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SEIDEL LAPLACIAN ENERGY OF INTUITIONISTIC FUZZY GRAPHS

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Abstract. A graph's energy is correlated with its spectrum,which is the sum of the latent values in the relevant adjacency matrix.We suggested various properties and the energy of the Seidel Laplacian of an intuitionistic fuzzy graph in this study effort. Furthermore, with appropriate illustrative cases, the lower and upper bounds on the energy of the Seidel Laplacian of an intuitionistic fuzzy graph were examined.

Keywords: Intuitionistic fuzzy graph, Seidel Laplacian energy, fuzzy graph and Intuitionistic fuzzy adjacency matrix

AMS Subject Classification: 03E72, 03B52

1. Introduction

Fuzzy sets are defined as functions of membership alone, as first presented by Zadeh [1] in 1965.In 1973, Kaufman [2] proposed the fuzzy graph to describe his fuzzy connection. In real-world situations where there was ambiguity, fuzzy sets were employed.Various applications, such as monitoring the highest power from solar power voltaic systems and controller circuit design, can benefit from the utilization of fuzzy logic. The notions of intuitionistic fuzzy relations and intuitionistic fuzzy graphs (IFG) were first presented by Atanassov [3]. The fuzzy graph's concept and design were initially proposed by [4] Rosenfeld. Unlike Gutman [5], who talked about the concepts underlying graph energy and how they connect to the total electron energy in certain compounds. In relation to the total electron energy of certain molecules, Balakrishnan[6] investigated the concepts of graph energy. Determining the boundaries of superior and inferior graph energy, Anjali and Mathew[7] examined the energy of a fuzzy graph. First proposed by Sharbaf and Fayazi [8], the concept of a fuzzy graph's Laplacian energy (LE).An IFG was introduced to the fuzzy graph notion by Parvathi and Karunambigai[9], how to interpret an IFG's Laplacian energy. Basha and Kartheek[10] extend the Laplacian energy concept to intuitionistic fuzzy graphs, defining adjacency matrix and Laplacian energy. Lower and

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upper bounds are derived and verified.Nageswari et al.[11] investigate the upper and lower bounds for the Seidel energy of connected and disconnected graphs, using the rank of the matrix. Intuitionistic fuzzy graphs are extensions of fuzzy graphs that can better describe ambiguity in decision-making issues.Akula et al.[13, 14, 15, 16, 17] compute the Laplacian energy and correlation coefficient of intuitionistic fuzzy graphs for group decision-making problems.It proposes a method for calculating comparative position loads and presents a collaborative decision-making technique in a money-investing scheme.N.R. Reddy et al. [19, 20, 21] improve the effectiveness of hesitancy fuzzy graphs in managing ambiguous and inconsistent input by introducing a new similarity metric. Based on expert preference data and hesitancy fuzzy preference relations, the researchers suggest a weighting method utilizing Laplacian energy and similarity measurements in a hesitancy fuzzy environment. By looking at the absolute departures of the eigenvalues of Seidel Signless Laplacian matrix (SL+) and Seidel Signless Laplacian energy (ESL+), respectively, from their mean, Ramane et al.[18] investigate the characteristics of the Seidel Laplacian matrix (SL) and the Seidel Signless Laplacian energy (ESL+).Sivaranjani et al.[22, 23] explore the energy of a graph, which is determined by its spectrum, which is the total latent values of the adjacency matrix. It proposes features and energy of the Seidel Laplacian of a fuzzy graph and studies the lower and upper bounds using suitable examples. The notion of the energy of an intuitionistic fuzzy graph is extended to encompass the idea of a fuzzy graph. To learn more about this topic, see [22]. This article extends the idea of an intuitionistic fuzzy graph's energy for Seidel Laplacian fuzzy graphs to another type of fuzzy graph.

2. Preliminaries

2.1. **Definition.** Fuzzy graphs with vertex sets V and edges sets E, fuzzy membership function u defined on V*V, and fuzzy non-membership function v,construct an IFG, G_i=(V,E,u,v).Such that

- $0 \leq u_{ij} + v_{ij} \leq 1$
- $0 \leq u_{ij}, v_{ij}, \pi_{ij} \leq 1$, where $\pi_{ij} = 1 - (u_{ij} + v_{ij})$.

