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NONLINEAR LANGEVIN FRACTIONAL DIFFERENTIAL EQUATION WITH NONLOCAL MIXED BOUNDARY CONDITIONS INVOLVING A CAPUTO-EXPONENTIAL

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Abstract. In this paper, the existence and uniqueness results for a nonlinear Langevin fractional differential equation with nonlocal mixed (multipoint, fractional integral and fractional derivative) boundary conditions involving a Caputo-exponential is studied. The uniqueness result is discussed via Banach's contraction mapping principle, and the existence of solutions is proved by using Schaefer's fixed point theorem. Finally, an example is also constructed to demonstrate the application of the main results.

Keywords: Langevin equation, Caputo's-exponential fractional derivative, implicit fractional differential, nonlocal mixed boundary conditions, fixed point theorems.

AMS Subject Classification: 35B40, 35L70.

1. Introduction

Fractional calculus has been utilized to represent physical and engineering phenomena, often best captured by fractional differential equations. It's important to highlight that traditional mathematical models using integer-order derivatives, even nonlinear ones, often fall short in various situations. This is due to the fact that fractional differential equations have various applications in engineering and scientific disciplines, for example, fluid dynamics, fractal theory, diffusion in porous media, fractional biological neurons, traffic flow, polymer rheology, neural network modeling, viscoelastic panel in supersonic gas flow, real system characterized by power laws, electrodynamics of complex medium, sandwich system identification, nonlinear oscillation of earthquake, models of population growth, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, nuclear reactors and theory of population dynamics. For more details about the theory of fractional calculus, fractional differential equations and their applications, we refer to the reader the monographs of Abbas et al. [1, 2], B. Ahmad et al. [3, 4], Agarwal et al. [5], Baleanu et al. [7], Benchohra et al. [9, 10], Fečkan et al. [15], Hilfer [19, 20], Kilbas et al. [23], Oldham et al. [32], Podlubny [33], Zhou et al. [44] and the reference therein.
Although the definition of the Riemann–Liouville type played an important role in the

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development of the theory of fractional calculus, real-world problems require fractional derivatives that contain physically interpretable initial conditions [1]-[5]. To overcome such problems, Caputo proposed a definition of the fractional derivative, the Caputo fractional derivative offers a significant advantage by enabling the integration of traditional initial and boundary conditions into problem formulation. Notably, conventional mathematical models employing integer-order derivatives often prove inadequate in diverse scenarios. However, some researchers have found it necessary to define new fractional derivatives with different singular or nonsingular kernels in order to provide more sufficient area to model more real-world problems in different fields of science and engineering [6, 12].
The question is whether there is a Caputo-exponential derivative in Fractional calculus?. The answer to this question is positive. There is indeed a fractional derivative, namely the Caputo/Riemann-Liouville fractional derivative with an exponential kernel (we call it the Caputo/Riemann-Liouville fractional exponential derivative for short), see [8, 14, 26]. The difference between the Caputo-exponential fractional derivative and the Caputo/Riemann-Liouville fractional derivative is that the solutions of classical differential equations or fractional differential equations with the Caputo/Riemann-Liouville derivative have "algebraic" asymptotics [22, 28] while the solutions of fractional equations with the Caputo-Hadamard derivative have "logarithmic" asymptotics [26] and the solutions of fractional differential equations with the Caputo exponential derivative have "exponential" asymptotics [27].
Langevin equation was introduced by Paul Langevin in 1908 [25], a brilliant French physicist in the early twentieth century, he proposed the nonlinear Langevin equation and created an accurate description of Brownian motion using his Langevin equation. The Langevin differential equation was used to explain the physical processes in oscillating domains. Analyzing the stock market [11], modelling evacuation processes [24], studying fluid suspensions [21], self organization in complex systems [16], photo-electron counting [42] and protein dynamics [36] are just some applications of this equation.
The virtually simultaneous development of fractional derivatives, various generalizations of the Langevin equation have been proposed and studied by various researchers during recent years. Despite the widespread use these applications, the fractional Langevin equation is extensively studied in the literature in both froms: theoretical and numerical points of view.
In 2020, Salem [34] have discussed existence and uniqueness results of solutions for anti-periodic fractional Langevin equation given by

$$\begin{cases} {}^cD^{\beta}({}^cD_0^{\alpha}+\lambda)x(t) = f(t, x(t), {}^cD^{\alpha}x(t)), \ t \in [0,1], \\ \\ x(0)+x(1)=0, x'(0)=0, {}^cD^{\alpha}x(1) = \dfrac{\mu}{\Gamma(\gamma)}\displaystyle\int_0^{\eta}(\eta-s)^{\gamma-1}x(s)ds, \end{cases}$$

where ${}^cD^{\beta}$, ${}^cD^{\alpha}$ are fractional derivatives in the Caputo sense with values of $\beta \in (1,2]$, $\alpha \in (0,1)$, $0 < \eta < 1$, $\gamma > 0$, $\mu \in \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function.
In 2021, A. Seemab et al. [37] investigated the existence, uniqueness and stability in the sense Ulam Hyers Rassias of solutions for Langevin equation with nonlocal boundary conditions involving a ψ-Caputo fractional operators of different orders given by

$$\begin{cases} {}^c_{a+,t}\mathfrak{D}^{\alpha}({}^c_{a+,t}\mathfrak{D}^{\alpha}+\lambda)[x] = \mathfrak{f}(t, x(t), {}^c_{a+,t}\mathfrak{D}^{\gamma}[x]), t \neq (a, T), \\ \\ x(a)=0, \ x(\eta)=0, \ x(T) = \mu({}^c_{a+,\xi}J^{\gamma,\psi})[x], \ \mu > 0, \end{cases}$$

where $\mu({}^{c}_{a+,\xi}J^{\gamma,\psi})$, ${}^{c}_{a+,t}\mathfrak{D}^{\theta}$ are Ψ-fractional integral of order γ, Ψ-Caputo fractional derivative of order $\theta \in \{\alpha, \beta, \gamma\}$ respectively, $0 \leq a < \eta < \xi < T < \infty$, $1 < \alpha \leq 2$, $0 < \beta < \gamma \leq 1$, λ is a real number and $\mathfrak{f} : [a, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{+}$ is a continuous function.

In 2022, Hamdy and Ahmed [18] investigated the existence and uniqueness results for a nonlinear coupled system of nonlinear fractional Langevin equations with a new kind of boundary conditions of the form

$$\begin{cases} {}^{c}\mathfrak{D}^{\kappa_1}_{0+}({}^{c}\mathfrak{D}^{\nu_1}_{0+} + \chi_1)x_1(t) = \Psi_1(t, x_1(t), x_2(t)),\ t \in J := [0, T], 1 < \nu_1 \leq 2, 1 < \kappa_1 \leq 2, \\ \\ {}^{c}\mathfrak{D}^{\kappa_2}_{0+}({}^{c}\mathfrak{D}^{\nu_2}_{0+} + \chi_2)x_2(t) = \Psi_2(t, x_1(t), x_2(t)),\ t \in J := [0, T], 1 < \nu_2 \leq 2, 1 < \kappa_2 \leq 2, \end{cases}$$

subject to the following coupled boundary conditions

$$\begin{cases} x_1(0) = 0, x_1(T) = \delta_1 x_1(\eta_1),\ x_1'(T) = \epsilon_1 x_1(\xi_1) \\ x_2(0) = 0, x_2(T) = \delta_2 x_1(\eta_2),\ x_2'(T) = \epsilon_2 x_1(\xi_2) \end{cases}$$

where ${}^{c}\mathfrak{D}^{\kappa_1}_{0+}$, ${}^{c}\mathfrak{D}^{\kappa_2}_{0+}$, ${}^{c}\mathfrak{D}^{\nu_1}_{0+}$, ${}^{c}\mathfrak{D}^{\nu_2}_{0+}$ denote the Caputo fractional derivative of order κ_1, κ_2, ν_1 and ν_2 respectively, Ψ_1, $\Psi_2 : [0, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, χ_1, χ_2 are the dissipative parameters and δ_i, ϵ_i and $0 < \eta_i, \xi_i < 1$ for $i = 1, 2$.

In 2023, Salem [35] studied to solve the following linear non-homogeneous fractional differential delay equations of the Hilfer type.

$$\begin{cases} \mathbb{D}^{\alpha,\beta}_{-\tau+}(\mathbb{D}^{\alpha,\beta}_{-\tau+}y)(x) = -B^2 y(x-\tau) + h(x), \quad B \in \mathbb{R}^{n\times n},\ x \in [0, T],\ \tau > 0 \\ y(x) = \phi(x), \quad \phi(x) \in \mathbb{R}^n,\ -r < x \leq 0 \\ \lim\limits_{x \longrightarrow \tau^+} (\mathbb{I}^{1-\gamma}_{-\tau+}y)(x) = b_1, \quad b_1 \in \mathbb{R}^n \\ \lim\limits_{x \longrightarrow \tau^+} \mathbb{I}^{1-\gamma}_{-\tau}(\mathbb{D}^{\alpha,\beta}_{-\tau+}y)(x) = Bb_2, \quad b_2 \in \mathbb{R}^n \end{cases}$$

where $h(x) \in C([0, T], \mathbb{R}^n)$, $\mathbb{D}^{\alpha,\beta}_{-\tau+}y$ denotes the Hilfer fractional derivative with type $\beta \in [0, 1]$ and of order $0 < \alpha < 1$, $\mathbb{I}^{\gamma}_{-\tau+}$ denotes γ-order of $R - L$ fractional integral, $\mathbb{D}^{\gamma+\alpha}_{-\tau+}\phi$ is the $\gamma + \alpha$-order of $R - L$ fractional derivative to the initial function $\phi(x), b_1, b_2 \in \mathbb{R}^n$ are constants vectors, B is nonsingular matrix, and $T = j\tau, j \in \mathbb{N}$ and τ is a fixed moment. That much is clear to observe $0 < \gamma = \alpha + \beta - \alpha\beta < 1$, $\gamma \geq \alpha$ and $\gamma \geq \beta$.

