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AN IMPROVEMENT AND A GENERALIZATION OF ANKENY AND RIVLIN'S RESULT ON THE MAXIMUM MODULUS OF POLYNOMIALS

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Abstract. For an arbitrary entire function $f(z)$, let

$$M(f,r) = \max_{|z|=r} |f(z)|.$$

By considering the polynomial of degree n having no zero in the interior of the unit circle $|z| = 1$, Ankeny and Rivlin obtained

$$M(p,R) \leq \frac{R^n + 1}{2} M(p,1), \ R \geq 1.$$

In this paper, we consider the polynomial of degree n having no zero in $|z| < k$, $k \geq 1$ and simultaneously considering the s^{th} derivative, $0 \leq s < n$, of the polynomial, we have obtained an improvement as well as a generalization of Ankeny and Rivlin's result.

Keywords: Polynomial, maximum modulus principle, generalization, s^{th} derivative.

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1. Introduction and statement of results

Approximation by polynomials is a fundamental concept in mathematics and applied sciences, offering a versatile tool for representing complex functions with simpler polynomial expressions. This approach involves the construction of polynomial functions that closely mimic the behavior of more intricate functions, facilitating easier analysis, computation, and problem-solving in various domains. Polynomial approximation plays a pivotal role in various disciplines such as numerical analysis, signal processing, computer-aided design, physics, and engineering.

Several approaches have been developed to address this challenge, each tailored to specific contexts and requirements. Least squares approximation, employing techniques like linear regression, focuses on minimizing the overall error between the polynomial and

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data points. Chebyshev approximation minimizes the maximum absolute error over an interval, ensuring robustness. Rational function approximation introduces flexibility by representing functions as ratios of polynomials. Each approach has its strengths and is chosen based on factors such as data characteristics, desired accuracy, and computational efficiency, making polynomial approximation a key tool across diverse disciplines. Researchers continue to refine and extend these techniques, ensuring that polynomial approximation remains a powerful and adaptable tool in the ever-expanding landscape of mathematical and computational sciences.

However, another approach was made through the applications of the Bernstein inequality, particularly the trigonometric version that holds significant importance in the literature for establishing inverse theorems in approximation theory (see Borwein and Erdélyi [3], Ivanov [9], Lorentz [12], Telyakovskii [18]) and, of course, have their own intrinsic interests. The first result in this area was connected with some investigation of the well-known Russian chemist Mendeleev [15]. In fact, Mendeleev's problem was to determine $\max_{-1\leq x\leq 1}|p'(x)|$, where $p(x)$ is a quadratic polynomial of real variable x with real coefficients and satisfying $-1 \leq p(x) \leq 1$ for $-1 \leq x \leq 1$. He himself was able to prove that if $p(x)$ is a quadratic polynomial and $|p(x)| \leq 1$ on $[-1, 1]$, then $|p'(x)| \leq 4$ on the same interval. A. A. Markov [14] generalized this result for a polynomial of degree n in the real line. In fact, he proved that if $p(x)$ is an algebraic polynomial of degree at most n with real coefficients, then

$$\max_{-1\leq x\leq 1}|p'(x)| \leq n^2 \max_{-1\leq x\leq 1}|p(x)|.$$

After about twenty years, Bernstein [2] needed the analogue of Markov's Theorem for the unit disc in the complex plane instead of the interval $[-1, 1]$ in order to prove inverse theorem of approximation (see Borwein and Erdélyi [3, p. 241]) to estimate how well a polynomial of a certain degree approximates a given continuous function in terms of its derivatives and Lipschitz constants. This leads to the famous well-known result known as Bernstein's inequality which states that if $t \in \tau_n$ (the set of all real trigonometric polynomials of degree at most n), then for $K := [0, 2\pi)$,

$$\max_{\theta\in K}|t^{(m)}(\theta)| \leq n^m \max_{\theta\in K}|t(\theta)|. \tag{1}$$

