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ESTIMATES OF TOEPLITZ DETERMINANTS FOR CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTION RELATED TO MODIFIED SIGMOID FUNCTION

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Abstract. The current comprehensive study aimed to determine upper bounds of Toeplitz determinants for some subclasses of bi-univalent functions. A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Modified sigmoid function play an important role in Geometric function theory and in this paper we derive the Sharp coefficient estimates, Fekete-Szegö inequality, second and third order Toeplitz determinants, for the subclasses $\mathcal{S}^*_\sigma(S)$, $\mathcal{C}_\sigma(S)$ of bi-univalent Sakaguchi type functions associated with the modified sigmoid function.

Keywords: Sakaguchi functions, Toeplitz determinants, modified sigmoid function, starlike functions, convex function.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ also let $\mathcal{S}$ denote the subclass of all functions in $\mathcal{A}$ which are univalent in Δ (see [5] for details). It is well known that every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \Delta) \text{ and}$$
$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq 1/4).$$

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The inverse function may have an analytic continuation to Δ, with

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 \\ -(5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions defined in the unit disk Δ.

Lewin [13] studied the class Σ of bi-univalent functions and derived the bound for the second coefficient. Lewin [13] also established that $|a_2| \leq 1.51$. Further, Brannan and Clunie conjectured [4] that $|a_2| \leq \sqrt{2}$.

Examples of bi-univalent functions are

$$\frac{z}{1-z}, \quad \frac{1}{2} log \frac{1+z}{1-z}, \quad -log(1-z).$$

(See also [22]). However the familiar Koebe function $\frac{z}{(1-z)^2}$ and its rotations are not members of Σ.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$ [5], provided there is an analytic function w defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Several authors have investigated similar problems in this direction (see [3, 15]). Srivastava et al. [22] introduced and studied subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Bounds for the initial coefficients of several classes of functions were also investigated in [1, 2, 17, 20, 21]. In the univalent function theory, an extensive focus has been given to estimate the bounds of Hankel matrices.

Pommerenke [18] defined the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & a_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

$(a_1 = 1)$.

In particular the second Hankel determinant is given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

One can easily observe that Fekete-Szegö inequality is $H_2(1)$. Many Researchers have obtained the upper bounds for the second Hankel determinant (see for details [7],[8],[10], [11], [16], [24]).

The Hankel determinants are more closely related to the Toeplitz determinants. One way to conceptualise a Toeplitz determinant is as a "upside-down" Hankel determinant, in which Hankel determinant have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal.

Thomas and Halim [23] defined the symmetric Toeplitz determinant $T_q(n)$ as follows:

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & & & \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}.$$

In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}.$$

A comprehensive overview of the various uses of Toeplitz matrices in pure and applied mathematics can be found in [25] . Many researchers have focussed on finding estimates for second and third order Toeplitz determinants [6, 9].

One may comprehend the significance of the modified sigmoid function from [12]. Geometric function theory relies heavily on special functions. Activation function is an example of a special function. In a neural network, the activation function serves as a squashing function, ensuring that a neuron's output falls between predetermined values (typically, 0 and 1 or -1 and 1). Piecewise-linear, sigmoid, and threshold functions are the three different forms of activation functions. The sigmoid function is the most often used activation function in artificial neural network hardware implementations. The sigmoid function, $g(z) = \frac{1}{1+e^{-z}}$ is useful because it is differentiable, which is important for the weight-learning algorithms. The sigmoid function will increase the size of the hypothesis space that the network can represent. Neural networks can be used for complex learning tasks. The sigmoid function has very important properties,

- It outputs real numbers between 0 and 1.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is a one-to- one function.
- It increases monotonically.

The normalized form of modified sigmoid function is given by $G(z) = \frac{2}{1+e^{-z}}$ for all $z \in \Delta$. In this paper we investigate the coefficient inequality, Toeplitz determinants and Fekete-Szegö inequality for the the subclasses $\mathcal{S}_\sigma^*$, $\mathcal{C}_\sigma$ of sakaguchi type function defined in modified sigmoid function.

2. Preliminaries

Let $\mathcal{P}$ denote the class of all functions $p(z)$ given by

$$p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i, \quad (z \in \Delta) \tag{3}$$

such that $\mathcal{R}\{p(z)\} > 0$ and $p(0) = 1$.

Lemma 2.1. [5] *Let $p \in \mathcal{P}$. Then*

$$|c_k| \leq 2, k = 1, 2, 3 \cdots \tag{4}$$

and the inequality is sharp.

