n}$, no edge forms from v_0 in upper deg-centric graph. Also, since each $deg(v_i) = 5 > e(v_i)$ in $CSF_{1,n}$, no edge forms from a v_i in upper deg-centric graph. However, since $deg(u_i) = 4$ in $CSF_{1,n}$, each u_i forms the edge with at least a distance of four vertices from u_i in $(CSF_{1,n})_{ud}$. By Definition 2.1, the n vertices $u_1, u_2, \ldots, u_n$ are adjacent with $n-7$ vertices that is $deg(u_n) = n-7$ in $(CSF_{1,n})_{ud}$. Finally,

$$\varepsilon((CSF_{1,n})_{ud}) = \frac{\sum\limits_{w_i \in V((CSF_{1,n})_{ud})} deg_(w_i)}{2} = \frac{n(n-7)}{2} = \frac{n^2-7n}{2}.$$

□

A *blossom graph*, denoted by $Bl_{1,n}$, is obtained by making each u_i adjacent to the central vertex of the closed sunflower graph. In view of Lemma 2.1, the upper deg-centric graph of a blossom graph Bl_n, $n \geq 3$, is the empty graph $\overline{K}_{2n+1}$.

Recall that a complete bipartite graph $K_{n,m}$, $n, m \geq 1$ is a graph whose vertex set can be partitioned into two independent sets X, $|X| = n$ and Y, $|Y| = m$ and each vertex in X is adjacent to all vertices in Y.

Proposition 2.12. *For a complete bipartite graph $K_{2,m}$, $m \geq 3$. Then, the upper deg-centric graph is the disjoint union of the empty graph $\overline{K}_2$ and the complete graph K_m.*

Proof. Consider a complete bipartite graph $K_{2,m}$, $m > 2$. Clearly, $K_{2,m}$ is a graph whose vertex set can be partitioned into two independent sets X, $|X| = 2$ and Y, $|Y| = m$. Let $X = v_1, v_2$, and $Y = u_{1,2}, \ldots, v_m,$. In accordance with Definition 2.1 construct $(K_{2,m})_{ud}$ as follows: since $deg_{K_{2,m}}(u_i) = 2$ and all pairs of vertices u_i, u_j have $d_{K_{2,m}}(u_i, u_j) = 2$ set Y yields a complete graph K_m. Clearly, set X yields the empty graph $\overline{K}_2$. Hence, the upper deg-centric graph is the disjoint union of the empty graph $\overline{K}_2$ and the complete graph K_m. □

Proposition 2.13. *Let G be a complete bipartite graph $K_{n,m}$, $n, m \geq 3$. Then, the upper deg-centric graph is the empty graph $\overline{K}_{n+m}$.*

Proof. The result is a direct consequence of Lemma 2.1. □

A tree denoted by T_n, $n \geq 1$ is a connected acyclic graph. It is known that a tree T_n has $n-1$ edges.

Proposition 2.14. *If $n \geq 3$, T_n, then in the upper deg-centric graph a vertex v_i has $deg_{(T_n)_{ud}}(v_i) \geq 2$.*

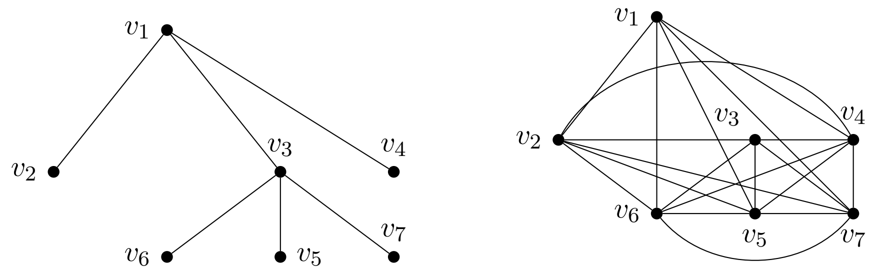
Proof. It is known that a tree T_n of order $n \geq 3$ has at least two pendant vertices. By Lemma 2.2, each pendant vertices forms an edge to all vertices. Therefore, each internal vertex v_i has $deg(v_i) \geq 2$ in $(T_n)_{ud}$. It is known from Lemma 2.2 that each pendant vertex has degree $n-1 \geq 3-1 \geq 2$. Hence, the result. □

An illustration of a proposition 2.14 is given in Figure 5.