The above inequality remains true for all $t \in \tau_n^c$ (the set of all complex trigonometric polynomials of degree at most n), which implies, as a particular case, the following algebraic polynomial version of Bernstein's inequality on the unit disk.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1}|p'(z)| \leq n \max_{|z|=1}|p(z)|. \tag{2}$$

Equality holds in (2) *if and only if $p(z)$ has all its zeros at the origin.*

It is really of interest both in theoretical and practical aspects that continuous functions are approximated by polynomials. In this regard, we have the following interesting result (Theorem 1.2) [3, p. 241, Part (a) of E.18] which approximates m times differentiable real-valued function on a half-closed interval $[0, 2\pi)$ by trigonometric polynomials. For the sake of convenience of the readers, we state the above result more precisely.

Let Lip_α, $\alpha \in (0, 1]$, denote the family of all real-valued functions g defined on K satisfying

$$\sup\left\{\frac{|g(x) - g(y)|}{|x - y|^\alpha} : x \neq y \in K\right\} < \infty.$$

If $C(K)$ denotes the set of all continuous functions on K, then for $f \in C(K)$, let

$$E_n(f) := \inf \left\{ \sup_{\theta \in K} |t - f| : t \in \tau_n \right\}.$$

Theorem 1.2. *(Direct theorem) Suppose f is m times differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$ for some $\alpha \in (0, 1]$. Then there is a constant C depending only on f so that*

$$E_n(f) \leq Cn^{-(m+\alpha)}, \qquad n = 1, 2, \dots .$$

On the other hand, the converse (inverse) of Theorem 1.2 is essentially of interest and is stated below.

Theorem 1.3. *(Inverse theorem) Suppose m is a non-negative integer, $\alpha \in (0, 1)$, and $f \in C(K)$. Suppose there is a constant $C > 0$ depending only on f such that*

$$E_n(f) \leq Cn^{-(m+\alpha)}, \qquad n = 1, 2, \dots .$$

Then f is m times continuously differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$.

The proof of Theorem 1.3 is done by the application of the well-known result due to Bernstein (inequality (1)) given in [3].

From the above discussion, it is worth to note that Bernstein and Markov-type inequalities play a significant role in approximation theory. Direct and inverse theorems of approximation and related matters may be found in many books on approximation theory, including Cheney [4], Lorentz [12], and DeVore and Lorentz [5].

Inequality (2) can be sharpened, if the zeros of $p(z)$ are restricted. In this direction, Erdös conjectured and later Lax [11] proved that if $p(z)$ has no zero in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{3}$$

As a partial generalization of (3), Malik [13] proved that if $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{4}$$

For the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, the precise estimate for maximum of $|p'(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable. For quite some time, it was believed that if $p(z) \neq 0$ in $|z| < k$, $k \leq 1$, then the inequality analogous to (4) should be

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|,$$

till Professor E. B. Saff gave the example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief.

Thus, the approximation does not seem to be known in general, and this problem is still open. However, two special cases in this direction have been considered by Govil giving extensions of (3) with strong restrictions. One such was established by him [6] in 1980 was that if $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|,$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, where $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

Govil and Rahman [8, Theorem 4] extended inequality (4) to the s^{th} derivative of the polynomial and proved under the same hypothesis for $1 \leq s < n$ that

$$\max_{|z|=1} |p^{(s)}(z)| \leq \frac{n(n-1)...(n-s+1)}{1+k^s} \max_{|z|=1} |p(z)|.$$

For an arbitrary entire function $f(z)$, let

$$M(f,r) = \max_{|z|=r} |f(z)|.$$

If $p(z)$ is a polynomial of degree n, as a consequence of the maximum modulus principle, we have the following result [16, 17],

$$M(p,R) \leq R^n M(p,1), \ R \geq 1. \tag{5}$$

Inequality (5) is best possible and equality holds for $p(z) = \lambda z^n$, $\lambda \neq 0$ being a complex number.

Ankeny and Rivlin [1] considered the class of polynomials having no zero in the interior of the unit circle and obtained the following refinement of inequality (5).