2.2. **Definition.** This definition of an intuitionistic fuzzy adjacency matrix of an intuitionistic fuzzy graph is the adjacency matrix of the associated intuitionistic fuzzy graph. That pertains to a fuzzy intuitionistic graph G=(V,E,u,v), an intuitionistic fuzzy adjacency matrix is called a M(G)=[u_{ij}],where a_{ij}=(u_{ij},v_{ij}). Keep in mind that u_{ij} indicates the degree of the non-connection between v_i and v_i, and v_{ij} depicts the strength of the relationship between v_i and v_j.

2.2.1. *Example.* In Fig.1, an intuitionistic fuzzy graph, the adjacency matrix is

$$A = \begin{bmatrix} (0,0) & (0.8,0.1) & (0.3,0.2) & (0.1,0.3) \\ (0.4,0.3) & (0,0) & (0.1,0.6) & (0.3,0.1) \\ (0.4,0.2) & (0.5,0.3) & (0,0) & (0.4,0.5) \\ (0.2,0.6) & (0.3,0.4) & (0.5,0.3) & (0,0) \end{bmatrix}$$

2.3. **Definition.** An intuitionistic fuzzy graphs adjacency matrix can be expressed as two matrix, one with entries representing membership values and the other with entries representing non-membership values i.e M(G)=[(u_{ij},v_{ij})],where

$$M(u_{ij}) = \begin{bmatrix} 0 & 0.8 & 0.3 & 0.1 \\ 0.4 & 0 & 0.1 & 0.3 \\ 0.4 & 0.5 & 0 & 0.4 \\ 0.2 & 0.3 & 0.5 & 0 \end{bmatrix} \qquad M(v_{ij}) = \begin{bmatrix} 0 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0 & 0.5 \\ 0.6 & 0.4 & 0.3 & 0 \end{bmatrix}$$

2.4. **Laplacian energy of an intuitionistic fuzzy graph.**

2.4.1. *Definition.* One may define the eigenvalues of an intuitionistic fuzzy adjacency matrix M(G) as (X, Y), where X represents the set of eigenvalues of M (u_{ij}) and Y represents the set of eigenvalues of M(v_{ij}).

2.4.2. *Definition.* Assume that G=(V,E,u, v) has an adjacency matrix M(G) and a degree matrix D(G) = $[a_{ij}]$.The matrix L(G)=D(G) − M(G). Is described as the G's intuitionistic fuzzy Laplacian matrix. In this case, the Laplacian matrix of an intuitionistic fuzzy graph may be expressed as two matrices, one with entries representing membership values and the other with entries representing non-membership values.

i.e $LE(G) = [LE(u_{ij}), LE(v_{ij})] = (\sum_{i=1}^{n} |S_i|, \sum_{i=1}^{n} |T_i|)$

Where,

$$S_i = \delta_i - \frac{2\sum_{1\le i<j\le n} u(v_i v_j)}{n}, \quad T_i = \omega_i - \frac{2\sum_{1\le i<j\le n} v(v_i v_j)}{n}.$$

2.5. **Seidel Laplacian energy of an intuitionistic fuzzy graph.**

2.5.1. *Definition.* Any graph's real symmetric matrix $S(G) = (s_{ij})$, where $s_{ij} = 1$ for vertices that are near to one another, $s_{ij} = -1$ for vertices that are not, and $s_{ij} = 0$ for nodes where $i = j$, is its Seidel matrix.

2.5.2. *Definition.* DS(G)=diagonal (m-1-2d_i), where i=1,2,. . . ..n

2.5.3. *Defination.* The specified Seidel Laplacian matrix of an intuitionistic fuzzy graph

$$SLe(G) = D(G) - S(G)$$

2.5.4. *Defination.* The Seidel Laplacian energy of an intuitionistic fuzzy graphs as follows. Let 1, 2, 3. . . m be the latent values of SL(G)

i.e $SLe(G) = [SLe(u_{ij}), SLe(v_{ij})] = (\sum_{i=1}^{n} |S_i|, \sum_{i=1}^{n} |T_i|)$

Where,

$$S_i = \delta_i - \frac{m(m-1) - 4\sum a_{ij}}{n}, \quad T_i = \omega_i - \frac{m(m-1) - 4\sum a_{ij}}{n}, where 1 \le i < j \le m$$

2.5.5. *Example.* Examine the fuzzy graph in Fig. 1 that follows 5 nodes and 6 edges. Figure 1 clarifies the IFAM.