Similarly in 2024, Cheng et al., [13] investigated the existence and uniqueness of solutions for the Langevin (k, φ)-Hilfer fractional Langevin differential equation having multipoint boundary conditions given by

$$\begin{cases} {}^{k,H}D^{\alpha_1,\mu_1;\varphi}({}^{k,H}D^{\alpha_2,\mu_2;\varphi} + \lambda)x(t) = g(t, x(t)),\ t \in (a, b], \\ x(a) = 0, x(b) = \sum\limits_{i=1}^{q} \int_a^{v_i} \varphi'(s)x(s)ds + \sum\limits_{j=1}^{p} \zeta_j^k I^{\phi_j,\varphi}x(x_j), \end{cases}$$

where ${}^{k,H}D^{\alpha_i,\mu_i;\varphi}$, $i = 1, 2$ is the (k, φ)-Hilfer fractional of order α_i, $0 < \alpha_i < 1$ and μ_i, $0 \leq \mu_i \leq 1$, $1 < \alpha_1 + \alpha_2 \leq 2$, $\lambda \in \mathbb{R}$ respectively, $g : (a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, ${}^{k,H}I^{\alpha_i;\varphi}$ are the (k, φ)-Riemann–Liouville fractional integrals of order $\phi_j > 0$, respectively, ξ_i, $\zeta_j \in \mathbb{R}$ and $a < v_i$, $\zeta_j < b$, $i = 1, 2, \ldots, q$, $j = 1, 2, \ldots, p$.

So, there are many studies by some researchers, including those mentioned above, on fractional an Langevin differential equations using different forms of the Hilfer derivative and Caputo-type derivative, such as Caputo, Caputo-Hadamard, Caputo-Katugampola and ψ-Cabuto. Unfortunately, there are few studies that have used in it Caputo-exponential derivatives of other equations this prompted us to study the previous type of equations in the framework of Caputo-exponential derivative. The second motivation is to prove the existence and uniqueness results of the following nonlinear fractional Langevin differential equation with mixed nonlocal boundary conditions (multipoint, fractional integral and fractional derivative) involving the Caputo-exponential derivative:

$$\begin{cases} {}^e_cD^\alpha_0({}^e_cD^\beta_0+\lambda)x(t)=f(t,x(t),{}^e_cD^\gamma_0x(t)),\ t\in J=[0,T],\ T>0, & (1)\\ \\ x(0)=0,\ {}^e_cD^\beta_0x(0)=0,\ a\,{}^eI^{q_1}x(\eta)+b\,{}^e_cD^{q_2}_0x(T)=c, & (2)\end{cases}$$

where ${}^e_cD^\theta$ are fractional derivatives in the sence of Caputo-exponential of order $\theta\in\{\alpha,\beta,\gamma,q_2\}$ and ${}^eI^{q_1}$ is the exponential fractional integral of order q_1 such that $1<\alpha\le 2$, $0<q_1,q_2<\gamma<\beta\le 1$, $f:J\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given function, a, b, c and λ are real constants, $\eta\in(0,T)$.
The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminary concepts about Caputo-exponential fractional derivatives and some auxiliary results. In Section 3, two results on the nonlinear langevin fractional differential equation with nonlocal mixed (multi-point, fractional integral and fractional derivative) boundary conditions (1)-(2) are presented, the first one is based on the Banach contraction principle and the second one on Schaefer's fixed point theorem. In the last section, two examples are given to illustrate the applicability of our main results.

2. Preliminaries

In this section, we introduce some notations and defenitions of Caputo-exponential type fractional calculus.
Let $[a,b]$, $(-\infty<a<b<+\infty)$ be interval. By $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|g(t)\|_{[a,b]}=\sup\{|g(t)|:\quad a\le t\le b\}.$$

First, let $AC([a,b],\mathbb{R})$ be the space of functions $g:[a,b]\longrightarrow\mathbb{R}$ that are absolutely continuous. We denote by AC^n_e the space

$$AC^n_e([a,b],\mathbb{R})=\Big\{g:[a,b]\longrightarrow\mathbb{R},{}^cD^{n-1}g(t)\in AC([a,b],\mathbb{R}),{}^eD=e^{-t}\frac{d}{dt}\Big\}.$$

where $n=[\alpha]+1$ with $[\alpha]$ is the integer part of α.

Definition 2.1 (See [31], [39]). *The exponential fractional integral of order $\alpha>0$ of a function $h\in L^1([a,b],E)$ is defined by*

$${}^eI^\alpha_ah(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(e^t-e^s)^{\alpha-1}h(s)e^sds,\ \textit{for each } t\in[a,b].$$

where $\Gamma(.)$ is the Euler's Gamma function defined by

$$\Gamma(\xi)=\int_0^\infty t^{\xi-1}e^{-t}dt,\ \xi>0.$$

Definition 2.2 (See [31], [39]). *Let $\alpha>0$ and $h\in AC^n_e([a,b],\mathbb{R})$. The exponential fractional derivatives of Caputo type of order α is defined by*

$$({}^e_cD^\alpha_a)h(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(e^t-e^s)^{n-\alpha+1}h(s)(e^{-s}\frac{d}{ds})^nh(s)\frac{ds}{e^{-s}},\ \textit{for each } t\in[a,b].$$

where $n=[\alpha]+1$, In particular, if $\alpha=0$ then $({}^e_cD^\alpha_a)h(t)=h(t)$.

Lemma 2.1 (See [31], [39]). *Let $\alpha > 0$, $n = [\alpha] + 1$ and $h \in AC_e^n([a,b],\mathbb{R})$. Then we have the formula*

$$^{e}I^{\alpha}(^{e}_{c}D^{\alpha}_{a})h(t) = h(t) - \sum_{k=0}^{n-1} \frac{(e^t - e^a)^k}{k!} {}^{e}D^k h(a).$$

Lemma 2.2 (See [39]). *Let $\alpha > 0$ and $h \in AC_e^n([a,b],\mathbb{R})$. Then the differential equation $(^{e}_{c}D^{\alpha}_{a})h(t) = 0$ has the solution*

$$h(t) = \eta_0 + \eta_1(e^t - e^a) + \eta_2(e^t - e^a)^2 + \ldots + \eta_{n-1}(e^t - e^a)^{n-1},$$

where $\eta_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$ and $n = [\alpha] + 1$.

Lemma 2.3 (See [31]). *Let $\alpha > 0$, $n = [\alpha] + 1$ and $h \in AC_e^n([a,b],\mathbb{R})$. Then*

$$^{e}I^{\alpha}_{a}(^{e}_{c}D^{\alpha}_{a})h(t) = h(t) + \eta_0 + \eta_1(e^t - e^a) + \eta_2(e^t - e^a)^2 + \ldots + \eta_{n-1}(e^t - e^a)^{n-1},$$

for some $\eta_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$ and $n = [\alpha] + 1$.

Proposition 2.1 (See [31]). *Let α, $\beta > 0$, then following relations hold for*

$$^{e}I^{\alpha}_{a}(e^t - e^a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(e^t - e^a)^{\alpha+\beta}$$

and

$$^{e}_{c}D^{\alpha}_{a}(e^t - e^a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(e^t - e^a)^{\beta-\alpha}, \quad t \in [a,b].$$

Remark 2.1. *By Proposition 2.1, when $\beta = 0$ we have*

$$^{e}I^{\alpha}_{a}[1] = \frac{1}{\Gamma(\alpha+1)}(e^t - e^a)^{\alpha}$$

and

$$^{e}_{c}D^{\alpha}_{a}[1] = \frac{1}{\Gamma(1-\alpha)}(e^t - e^a)^{-\alpha}, \ \alpha > 0, \ t \in [a,b].$$

Theorem 2.1 (See [31]). *(Semigroup property). If $\alpha, \beta > 0$, then the equation*

$$^{e}I^{\alpha}_{a}(^{e}I^{\beta}_{a}f)(t) = {}^{e}I^{\beta}_{a}(^{e}I^{\alpha}_{a}f)(t) = {}^{e}I^{\alpha+\beta}_{a}f(t),$$

are satisfied for all $t \in [a,b]$.

Note that the relation between the exponential fractional derivatives of Riemann-Liouville and Caputo types is given by

$$^{e}_{c}D^{\alpha}_{a}f(t) = {}^{e}D^{\alpha}_{a}\left[f(t) - \sum_{k=0}^{n-1} \frac{^{e}D^k}{k!}(e^t - e^a)^k\right]$$

where $^{e}D = e^{-t}\frac{d}{dt}$.