Theorem 1.4. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then*

$$M(p,R) \leq \frac{R^n+1}{2} M(p,1), \ R \geq 1. \tag{6}$$

The result is sharp for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

In an attempt to obtain a generalization of Theorem 1.4, Jain [10] considered polynomials with no zero in $|z| < k$, $k \geq 1$, and was able to prove the following result concerning the estimate of the maximum modulus of the s^{th} derivative, $0 \leq s < n$, of the polynomial, instead of the polynomial itself.

Theorem 1.5. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \geq 1$, then for $0 \leq s < n$,*

$$M(p^{(s)},R) \leq \begin{cases} \dfrac{1}{2}\left\{\dfrac{d^s}{dR^s}(R^n+k^n)\right\}\left(\dfrac{2}{1+k}\right)^n M(p,1), \ R \geq k, & (7) \\ \dfrac{1}{R^s+k^s}\left[\left\{\dfrac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]\left(\dfrac{R+k}{1+k}\right)^n M(p,1), \ 1 \leq R \leq k. & (8) \end{cases}$$

In this paper, under the same hypotheses, we obtain improved bounds of Theorem 1.5 which does not contain coefficients of the polynomial, proved by Jain [10]. More precisely, we prove

Theorem 1.6. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \geq 1$, then for $0 \leq s < n$,*

$$M(p^{(s)},R) \leq \begin{cases} \dfrac{1}{2}\left\{\dfrac{d^s}{dR^s}(R^n+k^n)\right\}\left(\dfrac{2}{1+k}\right)^n \{M(p,1)-m\}, \ 1 \leq s < n, \ R \geq k, & (9) \\ \dfrac{1}{2}(R^n+k^n)\left(\dfrac{2}{1+k}\right)^n M(p,1) - \left\{\dfrac{R^n+k^n}{2}\left(\dfrac{2}{1+k}\right)^n - 1\right\} m, & (10) \\ s=0, \ R \geq k, & \\ \dfrac{1}{R^s+k^s}\left[\left\{\dfrac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]\left[\left(\dfrac{R+k}{1+k}\right)^n \{M(p,1)-m\} + m\right], & (11) \\ 0 \leq s \leq n, \ 1 \leq R \leq k, & \end{cases}$$

where $m = \min_{|z|=k} |p(z)|$.

Equality holds in (10) *(with $k = 1$) for $p(z) = z^n + 1$ and in* (11) *(with $s = 1$) for $p(z) = (z + k)^n$.*

Remark 1.1. *By Lemma 2.6, it is evident that inequality* (10) *is an improvement of the corresponding inequality* (7) *for $s = 0$. Further, it is clear that inequalities* (9) *and* (11) *respectively of the theorem improve over the bounds given by* (7) *and* (8) *of Theorem 1.5 due to Jain* [10]*. Moreover, by means of example, we illustrate sharpness mathematically and bounds graphically as in Example 4.1.*

Remark 1.2. *Putting $s = 1$, $k = 1$, inequality* (10) *of our theorem reduces to the following result, which further sharpens the bound* (6) *given by Theorem 1.4 of Ankeny and Rivlin* [1]*.*

Corollary 1.1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \frac{R^n + 1}{2} M(p,1) - \frac{R^n - 1}{2} m,$$

where $m = \min\limits_{|z|=k} |p(z)|$.

2. Lemmas

We require the following lemmas to prove the theorem.

The following two lemmas are due to Jain [10].

Lemma 2.1. *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$. If $p(z)$ is a polynomial of degree at most n such that*

$$|p(z)| \leq |P(z)|, \ |z| = 1,$$

then for $0 \leq s < n$,

$$|p^{(s)}(z)| \leq |P^{(s)}(z)|, \ |z| \geq 1.$$

Lemma 2.2. *If $p(z)$ is a polynomial of degree at most n, then for $0 \leq s < n$,*

$$|p^{(s)}(z)| + |q^{(s)}(z)| \leq \left\{ \left| \frac{d^s}{dz^s}(1) \right| + \left| \frac{d^s}{dz^s}(z^n) \right| \right\} M(p,1), \ |z| \geq 1,$$

where $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

Lemma 2.3. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $1 \leq R \leq k^2$,*

$$M(p,R) \leq \left(\frac{R+k}{1+k}\right)^n M(p,1) - \left\{ \left(\frac{R+k}{1+k}\right)^n - 1 \right\} m, \tag{12}$$

where $m = \min\limits_{|z|=k} |p(z)|$.