Lemma 2.2. [14] *Let $p \in \mathcal{P}$ of the form (1.2). Then there exist some ξ, $\zeta \in \mathbb{C}$ with $|\xi| \leq 1$, $|\zeta| \leq 1$, such that*

$$\begin{aligned} 2c_2 &= c_1^2 + (4 - c_1^2)\xi, \\ 4c_3 &= c_1^3 + 2c_1\xi(4 - c_1^2) - (4 - c_1^2)c_1\xi^2 \\ &\quad + 2(4 - c_1^2)(1 - |\xi|^2)\zeta. \end{aligned}$$

Definition 2.1. [19] *Denote by S_S^* the subclass of $\mathcal{A}$ consisting of functions given by (1) and satisfying*

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0, \; z \in \Delta.$$

These functions introduced by Sakaguchi are called *functions starlike with respect to symmetric points*, and for a function $f \in \mathcal{A}$ the above inequality is a necessary and sufficient condition for f to be univalent and starlike with respect to symmetrical points in Δ (see [19, Theorem 1]).

Definition 2.2. *A function f given by (1) is said to be in the class $\mathcal{S}^*_\sigma(S)$ if the following subordination hold:*

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{2}{1+e^{-z}} \quad for\ z\ \in \Delta$$

and

$$\frac{2wg'(w)}{g(w)-g(-w)} \prec \frac{2}{1+e^{-w}} \quad for\ w\ \in \Delta,$$

where the function g is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{5}$$

Definition 2.3. *A function f given by (1) is said to be in the class $\mathcal{C}_\sigma(S)$ if the following subordination hold:*

$$\frac{(2zf'(z))'}{(f(z)-f(-z))'} \prec \frac{2}{1+e^{-z}} \quad for\ z\ \in \Delta$$

and

$$\frac{(2wg'(w))'}{(g(w)-g(-w))'} \prec \frac{2}{1+e^{-w}} \quad for\ w\ \in \Delta.$$

where the function g is given by (5).

Remark 2.1. *We shall illustrate with an example that the aforementioned classes $\mathcal{S}^*_\sigma(S)$, $\mathcal{C}_\sigma(S)$ are non empty.*

(1) *Taking $f_1(z) = z + az^2$, $a \in C$, then*

$$\Phi_*(z) = \frac{2zf_1'(z)}{f_1(z)-f_1(-z)} = 1 + 2za, \quad z \in \Delta.$$

For the values of $z = 0.9e^{i\theta}$, $a = 0.2$ it is clear from the Figure 1 that $\Phi_*(\Delta) \subset G(z) = \frac{2}{1+e^{-z}}$ and hence the class $\mathcal{S}^*_\sigma(S)$ is non empty.

(2) *Taking $f_1(z) = z + az^2$, $a \in C$, then*

$$\Psi_*(z) = \frac{(2zf_1'(z))'}{(f_1(z)-f_1(-z))'} = 1 + 4za, \quad z \in \Delta.$$

For the values of $z = 0.9e^{i\theta}$, $a = 0.01$ it is clear from the Figure 2 that $\Psi_*(\Delta) \subset G(z) = \frac{2}{1+e^{-z}}$ and hence the class $\mathcal{C}_\sigma(S)$ is non empty.

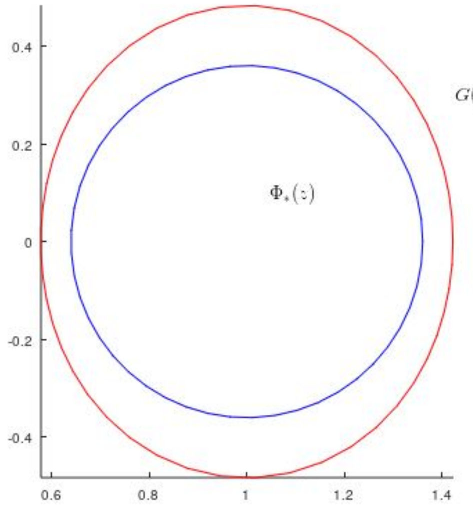

FIGURE 1. The image of $\Phi_*(z)$, (blue colour) is contained in $G(z)$(red colour)