Theorem 2.2 (See [31]). *If $0 < \beta < \alpha$ and $1 \le p < \infty$, then for $f \in L^p(a,b)$,*

$$^{e}D^{\beta}(^{e}I^{\alpha}_{a}f)(t) = {}^{e}I^{\alpha-\beta}_{a}f(t) \text{ and } {}^{e}_{c}D^{\beta}(^{e}I^{\alpha}_{a}f)(t) = {}^{e}I^{\alpha-\beta}_{a}f(t),$$

In addition, we have $^{e}D^{\alpha}(^{e}I^{\alpha}_{a}f)(t) = f(t)$ and $^{e}_{c}D^{\alpha}(^{e}I^{\alpha}_{a}f)(t) = f(t)$,

Theorem 2.3 (See [31]). *Let $\alpha > 0$ and $n = [\alpha] + 1$, then the following formulas are true:*

- $^{e}I^{\alpha}_{a}(^{e}D^{\beta}f)(t) = f(t) - \sum_{j=0}^{n} \frac{(e^x - e^a)^{\alpha-j}}{\Gamma(\alpha-j+1)} {}^{e}D^{n-j}(^{e}I^{n-\alpha}_{a}f)(a),$
- $^{e}I^{\alpha}_{a}(^{e}_{c}D^{\beta}f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(e^x - e^a)^{j}}{j!} {}^{e}_{c}D^{j}f(a).$

Theorem 2.4 (Banach's fixed point theorem, see [17]). *. Let D be a nonempty closed subset of a Banach space X. Then any contraction mapping T from D into itself has a unique fixed point.*

Theorem 2.5 (Schaefer's fixed point theorem, see [17]). *. Let X be a Banach space, let $T : X \longrightarrow X$ be a completely continuous operator, and let the set $D = \{x \in X : x = \delta Tx, 0 < \delta \leq 1\}$ be bounded. Then T has a fixed point in X.*

3. Main results

This section is devoted to the existence and uniqueness results for problem (1)-(2).

Definition 3.1. *A function $x \in C_e^2(J, \mathbb{R})$ is said to be a solution of the problem (1)-(2) if x satisfies the equation ${}_c^eD_0^\alpha({}_c^eD_0^\beta + \lambda)x(t) = f(t, x(t), {}_c^eD_0^\gamma)$ and satisfies the conditions $x(0) = 0$, ${}_c^eD_0^\beta x(0) = 0$, $a^eI^{q_1}x(\eta) + b_c^eD_0^{q_2}x(T) = c$ on J.*

To prove the existence of solutions to the problem (1)-(2), we need the following auxiliary lemma.

Lemma 3.1. *Let $1 < \alpha \leq 2$, be a continuous function. Then the linear problem*

$$\begin{cases} {}_c^eD_0^\alpha({}_c^eD_0^\beta + \lambda)x(t) = h(t), \ t \in J = [0, T], \\ x(0) = 0, \ {}_c^eD_0^\beta x(0) = 0, \ a^eI^{q_1}x(\eta) + b_c^eD_0^{q_2}x(T) = c, \end{cases}$$

has a unique solution given by

$$x(t) = ({}^eI_0^{\alpha+\beta})h(t) - \lambda({}^eI_0^\beta)x(t) + \frac{\Omega(t)}{\Lambda}\Big[c - a({}^eI_0^{\alpha+\beta+q_1})h(\eta) - b({}^eI_0^{\alpha+\beta-q_2})h(T) + \\ + a\lambda({}^eI_0^{\beta+q_1})x(\eta) + b\lambda({}^eI_0^{\beta-q_2})x(T)\Big], \tag{3}$$

where $\Lambda = \dfrac{a}{\Gamma(\beta + q_1 + 2)}(e^\eta - 1)^{\beta+q_1+1} + \dfrac{b}{\Gamma(\beta - q_2 + 2)}(e^T - 1)^{\beta-q_2+1} \neq 0$
and $\Omega(t) = \dfrac{(e^t - 1)^{\beta+1}}{\Gamma(\beta + 2)}$.

Proof. Assume that x satisfies (3.1) and (3.1). Applying the operator ${}^eI_0^\alpha$ to both sides of (3.1), and then using Lemma 2.1, we have

$$({}_c^eD_0^\beta + \lambda)x(t) = {}^eI_0^\alpha h(t) + c_0 + c_1(e^t - e^0), \tag{4}$$

or

$${}_c^eD_0^\beta x(t) = {}^eI_0^\alpha h(t) - \lambda x(t) + c_0 + c_1(e^t - 1). \tag{5}$$

Again, taking the integral operator ${}^eI_0^\beta$ to both sides of (5), and then using Lemma 2.1, we get

$$x(t) = ({}^eI_0^{\alpha+\beta})h(t) - \lambda({}^eI_0^\beta)x(t) + \frac{c_0}{\Gamma(\beta + 1)}(e^t - 1)^\beta + \frac{c_1}{\Gamma(\beta + 2)}(e^t - 1)^{\beta+1} + c_2, \tag{6}$$

where c_0, c_1 and c_2 are arbitrary constants.
Using the first condition ($x(0) = 0$) gives $c_2 = 0$, the second condition (${}_c^eD_0^\beta x(0) = 0$) gives $c_0 = 0$ and third conditions gives

$$c_1 = \frac{1}{\Lambda}\Big[c - a^eI_0^{\alpha+\beta+q_1}h(\eta) - b^eI_0^{\alpha+\beta-q_2}h(T) + a\lambda^eI_0^{\beta+q_1}x(\eta) + b\lambda^eI_0^{\beta-q_2}x(T)\Big],$$

where $\Lambda = \dfrac{a}{\Gamma(\beta + q_1 + 2)}(e^{\eta} - 1)^{\beta+q_1+1} + \dfrac{b}{\Gamma(\beta - q_2 + 2)}(e^T - 1)^{\beta-q_2+1} \neq 0.$
Finally, substituting the values of c_0, c_1 and c_2 in (6), we obtain (3).
Conversely, assume that x satisfies the fractional integral equation (3). In fact, using the operators ${}^e_cD^{\beta}_0$ and ${}^e_cD^{\alpha}_0$, gives we obtain

$$ {}^e_cD^{\alpha}_0({}^e_cD^{\beta}_0 + \lambda)x(t) = h(t), \ t \in J. $$

□

Let us now consider the space defined by:

$$ X = \{x : x \in C^2_e(J, \mathbb{R}), \ {}^e_cD^{\gamma}_0 x \in C_e(J, \mathbb{R})\}, $$

equipped with the norm

$$ \|x\|_X = \|x\|_\infty + \|{}^e_cD^{\gamma}_0 x\|_\infty = \sup_{t\in J} |x(t)| + \sup_{t\in J} |{}^e_cD^{\gamma}_0 x(t)|. $$

Clearly, $(X, \|.\|_X)$ is a Banach space, see [10].
We assume the following conditions to prove the existence of a solution of problem (1)-(2)

(H1): The function $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.
(H2): There exists constants $L \in \mathbb{R}_+$ such that

$$ |f(t, u, v) - f(t, \overline{u}, \overline{v})| \leq L(|u - \overline{u}| + |v - \overline{v}|), $$

for any u, v, $\overline{u}$ and $\overline{v} \in \mathbb{R}$, for a.e., $t \in J$.

We adopt the following notation.

$$ \Omega^* = \Omega^*_0 + \Omega^*_1, \tag{7} $$

$$ \kappa_1 = \kappa^0_1 + \kappa^1_1, \quad \kappa_2 = \kappa^0_2 + \kappa^1_2, \tag{8} $$

$$ \rho_0 = \rho^0_0 + \rho^1_0, \quad \rho_1 = \rho^0_1 + \rho^1_1, \tag{9} $$

where

$$ \Omega^*_0 = \sup_{t\in J} |\Omega(t)|, \quad \Omega^*_1 = \sup_{t\in J} |{}^e_cD^{\gamma}_0 \Omega(t)|, \tag{10} $$

$$ \kappa^0_1 = L\rho^0_1, \quad \kappa^0_2 = \kappa^0_1 + |\lambda|\rho^0_0, \quad \kappa^1_1 = L\rho^1_1, \quad \kappa^1_2 = \kappa^1_1 + |\lambda|\rho^1_0, \tag{11} $$

with

$$ \rho^0_0 = \sup_{t\in J} |\omega^0_0(t)|, \quad \rho^0_1 = \sup_{t\in J} |\omega^0_1(t)|, \quad \rho^1_0 = \sup_{t\in J} |\omega^1_0(t)|, \quad \rho^1_1 = \sup_{t\in J} |\omega^1_1(t)|, \tag{12} $$

$$ \omega^0_0(t) = ({}^eI^{\beta}_{0,t})[1] + \frac{|a|}{|\Lambda|}\Omega(t)({}^eI^{\beta+q_1}_{0,\eta})[1] + \frac{|b|}{|\Lambda|}\Omega(t)({}^eI^{\beta-q_2}_{0,T})[1], \tag{13} $$

$$ \omega^0_1(t) = ({}^eI^{\alpha+\beta}_{0,t})[1] + \frac{|a|}{|\Lambda|}\Omega(t)({}^eI^{\alpha+\beta+q_1}_{0,\eta})[1] + \frac{|b|}{|\Lambda|}\Omega(t)({}^eI^{\alpha+\beta-q_2}_{0,T})[1], \tag{14} $$

$$ \omega^1_0(t) = ({}^eI^{\beta-\gamma}_{0,t})[1] + \frac{|a|{}^e_cD^{\gamma}_0(\Omega(t))}{|\Lambda|}({}^eI^{\beta+q_1}_{0,\eta})[1] + \frac{|b|{}^e_cD^{\gamma}_0(\Omega(t))}{|\Lambda|}({}^eI^{\beta-q_2}_{0,T})[1], \tag{15} $$

and

$$ \omega^1_1(t) = ({}^eI^{\alpha+\beta-\gamma}_{0,t})[1] + \frac{|a|{}^e_cD^{\gamma}_0(\Omega(t))}{|\Lambda|}({}^eI^{\alpha+\beta+q_1}_{0,\eta})[1] + \frac{|b|{}^e_cD^{\gamma}_0(\Omega(t))}{|\Lambda|}({}^eI^{\alpha+\beta-q_2}_{0,T})[1]. \tag{16} $$

Our first result is based on the following Banach contraction mapping principle.