Proof. By hypothesis, $p(z)$ has no zero in $|z| < k$, $k \geq 1$. In case when $m = \min\limits_{|z|=k} |p(z)| \neq 0$, consider the polynomial $P(z) = p(z) + \alpha m$, where α is any real or complex number with $|\alpha| < 1$.

Now, for $|z| = k$,

$$|m\alpha| < m \leq |p(z)|.$$

Then it follows from Rouche's Theorem that $P(z)$ has no zero in $|z| \leq k$. The case for $m = 0$ is trivially true and hence we conclude that $P(z)$ has no zero in $|z| < k$.

Since all the zeros of $P(z)$ lie in $|z| \geq k$, $k \geq 1$, we can write $P(z) = a_n \prod_{j=1}^{n} \left(z - r_j e^{i\theta_j}\right)$, where $r_j \geq k$, $j = 1, 2, 3, ..., n$. Then

$$\begin{aligned} \left| \frac{P(Re^{i\theta})}{P(e^{i\theta})} \right| &= \prod_{j=1}^{n} \left| \frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}} \right| \\ &= \prod_{j=1}^{n} \left| \frac{Re^{i(\theta-\theta_j)} - r_j}{e^{i(\theta-\theta_j)} - r_j} \right| \\ &\leq \prod_{j=1}^{n} \left(\frac{R + r_j}{1 + r_j} \right) \\ &\leq \prod_{j=1}^{n} \left(\frac{R + k}{1 + k} \right) \\ &= \left(\frac{R + k}{1 + k} \right)^n . \end{aligned}$$

Substituting the value of $P(z) = p(z) + \alpha m$ in the above inequality, we have

$$\begin{aligned} |p(Re^{i\theta}) + \alpha m| &\leq \left(\frac{R + k}{1 + k} \right)^n |p(e^{i\theta}) + \alpha m| \\ &\leq \left(\frac{R + k}{1 + k} \right)^n \max_{\theta} |p(e^{i\theta}) + \alpha m|. \end{aligned} \tag{13}$$

Let θ_0 be such that

$$\max_{\theta} |p(e^{i\theta}) + \alpha m| = |p(e^{i\theta_0}) + \alpha m|. \tag{14}$$

We choose the argument of α such that

$$\begin{aligned} |p(e^{i\theta_0}) + \alpha m| &= |p(e^{i\theta_0})| - |\alpha| m \\ &\leq \max_{|z|=1} |p(z)| - |\alpha| m. \end{aligned} \tag{15}$$

Using (15) to (14), we have

$$\max_{\theta} |p(e^{i\theta}) + \alpha m| \leq \max_{|z|=1} |p(z)| - |\alpha| m. \tag{16}$$

From (13) and (16), we have

$$|p(Re^{i\theta}) + \alpha m| \leq \left(\frac{R + k}{1 + k} \right)^n \left\{ \max_{|z|=1} |p(z)| - |\alpha| m \right\},$$

which implies

$$|p(Re^{i\theta})| \leq \left(\frac{R + k}{1 + k} \right)^n \max_{|z|=1} |p(z)| - |\alpha| m \left\{ \left(\frac{R + k}{1 + k} \right)^n - 1 \right\},$$

which on taking limit as $|\alpha| \to 1$, we obtain inequality (12) and it proves Lemma 2.3 completely. □

Lemma 2.4. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then $1 \leq s < n$,*

$$\max_{|z|=1} |p^{(s)}(z)| \leq \frac{n(n-1)(n-2)...(n-s+1)}{1 + k^s} \max_{|z|=1} |p(z)|.$$

The above result is due to Govil [7].