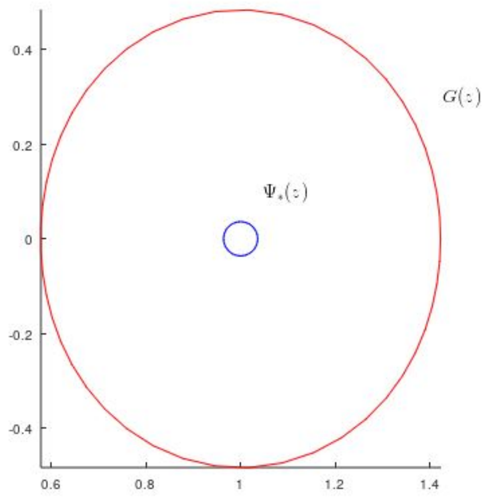

FIGURE 2. The image of $\Psi_*(z)$ (blue colour) is contained in $G(z)$(red colour)

3. Coefficient Estimates

In this section let us derive the coefficient estimates for the subclasses $\mathcal{S}_\sigma^*(S)$ and $\mathcal{C}_\sigma(S)$.

Theorem 3.1. *If $f \in \mathcal{S}_\sigma^*(S)$ is given by (1), then*

$$|a_2| \leq \frac{1}{2\sqrt{5}}, \tag{6}$$

$$|a_3| \leq \frac{5}{16}. \tag{7}$$

The inequalities are sharp.

Proof: Let $f \in \mathcal{S}_\sigma^*(S)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \Delta \to \Delta$, with $u(0) = v(0) = 0$, satisfying

$$\frac{2zf'(z)}{f(z) - f(-z)} = \phi(u(z)) \tag{8}$$

and

$$\frac{2wg'(w)}{g(w) - g(-w)} = \phi(v(w)) \tag{9}$$

where

$$\phi(u(z)) = \frac{2}{1 + e^{-u(z)}}$$

and

$$\phi(v(w)) = \frac{2}{1 + e^{-v(w)}}.$$

Define the functions p_1 and p_2 by

$$p_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

and

$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

or, equivalently,

$$\begin{aligned} u(z) &= \frac{p_1(z)-1}{p_1(z)+1} \\ &= \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} v(w) &= \frac{p_2(w)-1}{p_2(w)+1} \\ &= \frac{1}{2}\left(b_1 w + \left(b_2 - \frac{b_1^2}{2}\right) w^2 + \cdots\right). \end{aligned} \tag{11}$$

Then p_1 and p_2 are analytic in Δ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \Delta \to \Delta$, the functions p_1 and p_2 have positive real part in Δ, and $|b_i| \leq 2$ and $|c_i| \leq 2$. In view of (8), (10) clearly

$$\begin{aligned} \frac{2}{1+e^{-u(z)}} &= 1 + \frac{1}{4}c_1 z + \left(\frac{1}{4}c_2 - \frac{1}{8}c_1^2\right) z^2 \\ &\quad + \left(\frac{11}{192}c_1^3 - \frac{1}{4}c_2 c_1 + \frac{1}{4}c_3\right) z^3 + \cdots \end{aligned} \tag{12}$$

and from (9), (11), we have

$$\begin{aligned} \frac{2}{1+e^{-v(w)}} &= 1 + \frac{1}{4}b_1 w + \left(\frac{1}{4}b_2 - \frac{1}{8}b_1^2\right) w^2 \\ &\quad + \left(\frac{11}{192}b_1^3 - \frac{1}{4}b_2 b_1 + \frac{1}{4}b_3\right) w^3 + \cdots, \end{aligned} \tag{13}$$

while

$$\begin{aligned} \frac{2zf'(z)}{f(z)-f(-z)} &= 1 + 2a_2 z + 2a_3 z^2 \\ &\quad + 2z^3(2a_4 - a_2 a_3) + \cdots \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{2wg'(w)}{g(w)-g(-w)} &= 1 - 2a_2 w + 2(2a_2^2 - a_3)w^2 \\ &\quad + 2a_2(2a_2^2 - a_3)w^3 + \cdots \end{aligned} \tag{15}$$

From (12), (13), (14) and (15), we have

$$a_2 = \frac{c_1}{8}, \tag{16}$$

$$a_3 = \frac{1}{8}c_2 - \frac{1}{16}c_1^2, \tag{17}$$

$$a_2 = -\frac{b_1}{8}, \tag{18}$$

$$2a_2^2 - a_3 = \frac{b_2}{8} - \frac{b_1^2}{16}. \tag{19}$$

From (16) and (18), we have $c_1 = -b_1$ and

$$2a_2^2 = \frac{1}{64}(c_1^2 + b_1^2). \tag{20}$$

Using (17), (19) and (20) we obtain

$$\begin{aligned} 2a_2^2 &= \frac{1}{8}(b_2 + c_2) - \frac{1}{16}(b_1^2 + c_1^2) \\ &= \frac{1}{8}(b_2 + c_2) - 8a_2^2 \\ a_2^2 &= \frac{b_2 + c_2}{80}, \end{aligned} \tag{21}$$