A *sunlet graph*, denoted by Sl_n, $n \geq 3$, is a graph obtained by attaching a pendant vertex to every vertex of a cycle graph c_n, $n \geq 3$. In other words, a sunlet graph on $2n$ vertices is obtained by taking the corona product $C_n \circ K_1$.

Proposition 2.15. *For $n \geq 3$,*

$$\varepsilon((Sl_n)_{ud}) = \begin{cases} \frac{3n^2-n}{2} & \textit{if } n = 3, 4, 5. \\ n(2n\text{-}3) & \textit{if } n \geq 6. \end{cases}$$



(A) T_7 (B) $(T_7)_{ud}$

FIGURE 5. A tree of order seven and its upper deg-centric graph

Proof. (a) If $n = 3, 4, 5$. For a sunlet graph Sl_n, $n \geq 3$, is of the order $2n$. Let $V(Sl_n) = \{v_1, v_2, \dots, v_n, \underbrace{u_1, u_2, \dots, u_n}_{\text{pendant vertices}}\}$. Since all u_i are pendant vertices, each u_i forms the edge u_iv_i. Then, by Lemma 2.2, the n pendant vertices $u_1, u_2, \dots, u_n$ are adjacent to all other $2n-1$ vertices that is $deg(u_n) = 2n-1$ in $(Sl_n)_{ud}$. All other n vertices are adjacent with each u_n hence, $deg(v_n) = n$ in $(Sl_n)_{ud}$. Then we have,

$$\varepsilon((Sl_n)_{ud}) = \frac{\sum\limits_{w_i \in V((Sl_n)_{ud})} deg_(w_i)}{2} = \frac{3n^2 - n}{2}.$$

(b) If $n \geq 6$, by Lemma 2.2, the n pendant vertices $u_1, u_2, \dots, u_n$ are adjacent to all other $2n-1$ vertices that is $deg(u_n) = 2n-1$ in $(Sl_n)_{ud}$. Since $deg(v_n) = 3$ in Sl_n, these n vertices are adjacent with a distance of three or greater than three vertices in the cycle and with each u_n vertices. That is, $deg(v_n) = 2n-5$ in $(Sl_n)_{ud}$. Finally,

$$\varepsilon((Sl_n)_{ud}) = \frac{\sum\limits_{w_i \in V((Sl_n)_{ud})} deg(w_i)}{2} = n(2n-3).$$

□

The *ladder graph*, L_n, $n \geq 1$ is obtained by taking two copies of a path P_n with respective vertices say, $v_1, v_2, v_3, \dots, v_n$ and $u_1, u_2, u_3, \dots, u_n$ and adding the edges v_iu_i, $1 \leq i \leq n$. Note that $L_n \cong P_n \square K_2$ where $\square$ denotes the Cartesian product of two graphs.

Proposition 2.16. *For a ladder graph L_n, $n \geq 1$ it follows that:*

$$\varepsilon(L_{1_{ud}}) = 1,$$
$$\varepsilon(L_{2_{ud}}) = 2,$$
$$\varepsilon(L_{3_{ud}}) = 8,$$
$$\varepsilon(L_{4_{ud}}) = 16,$$
$$\varepsilon(G_{ud}) = \varepsilon(H_{ud}) + 4n - 10 \text{ where } H = L_{n-1} \text{ and } n \geq 5.$$