From Lemma 2.4, it follows readily

Lemma 2.5. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then $0 \leq s < n$,*

$$\max_{|z|=1} |p^{(s)}(z)| \leq \frac{1}{1+k^s} \left[\left\{ \frac{d^s}{dx^s}(1+x^n) \right\}_{x=1} \right] \max_{|z|=1} |p(z)|.$$

Lemma 2.6. *If $R \geq k \geq 1$, then for any positive integer n,*

$$\frac{R^n + k^n}{2} \left(\frac{2}{1+k} \right)^n - 1 \geq 0. \tag{17}$$

Proof. We prove this result by mathematical induction. For $n = 1$, inequality (17) becomes

$$\frac{R+k}{1+k} \geq 1,$$

which is true since $R \geq 1$.

Suppose inequality (17) is true for any positive integer n, that is

$$\frac{R^n + k^n}{2} \left(\frac{2}{1+k} \right)^n - 1 \geq 0. \tag{18}$$

(18) is equivalent to

$$\frac{R^n + k^n}{2} \geq \left(\frac{1+k}{2} \right)^n. \tag{19}$$

Since $R \geq k$, it can be easily verified that

$$\frac{R^{n+1} + k^{n+1}}{2} \geq \frac{R^n + k^n}{2} \frac{R+k}{2}. \tag{20}$$

Clearly,

$$\frac{R+k}{2} \geq \frac{1+k}{2}. \tag{21}$$

From (19) and (21), it follows that

$$\frac{R^n + k^n}{2} \frac{R+k}{2} \geq \left(\frac{1+k}{2} \right)^{n+1}. \tag{22}$$

Combining (20) and (22), we have

$$\frac{R^{n+1} + k^{n+1}}{2} \geq \left(\frac{1+k}{2} \right)^{n+1},$$

which is equivalent to

$$\frac{R^{n+1} + k^{n+1}}{2} \left(\frac{2}{1+k} \right)^{n+1} - 1 \geq 0,$$

and hence by mathematical induction, this lemma is proved. □

3. Proof of the theorem

Proof of Theorem 1.6 Since $p(z)$ has no zero in $|z| < k$, $k \geq 1$, the polynomial $p(z) + \alpha m$, where α is any real or complex number with $|\alpha| < 1$ and $m = \min_{|z|=k} |p(z)|$, has no zero in $|z| < k$, $k \geq 1$. The case for $m = 0$ is trivially true. For $m > 0$, we have for $|z| = k$, $|m\alpha| < m \leq |p(z)|$ and the conclusion follows from Rouche's Theorem. If $P(z) = p(kz) + \alpha m$, then the polynomial $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$ possesses the property that for $|z| = 1$,

$$|P(z)| \leq |Q(z)|$$

and it has all its zeros in $|z| \leq 1$. Applying Lemma 2.1 to the polynomials $P(z)$ and $Q(z)$, we get for any $0 \leq s < n$ and $t \geq 1$,

$$|P^{(s)}(te^{i\theta})| \leq |Q^{(s)}(te^{i\theta})|, \ 0 < \theta \leq 2\pi. \tag{23}$$

Further, by Lemma 2.2, we have for $t \geq 1$ and $0 \leq s < n$,

$$|P^{(s)}(te^{i\theta})| + |Q^{(s)}(te^{i\theta})| \leq \left\{ \frac{d^s}{dt^s}(1 + t^n) \right\} M(P,1), \ 0 < \theta \leq 2\pi,$$

which, by (23) implies that

$$|P^{(s)}(te^{i\theta})| \leq \frac{1}{2} \left\{ \frac{d^s}{dt^s}(1 + t^n) \right\} M(P,1). \tag{24}$$

Substituting the value of $P(z) = p(kz) + \alpha m$ in (24), we have

$$|p^{(s)}(kte^{i\theta})| \leq \frac{1}{2k^s} \left\{ \frac{d^s}{dt^s}(1 + t^n) \right\} \max_{|z|=k} |p(z) + \alpha m|, \ s \geq 1, \tag{25}$$

and

$$|p(kte^{i\theta}) + \alpha m| \leq \frac{1}{2}(1 + t^n) \max_{|z|=k} |p(z) + \alpha m|, \ s = 0. \tag{26}$$

Set $kt = R$. Since $t \geq 1$, $R \geq k$.