Using (4) together with triangle inequality, it follows that

$$|a_2| \leq \frac{1}{2\sqrt{5}}.$$

This inequality is best possible for the function given by (8) and (9) with $u(z) = z^2$ and $v(w) = w^2$.
By subtracting (19) from (17), and using further computations leads to

$$a_3 = \frac{c_2 - b_2}{16} + a_2^2 \tag{22}$$

from (20), we get

$$a_3 = \frac{c_1^2 + b_1^2}{128} + \frac{c_2 - b_2}{16},$$

Using triangle inequality and (4), we obtain

$$|a_3| \leq \frac{5}{16}.$$

The inequality is best possible for the function given by (8) and (9) with $u(z) = z$ and $v(w) = w$.

Theorem 3.2. *If $f \in \mathcal{C}_\sigma(S)$ is given by (1), then*

$$|a_2| \leq \frac{1}{2\sqrt{19}},$$

$$|a_3| \leq \frac{19}{192}.$$

The inequalities are sharp.

Proof: Let $f \in \mathcal{C}_\sigma(S)$ and $g = f^{-1}$.
Then there are analytic functions $u, v : \Delta \to \Delta$,
with $u(0) = v(0) = 0$, satisfying

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = \phi(u(z)) \tag{23}$$

and

$$\frac{(2wg'(w))'}{(g(w) - g(-w))'} = \phi(v(w)) \tag{24}$$

where

$$\phi(u(z)) = \frac{2}{1 + e^{-u(z)}}$$

and

$$\phi(v(w)) = \frac{2}{1 + e^{-v(w)}}$$

Simple computation yields,

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = 1 + 4a_2 z + 6a_3 z^2 + 4z^3(4a_4 - 3a_2 a_3) + \cdots \tag{25}$$

$$\frac{(2wg'(w))'}{(g(w) - g(-w))'} = 1 - 4a_2 w + 6(2a_2^2 - a_3)w^2 + 28(7a_2^3 - 6a_2 a_3 + a_4)w^3 + \cdots . \tag{26}$$

From (12), (13), (25) and (26), we have

$$a_2 = \frac{c_1}{16}, \tag{27}$$

$$a_3 = \frac{1}{24}c_2 - \frac{1}{48}c_1^2, \tag{28}$$

$$a_2 = -\frac{b_1}{16}, \tag{29}$$

$$2a_2^2 - a_3 = \frac{b_2}{24} - \frac{b_1^2}{48}. \tag{30}$$

From (27) and (29), we have $c_1 = -b_1$ and

$$2a_2^2 = \frac{1}{256}(c_1^2 + b_1^2). \tag{31}$$

Using (28), (30) and (31) we obtain

$$2a_2^2 = \frac{1}{24}(b_2 + c_2) - \frac{1}{48}(b_1^2 + c_1^2)$$

$$a_2^2 = \frac{b_2 + c_2}{304}, \tag{32}$$

Using (4) together with triangle inequality, it follows that

$$|a_2| \leq \frac{1}{2\sqrt{19}}.$$

The inequality is best possible for the function given by (23) and (24) with $u(z) = z^2$ and $v(w) = w^2$. By subtracting (30) from (28), and using further computations leads to

$$a_3 = \frac{c_2 - b_2}{48} + a_2^2 \tag{33}$$

Using triangle inequality and (4), we get

$$|a_3| \leq \frac{19}{192}.$$

The inequality is best possible for the function given by (23) and (24) with $u(z) = z$ and $v(w) = w$.

Remark 3.1. *Brannan and Clunie's conjecture* [4] *are verified for the subclasses* $\mathcal{S}_\sigma^*(S)$ *and* $\mathcal{C}_\sigma(S)$.

4. Fekete Szegö inequality

The Fekete Szegö inequality for the subclasses $\mathcal{S}^*_\sigma(S)$ and $\mathcal{C}_\sigma(S)$ will be determined in this section.