Theorem 3.1. *If the hypotheses (H1)-(H2) are satisfied and if*

$$ \kappa = \max(\kappa_1, \kappa_2) < 1, \tag{17} $$

then the problem (1)-(2) *has a unique solution in the space* X.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $\mathcal{F} : X \longrightarrow X$ defined by

$$(\mathcal{F}x)(t) = ({}^eI_0^{\alpha+\beta})\sigma_x(t) - \lambda({}^eI_0^{\beta})x(t) + \frac{\Omega(t)}{\Lambda}\Big[c - a({}^eI_0^{\alpha+\beta+q_1})\sigma_x(\eta) - b({}^eI_0^{\alpha+\beta-q_2})\sigma_x(T) + a\lambda({}^eI_0^{\beta+q_1})x(\eta) + b\lambda({}^eI_0^{\beta-q_2})x(T)\Big], \quad (18)$$

where $\sigma_x(t) = f(t, x(t), {}^e_cD_0^{\gamma}x(t))$,
$\Lambda = \dfrac{a}{\Gamma(\beta+q_1+2)}(e^{\eta}-1)^{\beta+q_1+1} + \dfrac{b}{\Gamma(\beta-q_2+2)}(e^{T}-1)^{\beta-q_2+1} \neq 0$ and $\Omega(t) = \dfrac{(e^t-1)^{\beta+1}}{\Gamma(\beta+2)}$.
Clearly, the fixed points of $\mathcal{F}$ are solutions of problem (1)-(2).
Setting $M_1 = \sup_{t\in J}|f(t,0,0)|$ and choosing

$$r_1 \geq \frac{M_1\rho_1 + \frac{|c|}{|\Lambda|}\Omega^*}{1-\kappa}.$$

Note that $B_{r_1} = \{x \in X : \|x\| \leq r_1\}$ is a subest of X which B_{r_1} bounded, closed and convex. The proof is divided into two steps as follows.
Step I: We show that $\mathcal{F}B_{r_1} \subset B_{r_1}$. For $x \in B_{r_1}$, by (H2), we have for each $t \in J$

$$\begin{aligned} |\sigma_x(t)| &= |f(t, x(t), {}^e_cD_0^{\gamma}x(t)) + f(t,0,0) - f(t,0,0)| \\ &\leq |f(t, x(t), {}^e_cD_0^{\gamma}) - f(t,0,0)| + |f(t,0,0)| \\ &\leq L(|x(t) + |{}^e_cD_0^{\gamma}x(t)|) + M_1 \\ &\leq L\|x\|_\infty + L\|{}^e_cD_0^{\gamma}x\|_\infty + M_1 \end{aligned} \quad (19)$$

Substituting (19) into (18), by using (10), (11), (12), (13) and (14) we have the follwing inequalities

$$|(\mathcal{F}x)(t)|$$

$$\begin{aligned} &\leq ({}^eI_0^{\alpha+\beta})|\sigma_x(t)| + |\lambda|({}^eI_0^{\beta})|x(t)| + \frac{\Omega(t)}{|\Lambda|}\Big[|a|({}^eI_0^{\alpha+\beta+q_1})|\sigma_x(\eta)| + |b|({}^eI_0^{\alpha+\beta-q_2})|\sigma_x(T)| \\ &+ |a\lambda|({}^eI_0^{\beta+q_1})|x(\eta)| + |b\lambda|({}^eI_0^{\beta-q_2})|x(T)| + |c|\Big], \\ &\leq ({}^eI_{0,t}^{\alpha+\beta})(L\|x\|_\infty + L\|{}^e_cD_0^{\gamma}x\|_\infty + M_1) + |\lambda|({}^eI_{0,t}^{\beta})\|x\| + \frac{\Omega(t)}{|\Lambda|}\Big[|a|({}^eI_{0,\eta}^{\alpha+\beta+q_1}) \\ &\times (L\|x\|_\infty + L\|{}^e_cD_0^{\gamma}x\|_\infty + M_1) + |b|({}^eI_{0,T}^{\alpha+\beta-q_2})(L\|x\|_\infty + L\|{}^e_cD_0^{\gamma}x\|_\infty + M_1) \\ &+ |a\lambda|({}^eI_{0,\eta}^{\beta+q_1})\|x\| + |b\lambda|({}^eI_{0,T}^{\beta-q_2})\|x\| + |c|\Big], \end{aligned}$$

$$
\begin{aligned}
&\leq \Bigg\{ L\Big(({}^{e}I_{0,t}^{\alpha+\beta})[1] + \frac{|a|\Omega(t)}{|\Lambda|}({}^{e}I_{0,\eta}^{\alpha+\beta+q_1})[1] + \frac{|b|\Omega(t)}{|\Lambda|}({}^{e}I_{0,T}^{\alpha+\beta-q_2})[1]\Big) + |\lambda|\Big(({}^{e}I_{0,t}^{\beta})[1] \\
&+ \frac{|a|\Omega(t)}{|\Lambda|}({}^{e}I_{0,\eta}^{\beta+q_1})[1] + \frac{|b|\Omega(t)}{|\Lambda|}({}^{e}I_{0,T}^{\beta-q_2})[1]\Big) \Bigg\} \|x\|_\infty + \Bigg\{ L\Big(({}^{e}I_{0,t}^{\alpha+\beta})[1] \\
&+ \frac{|a|\Omega(t)}{|\Lambda|}({}^{e}I_{0,\eta}^{\alpha+\beta+q_1})[1] + \frac{|b|\Omega(t)}{|\Lambda|}({}^{e}I_{0,T}^{\alpha+\beta-q_2})[1]\Big) \Bigg\} \|{}^{e}_{c}D_0^{\gamma}x\|_\infty + M_1\Big(({}^{e}I_{0,t}^{\alpha+\beta})[1] \\
&+ \frac{|a|\Omega(t)}{|\Lambda|}({}^{e}I_{0,\eta}^{\alpha+\beta+q_1})[1] + \frac{|b|\Omega(t)}{|\Lambda|}({}^{e}I_{0,T}^{\alpha+\beta-q_2})[1]\Big) + \frac{|c|\Omega(t)}{|\Lambda|}, \\
&= \Big(L\omega_1^0(t) + |\lambda|\omega_0^0(t) \Big)\|x\|_\infty + \Big(L\omega_1^0(t) \Big)\|{}^{e}_{c}D_0^{\gamma}x\|_\infty + M_1\omega_1^0(t) + \frac{|c|\Omega(t)}{|\Lambda|}.
\end{aligned}
$$

Consequently,

$$
\|(\mathcal{F}x)\|_\infty \leq \kappa_2^0\|x\|_\infty + \kappa_1^0\|{}^{e}_{c}D_0^{\gamma}x\|_\infty + M_1\rho_1^0 + \frac{|c|}{|\Lambda|}\Omega_0^*. \tag{20}
$$