As $t = \frac{R}{k}$, we have $\frac{d^s}{dt^s} \equiv k^s \frac{d^s}{dR^s}$ and hence for $R \geq k$, inequalities (25) and (26) are equivalent to

$$|p^s(Re^{i\theta})| \leq \frac{1}{2k^n} \left\{ \frac{d^s}{dR^s}(R^n + k^n) \right\} \max_{|z|=k} |p(z) + \alpha m|, \ s \geq 1, \tag{27}$$

and

$$|p(Re^{i\theta}) + \alpha m| \leq \frac{1}{2k^n}(R^n + k^n) \max_{|z|=k} |p(z) + \alpha m|, \ s = 0. \tag{28}$$

In both the inequalities (27) and (28), let z_0 on $|z| = k$ be such that

$$\max_{|z|=k} |p(z) + \alpha m| = |p(z_0) + \alpha m|. \tag{29}$$

Choosing the argument of α such that

$$|p(z_0) + \alpha m| = |p(z_0)| - |\alpha| m. \tag{30}$$

Also, we have

$$|p(z_0)| \leq \max_{|z|=k} |p(z)|. \tag{31}$$

Using (31) and (30) to (29), we have

$$\max_{|z|=k} |p(z) + \alpha m| \leq \max_{|z|=k} |p(z)| - |\alpha| m. \tag{32}$$

Using (32) to (27) and (28), we get for $R \geq k$,

$$|p^s(Re^{i\theta})| \leq \frac{1}{2k^n}\left\{\frac{d^s}{dR^s}(R^n+k^n)\right\}\left\{\max_{|z|=k}|p(z)|-|\alpha|m\right\},\ s\geq 1, \tag{33}$$

and

$$|p(Re^{i\theta})+\alpha m| \leq \frac{1}{2k^n}(R^n+k^n)\left\{\max_{|z|=k}|p(z)|-|\alpha|m\right\},\ s=0. \tag{34}$$

By Lemma 2.3, we have for $R \geq k$,

$$\max_{|z|=k}|p(z)| \leq \left(\frac{2k}{1+k}\right)^n\{M(p,1)-m\}+m.$$

Using the above inequality, (33) and (34) give for $R \geq k$,

$$|p^s(Re^{i\theta})| \leq \frac{1}{2}\left\{\frac{d^s}{dR^s}(R^n+k^n)\right\}\left[\left(\frac{2}{1+k}\right)^n\{M(p,1)-m\}+\frac{m}{k^n}-|\alpha|\frac{m}{k^n}\right], s\geq 1, \tag{35}$$

and

$$|p(Re^{i\theta})+\alpha m| \leq \frac{1}{2}(R^n+k^n)\left[\left(\frac{2}{1+k}\right)^n\{M(p,1)-m\}+\frac{m}{k^n}-|\alpha|\frac{m}{k^n}\right], s=0. \tag{36}$$

Since $|p(Re^{i\theta})|-|\alpha|m \leq |p(Re^{i\theta})+\alpha m|$, (36) implies

$$|p(Re^{i\theta})| \leq \frac{1}{2}(R^n+k^n)\left[\left(\frac{2}{1+k}\right)^n\{M(p,1)-m\}+\frac{m}{k^n}-|\alpha|\frac{m}{k^n}\right]-|\alpha|m, s=0. \tag{37}$$

Taking the limit as $|\alpha| \to 1$, and considering the maximum over θ, inequalities (35) and (37) become inequalities (9) and (10) of the Theorem.