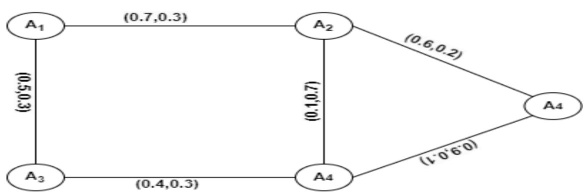

FIGURE 1. Intuitionistic fuzzy Graph

$$A(G) = \begin{bmatrix} (0,0) & (0.5,0.3) & (0.7,0.3) & (0,0) & (0,0) \\ (0.3,0.5) & (0,0) & (0.6,0.2) & (0,0) & (0,0) \\ (0,0) & (0.6,0.2) & (0,0) & (0.1,0.7) & (0.9,0.1) \\ (0.3,0.7) & (0,0) & (0.1,0.7) & (0,0) & (0.4,0.3) \\ (0,0) & (0,0) & (0.9,0.1) & (0.4,0.3) & (0,0) \end{bmatrix}$$

After that, d(G),DS(G) , and SL(G) are defined as follows.

$$d(G) = \begin{bmatrix} (1.2, 0.8) & (0, 0) & (0, 0) & (0, 0) & (0, 0) \\ (0, 0) & (0.9, 0.7) & (0, 0) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (0.7, 0.9) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (0, 0) & (0.8, 1.7) & (0, 0) \\ (0, 0) & (0, 0) & (0, 0) & (0, 0) & (1.3, 0.4) \end{bmatrix}$$

$$DS(G) = \begin{bmatrix} (1.6, 2.8) & (0, 0) & (0, 0) & (0, 0) & (0, 0) \\ (0, 0) & (2.2, 2.6) & (0, 0) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (2.6, 2.2) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (0, 0) & (2.4, 0.6) & (0, 0) \\ (0, 0) & (0, 0) & (0, 0) & (0, 0) & (1.4, 3.2) \end{bmatrix}$$

$$SL(G) = \begin{bmatrix} (0, 0) & (1, 1) & (1, 1) & (-1, -1) & (-1, -1) \\ (1, 1) & (0, 0) & (1, 1) & (-1, -1) & (-1, -1) \\ (-1, -1) & (1, 1) & (0, 0) & (1, 1) & (1, 1) \\ (1, 1) & (-1, -1) & (1, 1) & (0, 0) & (1, 1) \\ (-1, -1) & (-1, -1) & (1, 1) & (1, 1) & (0, 0) \end{bmatrix}$$

Then Seidel Laplacian matrix is define as

$$SLe(G) = D(G) - SL(G)$$

$$SL(G) = \begin{bmatrix} (1.6, 2.8) & (-1, -1) & (-1, -1) & (1, 1) & (1, 1) \\ (-1, -1) & (2.2, 2.6) & (-1, -1) & (1, 1) & (1, 1) \\ (1, 1) & (-1, -1) & (2.6, 2.2) & (-1, -1) & (-1, -1) \\ (-1, -1) & (1, 1) & (-1, -1) & (2.4, 0.6) & (-1, -1) \\ (1, 1) & (1, 1) & (-1, -1) & (-1, -1) & (1.4, 3.2) \end{bmatrix}$$

The latent membership values of SLe(G) is δ_i are

$$-0.7201, 4.0654, 2.7317, 2.7317 \text{and} 1.3952.$$

The latent non-membership values of SLe(G) is ω_i are

$$-0.6424, 4.4970, 3.7177, 2.2726 \text{and} 1.5550.$$

If G is an intuitionistic fuzzy graph, its Seidel Laplacian energy is

$$SLe(G, u) = \sum_{i=1}^{m} |\delta_i - \frac{m(m-1) - 4\sum a_{ij}}{n}|$$

$$SLe(G, u) = \sum_{i=1}^{m} |\delta_i - \frac{5(5-1) - 4\sum a_{ij}}{n}| = \sum_{i=1}^{m} |\delta_i - (-0.64)|$$

$$SLe(G,u) = |-0.7201+0.64|+|-0.40654+0.64|+|2.7391+0.64|+|2.7391+0.64|+|-0.7201+0.64|$$

$$SLe(G,u) = 13.5789$$

.