On the other hand, by using (10), (11), (12), (15) and (16), we can find that

$$
|{}^{e}_{c}D_0^{\gamma}(\mathcal{F}x)(t)|
$$

$$
\begin{aligned}
&\leq ({}^{e}I_0^{\alpha+\beta-\gamma})|\sigma_x(t)| + |\lambda|({}^{e}I_0^{\beta-\gamma})|x(t)| + \frac{{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}\Big[|a|({}^{e}I_0^{\alpha+\beta+q_1})|\sigma_x(\eta)| \\
&+ |b|({}^{e}I_0^{\alpha+\beta-q_2})|\sigma_x(T)| + |a\lambda|({}^{e}I_0^{\beta+q_1})|x(\eta)| + |b\lambda|({}^{e}I_0^{\beta-q_2})|x(T)| + |c| \Big], \\
&\leq ({}^{e}I_{0,t}^{\alpha+\beta-\gamma})(L\|x\|_\infty + L\|{}^{e}_{c}D_0^{\gamma}x\|_\infty + M_1) + |\lambda|({}^{e}I_{0,t}^{\beta-\gamma})\|x\| + \frac{{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|} \\
&\times \Big[|a|({}^{e}I_{0,\eta}^{\alpha+\beta+q_1})(L\|x\|_\infty + L\|{}^{e}_{c}D_0^{\gamma}x\|_\infty + M_1) + |b|({}^{e}I_{0,T}^{\alpha+\beta-q_2})(L\|x\|_\infty \\
&+ L\|{}^{e}_{c}D_0^{\gamma}x\|_\infty + M_1) + |a\lambda|({}^{e}I_{0,\eta}^{\beta+q_1})\|x\| + |b\lambda|({}^{e}I_{0,T}^{\beta-q_2})\|x\| + |c| \Big], \\
&\leq \Bigg\{ L\Big(({}^{e}I_{0,t}^{\alpha+\beta-\gamma})[1] + \frac{|a|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}({}^{e}I_{0,\eta}^{\alpha+\beta+q_1})[1] + \frac{|b|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|} \\
&\times ({}^{e}I_{0,T}^{\alpha+\beta-q_2})[1]\Big) + |\lambda|\Big(({}^{e}I_{0,t}^{\beta-\gamma})[1] + \frac{|a|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}({}^{e}I_{0,\eta}^{\beta+q_1})[1] + \frac{|b|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|} \\
&\times ({}^{e}I_{0,T}^{\beta-q_2})[1]\Big) \Bigg\} \|x\|_\infty + \Bigg\{ L\Big(({}^{e}I_{0,t}^{\alpha+\beta-\gamma})[1] + \frac{|a|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}({}^{e}I_{0,\eta}^{\alpha+\beta+q_1})[1] \\
&+ \frac{|b|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}({}^{e}I_{0,T}^{\alpha+\beta-q_2})[1]\Big) + |\lambda|\Big(({}^{e}I_{0,t}^{\beta-\gamma})[1] + \frac{|a|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}({}^{e}I_{0,\eta}^{\beta+q_1})[1] \\
&+ \frac{|b|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}({}^{e}I_{0,T}^{\beta-q_2})[1]\Big) \Bigg\} \|{}^{e}_{c}D_0^{\gamma}x\|_\infty + M_1\Big(({}^{e}I_{0,t}^{\alpha+\beta-\gamma})[1] + \frac{|a|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|} \\
&\times ({}^{e}I_{0,\eta}^{\alpha+\beta+q_1})[1] + \frac{|b|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}({}^{e}I_{0,T}^{\alpha+\beta-q_2})[1]\Big) + \frac{|c|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|} \\
&= \Big(L\omega_1^1(t) + |\lambda|\omega_0^1(t) \Big)\|x\|_\infty + \Big(L\omega_1^1(t) \Big)\|{}^{e}_{c}D_0^{\gamma}x\|_\infty + M_1\omega_1^1(t) + \frac{|c|{}^{e}_{c}D_0^{\gamma}(\Omega(t))}{|\Lambda|}.
\end{aligned}
$$

Consequently,

$$\|{}^e_cD^\gamma_0(\mathcal{F}x)\|_\infty \le \kappa^1_2\|x\|_\infty + \kappa^1_1\|{}^e_cD^\gamma_0 x\|_\infty + M_1\rho^1_1 + \frac{|c|}{|\Lambda|}\Omega^*_1. \tag{21}$$

Combining (20) and (21) by using (7), (8) and (9), we obtain

$$\begin{aligned}\|(\mathcal{F}x)\|_X &\le \kappa_2\|x\|_\infty + \kappa_1\|{}^e_cD^\gamma_0 x\|_\infty + M_1\rho_1 + \frac{|c|}{|\Lambda|}\Omega^* \\ &\le \kappa\|x\|_X + M_1\rho_1 + \frac{|c|}{|\Lambda|}\Omega^* \\ &\le \kappa r_1 + (1-\kappa)r_1, 0<\kappa<1 \\ &\le r_1. \end{aligned}\tag{22}$$

Hence, the operator $\mathcal{F}$ maps bounded sets into bounded sets in X.

Step II: To show that an operator $\mathcal{F} : X \longrightarrow X$ is contraction. Let $x,\, y \in X$ and $t \in J$, we have

$$\begin{aligned}&(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\\ &= ({}^eI^{\alpha+\beta}_0)(\sigma_x(t)-\sigma_y(t)) - \lambda({}^eI^{\beta}_0)(x(t)-y(t)) + \frac{\Omega(t)}{\Lambda}\Big[-a({}^eI^{\alpha+\beta+q_1}_0)\\ &\times(\sigma_x(\eta)-\sigma_y(\eta)) - b({}^eI^{\alpha+\beta-q_2}_0)(\sigma_x(T)-\sigma_y(T)) + a\lambda({}^eI^{\beta+q_1}_0)(x(\eta)-y(\eta))\\ &+ b\lambda({}^eI^{\beta-q_2}_0)(x(T)-y(T))\Big].\end{aligned}\tag{23}$$

By (H2), (11), (13) and (14), we can find that

$$\begin{aligned}&|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)|\\ &\le ({}^eI^{\alpha+\beta}_{0,t})(L\|x-y\|_\infty + L\|{}^e_cD^\gamma_0x - {}^e_cD^\gamma_0y\|_\infty) + |\lambda|({}^eI^{\beta}_{0,t})\|x-y\|\\ &+ \frac{\Omega(t)}{|\Lambda|}\Big[|a|({}^eI^{\alpha+\beta+q_1}_{0,\eta})(L\|x-y\|_\infty + L\|{}^e_cD^\alpha_0x - {}^e_cD^\gamma_0y\|_\infty)\\ &+ |b|({}^eI^{\alpha+\beta-q_2}_{0,T})(L\|x-y\|_\infty + L\|{}^e_cD^\gamma_0x - {}^e_cD^\gamma_0y\|_\infty)\\ &+ |a\lambda|({}^eI^{\beta+q_1}_{0,\eta})\|x-y\| + |b\lambda|({}^eI^{\beta-q_2}_{0,T})\|x-y\|\Big],\\ &\le \Bigg\{L\Big(({}^eI^{\alpha+\beta}_{0,t})[1] + \frac{|a|\Omega(t)}{|\Lambda|}({}^eI^{\alpha+\beta+q_1}_{0,\eta})[1] + \frac{|b|\Omega(t)}{|\Lambda|}({}^eI^{\alpha+\beta-q_2}_{0,T})[1]\Big)\\ &+ |\lambda|\Big(({}^eI^{\beta}_{0,t})[1] + \frac{|a|\Omega(t)}{|\Lambda|}({}^eI^{\beta+q_1}_{0,\eta})[1] + \frac{|b|\Omega(t)}{|\Lambda|}({}^eI^{\beta-q_2}_{0,T})[1]\Big)\Bigg\}\|x-y\|_\infty\\ &+ \Bigg\{L\Big(({}^eI^{\alpha+\beta}_{0,t})[1] + \frac{|a|\Omega(t)}{|\Lambda|}({}^eI^{\alpha+\beta+q_1}_{0,\eta})[1] + \frac{|b|\Omega(t)}{|\Lambda|}({}^eI^{\alpha+\beta-q_2}_{0,T})[1]\Big)\Bigg\}\|{}^e_cD^\gamma_0x - {}^e_cD^\gamma_0y\|_\infty\\ &\le \Bigg\{L\omega^0_1(t) + |\lambda|\omega^0_0(t)\Bigg\}\|x-y\|_\infty + \Bigg\{L\omega^0_1(t)\Bigg\}\|{}^e_cD^\gamma_0x - {}^e_cD^\gamma_0y\|_\infty.\end{aligned}$$

Therefore,

$$\|(\mathcal{F}x) - (\mathcal{F}y)\|_\infty \le \kappa^0_2\|x-y\|_\infty + \kappa^0_1\|{}^e_cD^\gamma_0x - {}^e_cD^\gamma_0y\|_\infty. \tag{24}$$

On the other hand by (H2), (11), (15) and (16), we can find that

$$|{}^e_cD^\gamma_0[(\mathcal{F}x)(t) - (\mathcal{F}y)(t)]|$$

$$
\begin{aligned}
&\leq \Bigg\{L\Big(({}^eI_{0,t}^{\alpha+\beta-\gamma})[1] + \frac{|a|{}_c^eD_0^\gamma(\Omega(t))}{|\Lambda|}({}^eI_{0,\eta}^{\alpha+\beta+q_1})[1] + \frac{|b|{}_c^eD_0^\gamma(\Omega(t))}{|\Lambda|}({}^eI_{0,T}^{\alpha+\beta-q_2})[1]\Big) \\
&+ |\lambda|\Big(({}^eI_{0,t}^{\beta-\gamma})[1] + \frac{|a|{}_c^eD_0^\gamma(\Omega(t))}{|\Lambda|}({}^eI_{0,\eta}^{\beta+q_1})[1] + \frac{|b|{}_c^eD_0^\gamma(\Omega(t))}{|\Lambda|}({}^eI_{0,T}^{\beta-q_2})[1]\Big)\Bigg\}\|x-y\|_\infty \\
&+ \Bigg\{L\Big(({}^eI_{0,t}^{\alpha+\beta-\gamma})[1] + \frac{|a|{}_c^eD_0^\gamma(\Omega(t))}{|\Lambda|}({}^eI_{0,\eta}^{\alpha+\beta+q_1})[1] + \frac{|b|{}_c^eD_0^\gamma(\Omega(t))}{|\Lambda|}({}^eI_{0,T}^{\alpha+\beta-q_2})[1]\Big) \\
&+ |\lambda|\Big(({}^eI_{0,t}^{\beta-\gamma})[1] + \frac{|a|{}_c^eD_0^\gamma(\Omega(t))}{|\Lambda|}({}^eI_{0,\eta}^{\beta+q_1})[1] + \frac{|b|{}_c^eD_0^\gamma(\Omega(t))}{|\Lambda|}({}^eI_{0,T}^{\beta-q_2})[1]\Big)\Bigg\}\|{}_c^eD_0^\gamma x - {}_c^eD_0^\gamma y\|_\infty, \\
&\leq \Bigg\{L\omega_1^1(t) + |\lambda|\omega_0^1(t)\Bigg\}\|x-y\|_\infty + \Bigg\{L\omega_1^1(t)\Bigg\}\|{}_c^eD_0^\gamma x - {}_c^eD_0^\gamma y\|_\infty.
\end{aligned}
$$