Proof. By applying Definition 2.1, it easily follows that $\varepsilon(L_{1_{ud}}) = 1$, $\varepsilon(L_{2_{ud}}) = 2$, $\varepsilon(L_{3_{ud}}) = 8$ and $\varepsilon(L_{4_{ud}}) = 16$. Now, besides the claimed result, it is valid that for any $n \geq 5$ and $H = L_{n-1}$ the size of H_{ud}; that is, $\varepsilon(H_{ud})$ can be determined by applying Definition 2.1. Consider $H = L_{n-1}$ and assume that both H_{ud} and $\varepsilon(H_{ud})$ has been determined. Now consider the extension from H to $G = L_n$. Some subgraph of H_{ud} is a subgraph of G_{ud}. Note that in G the degree of respectively v_{n-1}, u_{n-1} has increased to 3. Therefore, in G_{ud} the two edges $v_{n-1}u_{n-2}$ and $u_{n-1}v_{n-2}$ as well as the two edges $v_{n-1}v_{n-3}$ and $u_{n-1}u_{n-3}$

found in H_{ud} are not contributed in G_{ud}. All other edges formed from only amongst the vertices $V(H) \subset V(G)$ replicate exactly in G_{ud}. With regards to say, v_n the edges which contributes are v_nu_{n-1} together with v_nu_i, $1 \leq i \leq n-2$. A similar thing can be applied to vertex u_n. Hence,

$$\begin{aligned}\varepsilon(G_{ud}) &= \varepsilon(H_{ud}) + [2 \times 2(n-2) + 2 - 4] \\ &= = \varepsilon(H_{ud}) + 4n - 10.\end{aligned}$$

Finally, since an initial value, that is. $\varepsilon(L_{4_{ud}}) = 16$, is known, the result for $n \geq 5$ follows. □

3. Conclusions

From this study, it follows that for any vertex $v_i \in V(G)$ the open neighborhood $N_{G_{ud}}(v_i)$ can be partitioned into three sets that is:

(i) $N_{\overrightarrow{G_{ud}}}(v_i) = \{v_j : deg_G(v_i) \leq d_G(v_i, v_j) \text{ and } deg_G(v_j) > d_G(v_j, v_i)\}$.

(ii) $N_{\overleftarrow{G_{ud}}}(v_i) = \{v_j : deg_G(v_i) > d_G(v_i, v_j) \text{ and } deg_G(v_j) \leq d_G(v_j, v_i)\}$.

(iii) $N_{\overleftrightarrow{G_{ud}}}(v_i) = \{v_j : deg_G(v_i) \leq d_G(v_i, v_j) \text{ and } deg_G(v_j) \leq d_G(v_j, v_i)\}$.

Hence, (iii) represents the commutative initiation of edges.

Conjecture 3.1. *Let G and H be any pair of graphs of order n. If $deg_G(v_i) \leq deg_H(u_j)$, then $N_{\overrightarrow{G_{ud}}}(v_i) \geq N_{\overrightarrow{H_{ud}}}(u_j)$.*

Although this paper considers connected graphs, upper deg-centrication of a disconnected graph is possible. Note that in an empty graph $G = \mathfrak{N}_n$, $n \geq 1$ each vertex v_i has $deg_G(v_i) = 0$. If $n = 1$ then $G_{ud} = K_1$. If $n \geq 2$ then $d_G(v_i, v_j) = \infty$ for all pairs of distinct vertices. Hence, by Definition 2.1, it follows that $G_{ud} \cong K_n$. Let $G + H$ be the join of G and H. Let $G \cup H$ be the disjoint union of G and H. If graph $J = (((G_1 \cup G_2) \cup G_3) \cup \cdots \cup G_{t-1}) \cup G_t$ then clearly,

$$J_{ud} = (((G_{1_{ud}} + G_{2_{ud}}) + G_{3_{ud}}) + \cdots + G_{(t-1)_{ud}}) + G_{t_{ud}}. \quad (1)$$

The above facts highlight the wide scope for further research in this area.

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