$$SLe(G,v) = \sum_{i=1}^{m} |\omega_i - \frac{m(m-1) - 4\sum a_{ij}}{n}|$$

$$SLe(G,v) = \sum_{i=1}^{m} |\delta_i - \frac{5(5-1) - 4\sum a_{ij}}{n}| = \sum_{i=1}^{m} |\delta_i - (0.48)|$$

$$SLe(G,v) = |-0.6424-0.48|+|4.49670-0.48|+|3.7177-0.48|+|2.2726-0.48|+|1.5550-0.48|$$

$$SLe(G,v) = 11.2447.$$

$$Now, SLe(G,u) = 13.5789 \& SLe(G,v) = 11.2447.$$

$$Therefore, SLe(G) = (13.5789, 11.2447).$$

Theorem 2.1. *Let us assume that an intuitionistic fuzzy graph is one with V nodes and E edges. The latent values of the SLe matrix thus satisfy the following relations.*
i. $\sum_{i=1}^{m} \delta_i = (m^2 - m) - 4\sum a_{ij}$ *and* $\sum_{i=1}^{m} \omega_i = (m^2 - m) - 4\sum a_{ij}$
ii. $\sum_{i=1}^{m} \delta_i^2 = m^2(m-1) - 8(m-1)\sum a_{ij} + 4\sum d_i^2$ *and*
$\sum_{i=1}^{m} \omega_i^2 = m^2(m-1) - 8(m-1)\sum a_{ij} + 4\sum d_i^2$

Proof. i.

$$\sum_{i=1}^{m} \delta_i = trace(SLe(G)) = \sum_{i=1}^{m} m - 1 - 2d_i \sum_{i=1}^{m} \delta_i = (m^2 - m) - 4\sum a_{ij}.$$

Similarly, we may show that,

$$\sum_{i=1}^{m} \omega_i = trace(SLe(G)) = \sum_{i=1}^{m} m - 1 - 2d_i \sum_{i=1}^{m} \omega_i = (m^2 - m) - 4\sum a_{ij}.$$

ii.

$$\sum_{i=1}^{m} \delta_i^2 = trace(SLe(G))^2 = m^2(m-1) + 8(m-1)\sum a_{ij} + 4\sum d_i^2.$$

Similarly, we may show that,

$$\sum_{i=1}^{m} \omega_i^2 = trace(SLe(G))^2 = m^2(m-1) + 8(m-1)\sum a_{ij} + 4\sum d_i^2.$$

Since
$\sum_{i=1}^{m} \delta_i^2$=trace(SLe(G))= the sum of the latent membership values of SLe(G)

and

$\sum_{i=1}^{m} \omega_i^2$=trace(SLe(G))= the sum of the latent membership values of SLe(G) □

Theorem 2.2. *Let us assume that there are E edges and V nodes in an intuitionistic fuzzy graphs. Then* $\sum_{i=1}^{m} \delta_i = 0$ *and* $\sum_{i=1}^{m} \omega_i = 0$

Proof. Since

$$\sum_{i=1}^{m} S_i = \delta_i - \frac{m(m-1) - 4\sum a_{ij}}{m}$$

$$\sum_{i=1}^{m} S_i = (m^2 - m) - 4\sum a_{ij} = (m^2 - m) - 4\sum a_{ij} - (m^2 - m) - 4\sum a_{ij}$$

$$\sum_{i=1}^{m} \delta_i = 0$$

and similarity,we may show that

$$\sum_{i=1}^{m} \omega_i = 0$$

□

Theorem 2.3. *Let us assume that there are E edges and V nodes in an intuitionistic fuzzy graphs. Then*$\sum_{i=1}^{m} S_i = (m^2 - m) - 16\frac{\sum a_{ij}}{m} + 4\sum d_i^2$ *and*
$\sum_{i=1}^{m} S_i = (m^2 - m) - 16\frac{\sum a_{ij}}{m} + 4\sum d_i^2$