Hence,

$$
\||{}_c^eD_0^\gamma[(\mathcal{F}x) - (\mathcal{F}y)]\|_\infty \leq \kappa_2^1\|x-y\|_\infty + \kappa_1^1\|{}_c^eD_0^\gamma x - {}_c^eD_0^\gamma y\|_\infty. \tag{25}
$$

Combining (24) and (25) by using (8), we can write

$$
\begin{aligned}
\|(\mathcal{F}x) - (\mathcal{F}y)\|_X &= \|(\mathcal{F}x) - (\mathcal{F}y)\|_\infty + \||{}_c^eD_0^\gamma[(\mathcal{F}x) - (\mathcal{F}y)]\|_\infty \\
&\leq (\kappa_2^0 + \kappa_2^1)\|x-y\|_\infty + (\kappa_1^0 + \kappa_1^1)\|{}_c^eD_0^\gamma x - {}_c^eD_0^\gamma y\|_\infty \\
&\leq \kappa_2\|x-y\|_\infty + \kappa_1\|{}_c^eD_0^\gamma x - {}_c^eD_0^\gamma y\|_\infty \\
&\leq \kappa\|x-y\|_X.
\end{aligned} \tag{26}
$$

Consequently by (17), $\mathcal{F}$ is a contraction. As a consequence of Banach fixed point theorem, we deduce that $\mathcal{F}$ has a fixed point which is a solution of the problem (1)-(2). □

The second result is based on Schaefer's fixed point theorem. Let us introduce the following condition

(H3): There exists a constant $M_2 > 0$ such that $|f(t,u,v)| \leq M_2$, for any u, $v \in \mathbb{R}$ for a.e., $t \in J$.

Theorem 3.2. *Assume that conditions (H1), (H3) hold. Then the problem* (1)-(2) *has at least one solution on J.*

Proof. We shall use Schaefer's fixed point theorem to prove that operator $\mathcal{F}$, defined in (18) has at least one fixed point in X. The proof is divided into four steps:
Step 1: The operator $\mathcal{F}$ is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in X, then for each $t \in J$, and by (12), (13), and (14), we can find that

$$
|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)|
$$

$$\leq ({}^{e}I_{0}^{\alpha+\beta})|\sigma_{x_n}(t)-\sigma_x(t)|+|\lambda|({}^{e}I_{0}^{\beta})|x_n(t)-x(t)|+\frac{\Omega(t)}{|\Lambda|}\Big[|a|({}^{e}I_{0}^{\alpha+\beta+q_1})$$
$$\times|\sigma_{x_n}(\eta)-\sigma_x(\eta)|+|b|({}^{e}I_{0}^{\alpha+\beta-q_2})|(\sigma_{x_n}(T)-\sigma_x(T)|+|a\lambda|({}^{e}I_{0}^{\beta+q_1})|x_n(\eta)-x(\eta)|$$
$$+|b\lambda|({}^{e}I_{0}^{\beta-q_2})|x_n(T)-x(T)|\Big]$$
$$\leq\left\{({}^{e}I_{0,t}^{\alpha+\beta})[1]+\frac{\Omega(t)}{|\Lambda|}\Big[|a|({}^{e}I_{0}^{\alpha+\beta+q_1})[1]+|b|({}^{e}I_{0}^{\alpha+\beta-q_2})[1]\Big]\right\}\|\sigma_{x_n}-\sigma_x\|_\infty$$
$$+\left\{|\lambda|({}^{e}I_{0,t}^{\beta-\gamma})[1]+\frac{\Omega(t)}{|\Lambda|}\Big[|a|({}^{e}I_{0}^{\beta+q_1})[1]+|b|({}^{e}I_{0}^{\beta-q_2})[1]\Big]\right\}\|x_n-x\|_\infty$$
$$\leq\omega_1^0(t)\|\sigma_{x_n}-\sigma_x\|_\infty+|\lambda|\omega_0^0(t)\|x_n-x\|_\infty.$$

Therefore,

$$\|(\mathcal{F}x_n)(t)-(\mathcal{F}x)(t)\|_\infty\leq\rho_1^0\|\sigma_{x_n}-\sigma_x\|_\infty+|\lambda|\rho_0^0\|x_n-x\|_\infty. \tag{27}$$

On the other hand by using (12), (15) and (16), we obtain

$$|{}_{c}^{e}D_0^{\gamma}[(\mathcal{F}x_n)(t)-(\mathcal{F}x)(t)]|$$

$$\leq({}^{e}I_{0}^{\alpha+\beta-\gamma})|\sigma_{x_n}(t)-\sigma_x(t)|+|\lambda|({}^{e}I_{0}^{\beta-\gamma})|x_n(t)-x(t)|+\frac{{}_{c}^{e}D_0^{\gamma}(\Omega(t))}{|\Lambda|}$$
$$\times\Big[|a|({}^{e}I_{0}^{\alpha+\beta+q_1})|\sigma_{x_n}(\eta)-\sigma_x(\eta)|+|b|({}^{e}I_{0}^{\alpha+\beta-q_2})|(\sigma_{x_n}(T)-\sigma_x(T)|$$
$$+|a\lambda|({}^{e}I_{0}^{\beta+q_1})|x_n(\eta)-x(\eta)|+|b\lambda|({}^{e}I_{0}^{\beta-q_2})|x_n(T)-x(T)|\Big]$$
$$\leq\left\{({}^{e}I_{0,t}^{\alpha+\beta-\gamma})[1]+\frac{{}_{c}^{e}D_0^{\gamma}(\Omega(t))}{|\Lambda|}\Big[|a|({}^{e}I_{0}^{\alpha+\beta+q_1})[1]+|b|({}^{e}I_{0}^{\alpha+\beta+q_2})[1]\Big]\right\}\|\sigma_{x_n}-\sigma_x\|_\infty$$
$$+\left\{|\lambda|({}^{e}I_{0,t}^{\beta-\gamma})[1]+\frac{{}_{c}^{e}D_0^{\gamma}(\Omega(t))}{|\Lambda|}\Big[|a|({}^{e}I_{0}^{\beta+q_1})[1]+|b|({}^{e}I_{0}^{\beta-q_2})[1]\Big]\right\}\|x_n-x\|_\infty$$
$$\leq\omega_1^1(t)\|\sigma_{x_n}-\sigma_x\|_\infty+|\lambda|\omega_0^1(t)\|x_n-x\|_\infty.$$

Therefore,

$$\|{}_{c}^{e}D_0^{\gamma}[(\mathcal{F}x_n)(t)-(\mathcal{F}x)(t)]\|_\infty\leq\rho_1^1\|\sigma_{x_n}-\sigma_x\|_\infty+|\lambda|\rho_0^1\|x_n-x\|_\infty. \tag{28}$$

We remark that the continuity of the functional σ (i.e f is continuous), confirms the continuity of ${}_{c}^{e}D_0^{\gamma}(\mathcal{F})$ and $\mathcal{F}$, for each $t\in J$. Then

$$\|(\mathcal{F}x_n)(t)-(\mathcal{F}x)(t)\|_X\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Hence, the operator $\mathcal{F}$ is continuous on X.
Step 2: The operator $\mathcal{F}$ maps bounded sets into bounded sets in X.
For $r_2>0$, there exists constants $l>0$, for each $x\in\{x\in X:\|x\|_X=\|x\|_\infty+\|{}_{c}^{e}D_0^{\gamma}\|_\infty\}$.
Then, for any $t\in J$ and $x\in B_{r_2}$, and by using (H3), (10), (12), (13), and (14), we have

$$|(\mathcal{F}x)(t)|$$

$$\leq ({}^eI_0^{\alpha+\beta})|\sigma_x(t)| + |\lambda|({}^eI_0^{\beta})|x(t)| + \frac{\Omega(t)}{|\Lambda|}\Big[|a|({}^eI_0^{\alpha+\beta+q_1})|\sigma_x(\eta)| + |b|({}^eI_0^{\alpha+\beta+q_2})|\sigma_x(T)|$$
$$+ |a\lambda|({}^eI_0^{\beta+q_1})|x(\eta)| + |b\lambda|({}^eI_0^{\beta-q_2})|x(T)| + |c|\Big],$$
$$\leq M_2({}^eI_{0,t}^{\alpha+\beta})[1] + |\lambda|r_2({}^eI_{0,t}^{\beta})[1] + \frac{\Omega(t)}{|\Lambda|}\Big[|a|M_2({}^eI_{0,\eta}^{\alpha+\beta+q_1})[1] + |b|M_2({}^eI_{0,T}^{\alpha+\beta-q_2})[1]$$
$$+ |a\lambda|r_2({}^eI_0^{\beta+q_1}) + |b\lambda|r_2({}^eI_0^{\beta-q_2}) + |c|\Big],$$
$$\leq M_2\left[({}^eI_{0,t}^{\alpha+\beta})[1] + \frac{\Omega(t)}{|\Lambda|}\Big[|a|({}^eI_{0,\eta}^{\alpha+\beta+q_1})[1] + |b|({}^eI_{0,T}^{\alpha+\beta-q_2})[1]\Big]\right]$$
$$+ |\lambda|r_2\left[\frac{\Omega(t)}{|\Lambda|}\Big[+({}^eI_{0,t}^{\beta})[1] + |a|({}^eI_{0,\eta}^{\beta+q_1}) + |b|({}^eI_{0,T}^{\beta-q_2})\Big]\right] + \frac{|c|\Omega(t)}{|\Lambda|},$$
$$\leq M_2\omega_1^0(t) + |\lambda|r_2\omega_0^0(t) + \frac{|c|}{|\Lambda|}\Omega_0^*.$$