Theorem 4.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{S}^*_\sigma(S)$. Then*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{1}{4}, & \text{if } |g(\lambda)| < \frac{1}{8} \\ 4|g(\lambda)|, & \text{if} |g(\lambda)| \geq \frac{1}{8} \end{cases} \tag{34}$$

where $\lambda \in R$ and $g(\lambda) = \frac{1-\lambda}{40}$.

Proof: From (22) for $\lambda \in R$, we have

$$a_3 - \lambda a_2^2 = \frac{c_2 - b_2}{16} + (1-\lambda)a_2^2. \tag{35}$$

By substituting (21) in (35), we get

$$a_3 - \lambda a_2^2 = \frac{c_2 - b_2}{16} + (1-\lambda)\frac{b_2 + c_2}{80} \tag{36}$$

$$= \left(g_1(\lambda) + \frac{1}{16}\right) c_2 \tag{37}$$

$$+ \left(g_1(\lambda) - \frac{1}{16}\right) b_2, \tag{38}$$

where $g_1(\lambda) = \frac{1-\lambda}{80}$.
Taking modulus,

$$|a_3 - \lambda a_2^2| \leq \left|\left(g(\lambda) + \frac{1}{8}\right)\right| + \left|\left(g(\lambda) - \frac{1}{8}\right)\right|, \tag{39}$$

where $g(\lambda) = \frac{1-\lambda}{40}$. Thus we conclude that,

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{1}{4}, & \text{if } |g(\lambda)| < \frac{1}{8} \\ 4|g(\lambda)|, & \text{if} |g(\lambda)| \geq \frac{1}{8} \end{cases}.$$

In similar lines, we can state the theorem for the class $\mathcal{C}_\sigma(S)$ as follows:

Theorem 4.2. *Let the function $f(z)$ given by (1) be in the class $\mathcal{C}_\sigma(S)$. Then*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{1}{12}, & \text{if } |h(\lambda)| < \frac{1}{24} \\ 4|h(\lambda)|, & \text{if} |h(\lambda)| \geq \frac{1}{24} \end{cases}$$

where $\lambda \in R$ and $h(\lambda) = \frac{1-\lambda}{152}$.

5. Second and Third order Toeplitz Determinants

In this section we find the second and third order Toeplitz determinants for the classes $\mathcal{S}^*_\sigma(S)$ and $\mathcal{C}_\sigma(S)$.

Theorem 5.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{S}^*_\sigma(S)$. Then*

$$|T_2(2)| \leq 0.1476,$$

$$|T_3(1)| \leq 1.1781.$$

Proof:
If $f \in \mathcal{S}$ is of the form (1) then

$$|T_3(1)| = |1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2|$$

Applying triangle inequality, we get

$$\leq 1 + 2|a_2^2| + |a_3||a_3 - 2a_2^2|$$

Using (6) , (7) and (34), we have

$$\begin{aligned} |T_3(1)| &\leq 1 + \frac{1}{10} + \frac{5}{64} \\ &\leq 1.1781. \end{aligned}$$

and

$$\begin{aligned} |T_2(2)| &= |a_2^2 - a_3^2| \\ &\leq |a_2^2| + |a_3^2| \\ &\leq 0.1476. \end{aligned}$$

In similar lines we can state the next theorem as follows:

Theorem 5.2. *Let the function $f(z)$ given by (1) be in the class $\mathcal{C}_\sigma(S)$. Then*

$$|T_2(2)| \leq 0.0229$$

$$|T_3(1)| \leq 1.0345.$$

Remark 5.1. *The determination of the sharp estimates for the second and third order Toeplitz determinants for the subclasses $\mathcal{S}_\sigma^*(S)$ and $\mathcal{C}_\sigma(S)$ remain to be explored.*

6. Conclusions

Artificial neurons have been activated using a broad range of sigmoid functions, such as the logistic and hyperbolic tangent functions. In statistics, sigmoid curves are also frequently used as cumulative distribution functions (that is, functions that range from 0 to 1), such as the logistic density integral, normal density integral, Student's t probability density functions and in this paper we made a connections with some subclasses of analytic functions. The current study dealt with the upper bounds of Toeplitz determinants of symmetric functions and we obtained the Sharp coefficient estimates, Fekete-Szegö inequality, second and third order Toeplitz determinants, for the subclasses $\mathcal{S}_\sigma^*(S)$, $\mathcal{C}_\sigma(S)$ of bi-univalent functions associated with the modified sigmoid function. We anticipate great applications of these findings in the domains of mathematics, engineering, science, and technology, we also motivate further research into the determination of sharp estimates of second and third-order Toeplitz determinants for these subclasses.

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