Proof. Consider

$$\sum_{i=1}^{m} S_i^2 = \sum_{i=1}^{m} \left(\delta_i - \frac{(m^2 - m - 4\sum a_{ij})}{m}\right)^2$$

$$= \sum_{i=1}^{m} \delta_i^2 - 2\left(\frac{(m^2 - m - 4\sum a_{ij})}{m}\right)\sum_{i=1}^{m} \delta_i + \left(\frac{(m^2 - m - 4\sum a_{ij})}{m}\right)^2$$

$$= \sum_{i=1}^{m} \delta_i^2 = (m^2 - m) - 16\frac{\sum a_{ij}}{m} + 4\sum d_i^2$$

and Similarly, we may show that

$$= \sum_{i=1}^{m} \omega_i^2 = (m^2 - m) - 16\frac{\sum a_{ij}}{m} + 4\sum d_i^2$$

□

Theorem 2.4. *Let us assume that there are E edges and V nodes in an intuitionistic fuzzy graphs. Then* $\sqrt{m\left((m^2 - m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_i j^2}{m}\right)} \leq SLe(G_u)$ *and*
$m\left((m^2 - m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_i j^2}{m}\right) \leq SLe(G_v)$

Proof. Cauchy-Schwarz inequality states

$$\left(\sum_{i=1}^{n} g_i h_i\right)^2 \leq \left(\sum_{i=1}^{n} g_i^2\right)\left(\sum_{i=1}^{n} h_i^2\right)$$

Assume g_i and h_i=$|S_i|^2$,where i=1,2,...,n.
Then

$$\left(\sum_{i=1}^{n}(1)(|S_i|)\right)^2 \leq \left(\sum_{i=1}^{n}(1)\right)\left(\sum_{i=1}^{n}|S_i|^2\right)$$

$$\left(\sum_{i=1}^{n}(|S_i|)\right)^2 = m\left(\sum_{i=1}^{n}|S_i|^2\right)$$

$$m(m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_ij^2}{m} \leq SLe(G_u)^2$$

That implies that $\sqrt{m\left((m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_ij^2}{m}\right)} \leq SLe(G_u)$. Likewise,we might demonstrade that non-membership

$$\sqrt{m\left((m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_ij^2}{m}\right)} \leq SLe(G_v)$$

□

Theorem 2.5. *Let us assume that there are E edges and V nodes in an intuitionistic fuzzy graphs. Then* $\sqrt{2\left((m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_ij^2}{m}\right)} \leq SLe(G_u)$ *and* $\sqrt{2\left((m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_ij^2}{m}\right)} \leq SLe(G_v)$

Proof. Consider

$$(SLe(G,u))^2 = \sum_{i=m}^{2} +2\sum_{i<j}^{m}|S_i||S_j|$$

$$(SLe(G,u))^2 \leq \sum_{i=1}^{m}|S_i|^2+\sum_{i<j}^{m}|S_iS_j|$$

$$(SLe(G,u))^2 = (m^2-m)+4\sum_{i=1}^{2}d_i^2-\frac{16\sum a_{ij}}{m}+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}$$

$$(SLe(G,u))^2 = 2(m^2-m)+4\sum_{i=1}^{2}d_i^2-\frac{16\sum a_{ij}^2}{m}$$

$$(SLe(G,u)) \geq \sqrt{2\left((m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}\right)}$$

Likewise,we might demonstrate that non-membership

$$(SLe(G,u)) \geq \sqrt{2\left((m^2-m)+4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right)}$$

□

Theorem 2.6. *Let us assume that there are E edges and V nodes in an intuitionistic fuzzy graphs then*
$i.S_i = \delta_i - \frac{(m^2-m)-4\sum a_{ij}}{m}$*.Let* $S_{mini} + mini|S_i|$ *and* $S_{maxi} + maxi|S_i|$*,where* $i = 1 \leq i \leq m$, $\sqrt{m\left[(m^2-m)+4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right] - \frac{m^2}{4}(S_{maxi}-S_{mini})^2} \leq SLe(G)$
$ii.T_i = \delta_i - \frac{(m^2-m)-4\sum a_{ij}}{m}$*.Let* $T_{mini} + mini|T_i|$ *and* $T_{maxi} + maxi|T_i|$*,where* $i = 1 \leq i \leq m$, $\sqrt{m\left[(m^2-m)+4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right] - \frac{m^2}{4}(T_{maxi}-T_{mini})^2} \leq SLe(G)$