Therefore,

$$\|(\mathcal{F}x)\|_\infty \leq M_2\rho_1^0 + |\lambda|r_2\rho_0^0 + \frac{|c|}{|\Lambda|}\Omega_0^*. \tag{29}$$

On the other hand by using (H3), (10), (12), (15) and (16), we obtain

$$|{}_c^eD_0^{\gamma}(\mathcal{F}x)(t)|$$

$$\leq ({}^eI_0^{\alpha+\beta-\gamma})|\sigma_x(t)| + |\lambda|({}^eI_0^{\beta-\gamma})|x(t)| + \frac{|{}_c^eD_0^{\gamma}(\Omega(t))}{|\Lambda|}\Big[|a|({}^eI_0^{\alpha+\beta+q_1})|\sigma_x(\eta)|$$
$$+ |b|({}^eI_0^{\alpha+\beta+q_2})|\sigma_x(T)| + |a\lambda|({}^eI_0^{\beta+q_1})|x(\eta)| + |b\lambda|({}^eI_0^{\beta-q_2})|x(T)| + |c|\Big],$$
$$\leq M_2({}^eI_{0,t}^{\alpha+\beta-\gamma})[1] + |\lambda|r_2({}^eI_{0,t}^{\beta-\gamma})[1] + \frac{|{}_c^eD_0^{\gamma}(\Omega(t))}{|\Lambda|}\Big[|a|M_2({}^eI_{0,\eta}^{\alpha+\beta+q_1})[1]$$
$$+ |b|M_2({}^eI_{0,T}^{\alpha+\beta+q_2})[1] + |a\lambda|r_2({}^eI_0^{\beta+q_1}) + |b\lambda|r_2({}^eI_0^{\beta-q_2}) + |c|\Big],$$
$$\leq M_2\left[({}^eI_{0,t}^{\alpha+\beta-\gamma})[1] + \frac{|{}_c^eD_0^{\gamma}(\Omega(t))}{|\Lambda|}\Big[|a|({}^eI_{0,\eta}^{\alpha+\beta+q_1})[1] + |b|({}^eI_{0,T}^{\alpha+\beta+q_2})[1]\Big]\right]$$
$$+ |\lambda|r_2\left[\frac{|{}_c^eD_0^{\gamma}(\Omega(t))}{|\Lambda|}\Big[+({}^eI_{0,t}^{\beta-\gamma})[1] + |a|({}^eI_{0,\eta}^{\beta+q_1}) + |b|({}^eI_{0,T}^{\beta-q_2})\Big]\right] + \frac{|c|{}_c^eD_0^{\gamma}\Omega(t)}{|\Lambda|},$$
$$\leq M_2\omega_1^1(t) + |\lambda|r_2\omega_0^1(t) + \frac{|c|}{|\Lambda|}\Omega_1^*.$$

Therefore,

$$\|{}_c^eD_0^{\gamma}(\mathcal{F}x)\|_\infty \leq M_2\rho_1^1 + |\lambda|r_2\rho_0^1 + \frac{|c|}{|\Lambda|}\Omega_1^*. \tag{30}$$

Combining (29) and (30), by using (7), (9), we can write

$$\|(\mathcal{F}x)\|_X \leq M_2\rho_1 + |\lambda|r_2\rho_0 + \frac{|c|}{|\Lambda|}\Omega^* := l. \tag{31}$$

Step 3: The operator $\mathcal{F}$ maps bounded sets into equicontinuous sets of X.
As in step 2, let $t_1,\ t_2 \in J$, $t_1 < t_2$ and let B_{r_2} be a bounded set of X and let $x \in B_{r_2}$.

Then

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)|$$

$$= \Bigg|({}^eI_0^{\alpha+\beta})(\sigma_x(t_2) - \sigma_x(t_1)) + \lambda({}^eI_0^{\beta})(x(t_2) - x(t_1)) + \frac{\Omega(t_2) - \Omega(t_1)}{\Lambda}$$
$$\times \Big[a({}^eI_0^{\alpha+\beta+q_1})\sigma_x(\eta) + b({}^eI_0^{\alpha+\beta-q_2})\sigma_x(T) + a\lambda({}^eI_0^{\beta+q_1})x(\eta) + b\lambda({}^eI_0^{\beta-q_2})x(T) + c\Big]\Bigg|,$$
$$\leq \frac{1}{\Gamma(\alpha+\beta)}\Big|\int_0^{t_1}[(e^{t_2} - e^s)^{\alpha+\beta-1} - (e^{t_1} - e^s)^{\alpha+\beta-1}]e^s\sigma_x(s)ds$$
$$+ \int_{t_1}^{t_2}(e^{t_2} - e^s)^{\alpha+\beta-1}e^s\sigma_x(s)ds\Big| + \frac{1}{\Gamma(\beta)}\Big|\int_0^{t_1}[(e^{t_2} - e^s)^{\beta-1} - (e^{t_1} - e^s)^{\beta-1}]e^s x(s)ds$$
$$+ \int_{t_1}^{t_2}(e^{t_2} - e^s)^{\beta-1}e^s x(s)ds\Big| + \frac{|\Omega(t_2) - \Omega(t_1)|}{|\Lambda|}\Bigg|\Big[a({}^eI_0^{\alpha+\beta+q_1})\sigma_x(\eta) + b({}^eI_0^{\alpha+\beta-q_2})\sigma_x(T)$$
$$+ a\lambda({}^eI_0^{\beta+q_1})x(\eta) + b\lambda({}^eI_0^{\beta-q_2})x(T) + c\Big]\Bigg|,$$
$$\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}[(e^{t_2} - e^s)^{\alpha+\beta-1} - (e^{t_1} - e^s)^{\alpha+\beta-1}]e^s|\sigma_x(s)|ds$$
$$+ \int_{t_1}^{t_2}(e^{t_2} - e^s)^{\alpha+\beta-1}e^s|\sigma_x(s)|ds\Big| + \frac{1}{\Gamma(\beta)}\int_0^{t_1}[(e^{t_2} - e^s)^{\beta-1} - (e^{t_1} - e^s)^{\beta-1}]e^s\|x\|_\infty ds$$
$$+ \int_{t_1}^{t_2}(e^{t_2} - e^s)^{\beta-1}e^s\|x\|_\infty ds + \frac{|\Omega(t_2) - \Omega(t_1)|}{|\Lambda|}\Big[a({}^eI_0^{\alpha+\beta+q_1})|\sigma_x(\eta)|$$
$$+ b({}^eI_0^{\alpha+\beta-q_2})|\sigma_x(T)| + ak({}^eI_0^{\beta+q_1})\|x\|_\infty + bk({}^eI_0^{\beta-q_2})\|x\|_\infty + c\Big],$$
$$\leq \frac{M_2}{\Gamma(\alpha+\beta+1)}\left[(e^{t_1} - 1)^{\alpha+\beta} - (e^{t_2} - 1)^{\alpha+\beta} + 2(e^{t_2} - e^{t_1})^{\alpha+\beta}\right] + \frac{r_2}{\Gamma(\beta+1)}$$
$$\times \left[(e^{t_1} - 1)^{\beta} - (e^{t_2} - 1)^{\beta} + 2(e^{t_2} - e^{t_1})^{\beta}\right] + \Big[\frac{M_2}{|\Lambda|}\Big(|a|({}^eI_{0,\eta}^{\alpha+\beta+q_1})[1] + |b|({}^eI_{0,T}^{\alpha+\beta-q_2})[1]\Big)\Big]$$
$$+ \frac{|\lambda|r_2}{|\Lambda|}\Big(|a|({}^eI_{0,\eta}^{\beta+q_1})[1] + |b|({}^eI_{0,T}^{\beta-q_2})[1]\Big) + |c|\Big]|\Omega(t_1) - \Omega(t_2)|.$$

This inequality is independent on x and tends to zero as $t_2 \longrightarrow t_1$, which implies that

$$\|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)\|_\infty \longrightarrow 0, \quad t_2 \longrightarrow t_1. \tag{32}$$