Since $p(z)$ has no zero in $|z|<k$, $k\geq 1$, the polynomial $p(Rz)$, where $R\leq k$, has no zero in $|z|<\frac{k}{R}$, $\frac{k}{R}\geq 1$. Applying Lemma 2.5 to $p(Rz)$, we have for $0\leq s<n$,

$$M(p^{(s)},R) \leq \frac{1}{R^s+k^s}M(p,R)\left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]. \tag{38}$$

Using Lemma 2.3 for $R\leq k$, inequality (38) becomes

$$\begin{aligned} M(p^{(s)},R) \ \leq\ & \frac{1}{R^s+k^s}\left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]\left[\left(\frac{R+k}{r+k}\right)^n M(p,1)\right. \\ & \left. -m\left\{\left(\frac{R+k}{r+k}\right)^n-1\right\}\right], \end{aligned}$$

which is inequality (11). This completes the proof of the Theorem.

4. Numerical example and graphical representations

It is clear that the bounds of Theorem 1.6, in general, improve over that of Theorem 1.5 due to Jain [10]. Below, we consider an example which shows that the improvement is significant.

Example 4.1. *Let $p(z)=z^4+4^4$ with all zeros $4e^{\frac{i\pi}{4}(1+2l)}$, $l=0,1,2,3$ on $|z|=4$, so that Theorem 1.6 holds for $0<k\leq 4$. If we take $k=3$, so that $p(z)$ has all its zeros in $|z|\geq k=3$ and $s=2$, then on $|z|=R$, we have*

$$|p(Re^{i\theta})| = \sqrt{R^8+65536+512R^4\cos 4\theta}$$

and their graphics for $0\leq\theta<2\pi$ and $R=2,3,4$ are presented below.

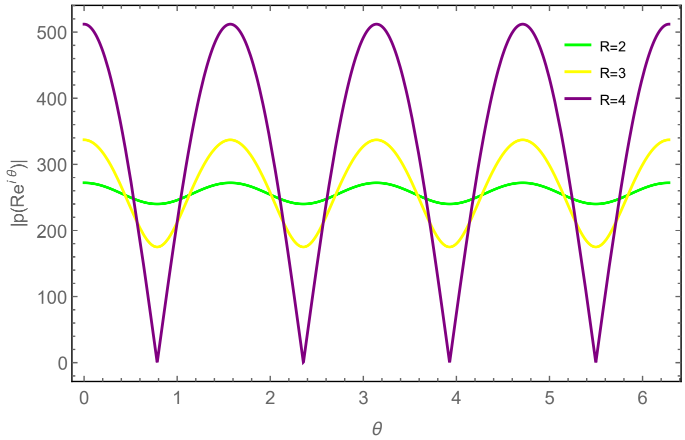

Fig. 1 *Graphics of the periodic functions $\theta \mapsto |p(Re^{i\theta})|$ with period $\frac{\pi}{2}$ for $0 \le \theta < 2\pi$ and $R = 2, 3, 4$, clearly showing the extremals*

Clearly, we have

$$M(p, R) = \max_{0 \le \theta < 2\pi} |p(Re^{i\theta})| = R^4 + 4^4.$$

Case 1. *For $R \ge k$, let $R = 4$.*
Since

$$M(p, 1) = 257 \text{ and } M(p'', 4) = 192,$$

as well as $m = \min_{|z|=k} |p(k)| = p(k) = 4^4 - k^4$, $1 \le k \le 4$, we can consider the difference $k \mapsto \Delta(k)$ between the right and the left hand sides of the inequalities (7) *of Theorem 1.5, and* (9) *of Theorem 1.6 as*

$$\Delta(k) = \begin{cases} 96\left(\dfrac{2}{1+k}\right)^4 M(p,1) - M(p'',4), & \text{inequality (7)}, \\[2ex] 96\left(\dfrac{2}{1+k}\right)^4 \{M(p,1) - m\} - M(p'',4), & \text{inequality (9)}, \end{cases}$$

and their graphics for $1 \le k \le 4$ are presented below.