Proof. i.Using the outcome [17], let an intuitionistic fuzzy graphs consist of V nodes and E edges.Which

$$\sum_{i=1}^{m} g_i^2 \sum_{i=1}^{m} h_i^2 = \left(\sum_{i=1}^{m} g_i^2 h_i^2\right) \leq \frac{m^2}{4}(A_1A_2 - a_1a_2)^2.$$

Where $A_1 = maxi(g_i), A_2 = maxi(h_i), a_1 = mini(g_i), a_2 = mini(h_i), 1 \leq i \leq m$,
$i = 1, 2$.Let $g_i = 1$ and $h_i = |S_i|$.Then
$\sum_{i=1}^{2} 1 \sum_{i=1}^{m} |S_i|^2 - (\sum_{i=1}^{m} |S_i|)^2 \leq \frac{m^2}{4}(S_{maxi} - S_{mini})$
$m\left[(m^2-m)+4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]$, this impliees that

$$\sqrt{m\left[(m^2-m)+4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right] - \frac{m^2}{4}(S_{maxi}-S_{mini})^2} \leq SLe(G)$$

ii. Similarly, we may show that

$$\sqrt{m\left[(m^2-m)+4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right] - \frac{m^2}{4}(T_{maxi}-T_{mini})^2} \leq SLe(G)$$

□

Theorem 2.7. *Let us assume that there are E edges and V nodes in an intuitionistic fuzzy graphs and* S_{mini}, S_{maxi} *and* T_{mini}, T_{maxi} *be same as theorem 6. Then*
$i.SLe(G) \geq -\frac{\sqrt{S_{mini}S_{maxi}}\sqrt{m\left[(m^2-m)+4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]}}{S_{mini}S_{maxi}}$,
$ii.SLe(G) \geq -\frac{\sqrt{T_{mini}T_{maxi}}\sqrt{m\left[(m^2-m)+4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]}}{T_{mini}T_{maxi}}$

Proof. Using the result [17],

$$\sum_{i=1}^{m} g_i^2 \sum_{i=1}^{m} h_i^2 \leq \frac{m^2}{4}\left(\sqrt{\frac{A_1A_2}{a_1a_2}} - \sqrt{\frac{a_1a_2}{A_1A_2}}\right)^2 \left(\sum_{i=1}^{m} g_ih_i\right)^2$$

Let $g_i = 1$ and $h_i = |S_i|$. Then $\sum_{i=1}^{m} 1 \sum_{i=1}^{m} |S_i|^2 - (\sum_{i=1}^{m} |S_i|)^2$

$$\leq \frac{1}{4}\left(\frac{\sqrt{S_{maxi-}}}{\sqrt{S_{mini}}} + \frac{\sqrt{S_{mini}}}{\sqrt{S_{maxi-}}}\right)^2 \left(\sum_{i=1}^{m} |S_i|\right)^2$$

$$m\left[(m^2-m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right] \leq \frac{1}{4}\left(\frac{\sqrt{S_{maxi-}}}{\sqrt{S_{mini}}} + \frac{\sqrt{S_{mini}}}{\sqrt{S_{maxi-}}}\right)^2 \left(\sum_{i=1}^{m} |S_i|\right)^2$$

$$= \frac{1}{4}\frac{(S_{maxi} + S_{mini})^2}{S_{maxi}S_{mini}} SLe(G)^2$$

This implies that ,

$$\text{SLe}(G) \geq -\frac{\sqrt{S_{\text{mini}}S_{\text{maxi}}}\sqrt{m\left[(m^2-m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]}}{S_{\text{mini}}S_{\text{maxi}}}$$

Similarity,we may show that.

$$\text{SLe}(G) \geq -\frac{\sqrt{T_{\text{mini}}T_{\text{maxi}}}\sqrt{m\left[(m^2-m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]}}{T_{\text{mini}}T_{\text{maxi}}}$$

□

Theorem 2.8. *Let us assume that there are E edges and V nodes in an intuitionistic fuzzy graphs and λ(n) is defined as* $\lambda(m) = m(\frac{m}{2})(1 - (\frac{1}{m})(\frac{m}{2}))$[17]*.Then*

$$\sqrt{m\left[(m^2-m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]} - \lambda(n)(S_{mini} - S_{maxi})^2 \leq SLe(G), and$$

$$\sqrt{m\left[(m^2-m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]} - \lambda(n)(T_{mini} - T_{maxi})^2 \leq SLe(G)$$

Proof. Using the result [17],

$$|\sum_{i=1}^{m} g_i h_i \sum_{i=1}^{m} g_i^2 \sum_{i=1}^{m} h_i^2| \leq \lambda(n)(A-a)(B-b)$$

in which the conditions $a \leq g_i \leq A, b \leq h_i \leq b$ are satisfied.
Let $g_i = h_i = |S_i|, a = b = S_{mini} + S_{maxi}$.This suggests that

$$|m\sum_{i=1}^{m} |S_i|^2 \sum_{i=1}^{m} |S_i| \sum_{i=1}^{m} |S_i|| \leq \lambda(n)(S_{mini} - S_{maxi})$$

$$\sqrt{m\left[(m^2-m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]} SLe(G)^2 \leq \lambda(n)(S_{\text{mini}} - S_{\text{maxi}})^2$$

$$SLe(G)^2 + \lambda(n)(S_{mini} - S_{maxi})^2 \geq$$

$$\sqrt{m\left[(m^2-m) + 4\sum_{i=1}^{m} d_i^2 - \frac{16\sum a_{ij}^2}{m}\right]} - \lambda(n)(S_{\text{mini}} - S_{\text{maxi}})^2 \leq SLe(G)$$

.Similarity,we may show that

$$\sqrt{m\left[(m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}\right]-\lambda(n)(T_{\text{mini}}-T_{\text{maxi}})^2}\leq SLe(G)$$

□

Theorem 2.9. *Let us assume that there are E edges and V nodes in an intuitionistic fuzzy graph. Then*

$$SLe(G)\geq\frac{mS_{mini}S_{maxi}\left[(m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}\right]}{S_{mini}+S_{maxi}}, and$$

$$SLe(G)\geq\frac{mT_{mini}T_{maxi}\left[(m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}\right]}{T_{mini}+T_{maxi}}$$

Proof. Using the result[17],

$$\sum_{i=1}^{n}g_i^2+xX\sum_{i=1}^{n}h_i^2\leq(x+X)\sum_{i=1}^{n}g_ih_i.$$

In which x,X are real constants g_i, h_i are real constants that for each I,$1\leq i\leq p$ the conditions $xg_i\leq h_i\leq Xg_i$ are satisfied.
Let $g_i=1,\ h_i=|S_i|, x=S_{mini}, X=S_{maxi}$ this implies that

$$\sum_{i=1}^{n}\left(|S_i|^2+S_{\min i}S_{\max i}\sum_{i=1}^{m}(1)\right)\leq\sum_{i=1}^{m}|S_i|$$

$$(m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}+mS_{\min i}S_{\max i}\leq(S_{\min i}+S_{\max i})SLe(G)^m$$

$$(S_{\min i}+S_{\max i})SLe(G)\geq mS_{\min i}S_{\max i}+\left\{(m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}\right\}$$

$$SLe(G)\geq\frac{mS_{\min i}S_{\max i}+\left\{(m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}\right\}}{(S_{\min i}+S_{\max i})},$$

similarly,we may show that

$$SLe(G)\geq\frac{mT_{\min i}T_{\max i}+\left\{(m^2-m)+4\sum_{i=1}^{m}d_i^2-\frac{16\sum a_{ij}^2}{m}\right\}}{(T_{\min i}+T_{\max i})}$$

□

3. Conclusions

The idea of Seidel Laplacian energy (SLe) for an intuitionistic fuzzy graph is extended from that of SLe for a fuzzy graph.In this work, we defined the SLe of an intuitionistic fuzzy graphs with the help of its adjacent matrix. We provide lower and upper limits for the SLe of intuitionistic fuzzy graphs.We want to investigate more types of fuzzy graph in the future, and it is expected that these investigations may produce similar limitations on SLe. Future research on group decision-making under Seidel Laplacian and Signless Laplacian energies may be the main focus. Additionally, it may be used for decision-making challenges.

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