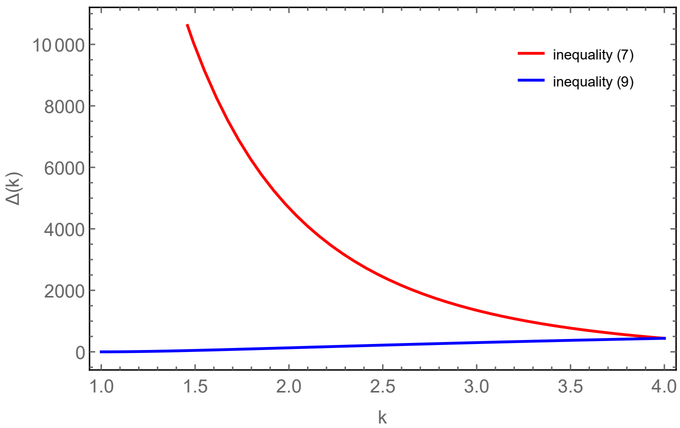

Fig. 2 *Comparison of the differences $k \mapsto \Delta(k)$, $1 \le k \le 4$ in the inequalities* (7) *and* (9)

Also, by inequality (7) *of Theorem* 1.5 *for* $k = 3$*, we have* $M(p'', 4) \leq 1542$*, while inequality* (9) *of Theorem* 1.6 *gives* $M(p'', 4) \leq 492$*, a significant improvement of* 68.09% *over the bound obtained from* (7) *is seen.*

Case 2. *For* $1 \leq R \leq k$*, let* $R = 2$*.*

Since

$$M(p, 1) = 257 \text{ and } M(p'', 2) = 48,$$

we consider the difference $k \mapsto \delta(k)$ *between the right and the left-hand sides of the inequalities* (8) *of the Theorem 1.5, and* (11) *of the Theorem 1.6 as*

$$\delta(k) = \begin{cases} \left(\dfrac{12}{4+k^2}\right)\left(\dfrac{2+k}{1+k}\right)^4 M(p,1) - M(p'',2), & \text{inequality (8)}, \\ \left(\dfrac{12}{4+k^2}\right)\left[\left(\dfrac{2+k}{1+k}\right)^4 \{M(p,1) - m\} + m\right] - M(p'',2), & \text{inequality (11)}. \end{cases}$$

and their graphics for $1 \leq k \leq 4$ *are presented below.*

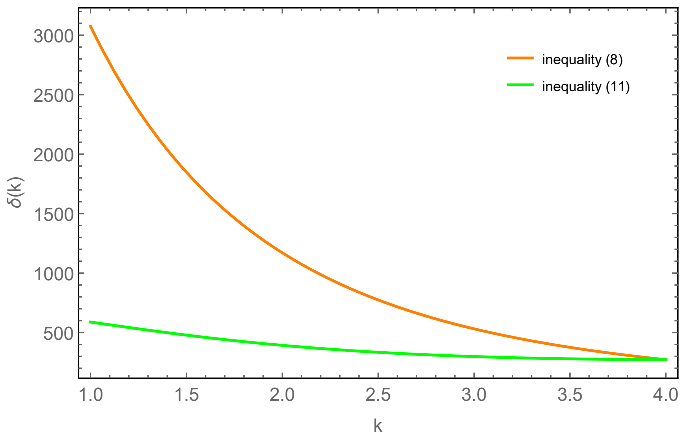

Fig. 3 *Comparison of the differences* $k \mapsto \delta(k)$*,* $1 \leq k \leq 4$ *in the inequalities* (8) *and* (11)

By inequality (8) *of Theorem* 1.5 *for* $k = 3$*, we have* $M(p'', 2) \leq 579.177$*, while inequality* (11) *of Theorem* 1.6 *gives* $M(p'', 2) \leq 346.334$*, a considerable improvement of* 40.20% *over the bound obtained from* (8) *happens.*

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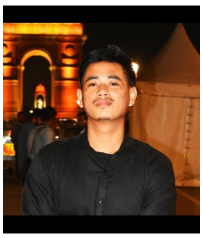

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