On the other hand,

$$|({}_c^eD_0^\gamma\mathcal{F}x)(t_2) - ({}_c^eD_0^\gamma\mathcal{F}x)(t_1)|$$

$$
\begin{aligned}
&= \Bigg| ({}^{e}I_0^{\alpha+\beta-\gamma})(\sigma_x(t_2)-\sigma_x(t_1)) + \lambda({}^{e}I_0^{\beta-\gamma})(x(t_2)-x(t_1)) \\
&+ \frac{{}^{e}_{c}D_0^{\gamma}(\Omega(t_2)) - {}^{e}_{c}D_0^{\gamma}\Omega((t_1))}{\Lambda}\Big[a({}^{e}I_0^{\alpha+\beta+q_1})\sigma_x(\eta) + b({}^{e}I_0^{\alpha+\beta-q_2})\sigma_x(T) \\
&+ a\lambda({}^{e}I_0^{\beta+q_1})x(\eta) + b\lambda({}^{e}I_0^{\beta-q_2})x(T) + c\Big]\Bigg|, \\
&\le \frac{1}{\Gamma(\alpha+\beta-\gamma)}\Big|\int_0^{t_1}[(e^{t_2}-e^s)^{\alpha+\beta-\gamma-1} - (e^{t_1}-e^s)^{\alpha+\beta-\gamma-1}]e^s\sigma_x(s)ds \\
&+ \int_{t_1}^{t_2}(e^{t_2}-e^s)^{\alpha+\beta-\gamma-1}e^s\sigma_x(s)ds\Big| + \frac{|\lambda|}{\Gamma(\beta-\gamma)} \\
&\times \Big|\int_0^{t_1}[(e^{t_2}-e^s)^{\beta-\gamma-1} - (e^{t_1}-e^s)^{\beta-\gamma-1}]e^s x(s)ds \\
&+ \int_{t_1}^{t_2}(e^{t_2}-e^s)^{\beta-\gamma-1}e^s x(s)ds\Big| + \frac{|{}^{e}_{c}D_0^{\gamma}(\Omega(t_2)) - {}^{e}_{c}D_0^{\gamma}(\Omega(t_1))|}{|\Lambda|}\Bigg| \\
&\times \Big[a({}^{e}I_0^{\alpha+\beta+q_1})\sigma_x(\eta) + b({}^{e}I_0^{\alpha+\beta-q_2})\sigma_x(T) + a\lambda({}^{e}I_0^{\beta+q_1})x(\eta) \\
&+ b\lambda({}^{e}I_0^{\beta-q_2})x(T) + c\Big]\Bigg|, \\
&\le \frac{1}{\Gamma(\alpha+\beta-\gamma)}\int_0^{t_1}[(e^{t_2}-e^s)^{\alpha+\beta-\gamma-1} - (e^{t_1}-e^s)^{\alpha+\beta-\gamma-1}]e^s|\sigma_x(s)|ds \\
&+ \int_{t_1}^{t_2}(e^{t_2}-e^s)^{\alpha+\beta-\gamma-1}e^s|\sigma_x(s)|ds\Big| \\
&+ \frac{|\lambda|}{\Gamma(\beta-\gamma)}\int_0^{t_1}[(e^{t_2}-e^s)^{\beta-\gamma-1} - (e^{t_1}-e^s)^{\beta-\gamma-1}]e^s\|x\|_\infty ds \\
&+ \int_{t_1}^{t_2}(e^{t_2}-e^s)^{\beta-\gamma-1}e^s\|x\|_\infty ds + \frac{|{}^{e}_{c}D_0^{\gamma}(\Omega(t_2)) - {}^{e}_{c}D_0^{\gamma}(\Omega(t_1))|}{|\Lambda|}\Big[a({}^{e}I_0^{\alpha+\beta+q_1}) \\
&\times |\sigma_x(\eta)| + b({}^{e}I_0^{\alpha+\beta-q_2})|\sigma_x(T)| + a\lambda({}^{e}I_0^{\beta+q_1})\|x\|_\infty + b\lambda({}^{e}I_0^{\beta-q_2})\|x\|_\infty + c\Big], \\
&\le \frac{M_2}{\Gamma(\alpha+\beta-\gamma+1)}\Big[(e^{t_1}-1)^{\alpha+\beta-\gamma} - (e^{t_2}-1)^{\alpha+\beta-\gamma} + 2(e^{t_2}-e^{t_1})^{\alpha+\beta-\gamma}\Big] \\
&+ \frac{r_2}{\Gamma(\beta-\gamma+1)}\Big[(e^{t_1}-1)^{\beta-\gamma} - (e^{t_2}-1)^{\beta-\gamma} + 2(e^{t_2}-e^{t_1})^{\beta-\gamma}\Big] \\
&+ \Bigg[\frac{M_2}{|\Lambda|}\Big(|a|({}^{e}I_{0,\eta}^{\alpha+\beta+q_1})[1] + |b|({}^{e}I_{0,T}^{\alpha+\beta-q_2})[1]\Big) + \frac{|\lambda|r_2}{|\Lambda|}\Big(|a|({}^{e}I_{0,\eta}^{\beta+q_1})[1] \\
&+ |b|({}^{e}I_{0,T}^{\beta-q_2})[1]\Big) + |c|\Bigg]|\Omega(t_1)-\Omega(t_2)|.
\end{aligned}
$$

This inequality is independent of x and tends to zero as $t_2 \longrightarrow t_1$, which implies that

$$
\|({}^{e}_{c}D_0^{\gamma}\mathcal{F}x)(t_2) - ({}^{e}_{c}D_0^{\gamma}\mathcal{F}x)(t_1)\|_\infty \longrightarrow 0, \quad t_2 \longrightarrow t_1. \tag{33}
$$

Thus, it follows from (32) and (33) that

$$\|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)\|_X \longrightarrow 0, \quad t_2 \longrightarrow t_1. \tag{34}$$

This steps 1 to 3, together with the Arzela–Ascoli Theorem, we conclude the operator $\mathcal{F}$ is compltely continuous.

Step 4: Now, it remains to show that the set $\chi = \{x \in X : \ x = \delta \mathcal{F}x, \text{ for some } \delta \in (0,1)\}$ is bounded.

Let $x \in \chi$, then $x = \delta\mathcal{F}x$ for some $0 < \delta < 1$. Thus for each $t \in J$, we have

$$x(t) = \delta\Bigg[({}^eI_0^{\alpha+\beta})\sigma_x(t) - k({}^eI_0^{\beta})x(t) + \frac{\Omega(t)}{\Lambda}\Big[c - a({}^eI_0^{\alpha+\beta+q_1})\sigma_x(\eta) - b({}^eI_0^{\alpha+\beta+q_2})\sigma_x(T)$$
$$+ a\lambda({}^eI_0^{\beta+q_1})x(\eta) + b\lambda({}^eI_0^{\beta-q_2})x(T)\Big]\Bigg],$$

It follows from (31) that for each $t \in J$,

$$\|(\mathcal{F}x)\|_X \le \delta\left(M_2\rho_1 + |\lambda| r_2\rho_0 + \frac{|c|}{|\Lambda|}\Omega^*\right),$$

Thus $\|x\|_X < \infty$. Then χ is bounded.

As a consequence of Schaefer's fixed point theorem. $\mathcal{F}$ has a fixed point which is a solution of problem (1)-(2). ☐

4. Example

Consider the following nonlinear problem

$$\begin{cases} {}^e_cD_0^{\frac{3}{2}}({}^e_cD_0^{\frac{4}{5}} + \frac{1}{9})x(t) = \dfrac{|x(t)|}{(t^2+9)(1+|x(t)|)} + \dfrac{1}{10(2-t)}(|{}^e_cD_0^{\frac{1}{2}}x(t)|) + \dfrac{1}{5}, \ t \in [0,1], & (35) \\ x(0) = 0, \ {}^e_cD_0^{\frac{4}{5}}x(0) = 0, \ \dfrac{3}{4}({}^eI^{\frac{1}{3}}x(\dfrac{1}{2})) + \dfrac{4}{7}({}^e_cD_0^{\frac{1}{4}}x(1)) = 1, & (36) \end{cases}$$

We see that, $\alpha = \frac{3}{2}$, $\beta = \frac{4}{5}$, $\gamma = \frac{1}{2}$, $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{4}$, $\lambda = \dfrac{1}{25}$, $a = \frac{3}{4}$, $b = \frac{4}{7}$, $c = 1$, $T = 1$, $\eta = \frac{1}{2}$ and

$$f(t,u,v) = \frac{|u|}{(t^2+19)(1+|u|)} + \frac{|v|}{20(2-t)} + \frac{1}{5},$$

Clearly, the function f is continuous and for u, $\overline{u}$, v, $\overline{v} \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\begin{aligned}|f(t,u,v) - f(t,\overline{u},\overline{v})| &\le \frac{1}{t^2+19}|u-\overline{u}| + \frac{1}{20}|v-\overline{v}|, \\ &\le \frac{1}{20}(|u-\overline{u}| + |v-\overline{v}|).\end{aligned}$$

Hence, condition (H2) is satisfied with $L = \frac{1}{20}$.

A simple computation shows that $\Lambda = 1.0911 \neq 0$. By using (9) and (12), we get

$$\rho_0^0 = 4.7955, \ \rho_1^0 = 3.6629, \ \rho_0^1 = 4.4868, \ \rho_1^1 = 4.2682, \ \rho_0 = 9.2824, \ \rho_1 = 7.9311,$$

by using (8) and (11), we get

$$\kappa_1^0 = 0.1831, \ \kappa_2^0 = 0.3750, \ \kappa_1^1 = 0.2134, \ \kappa_2^1 = 0.3929, \ \kappa_1 = 0.3966, \ \kappa_2 = 0.7679.$$

Thus,

$$\kappa = \max(\kappa_1, \kappa_2) = 0,7679 < 1.$$

Hence, all conditions of Theorem 3.1 are satisfied, from which it follows that the problem (35)-(36) has a unique solution. So, all the assumptions of Theorem 3.2 are satisfied, then the problem (35)-(36) has at least one solution on $[0, 1]$.

5. Conclusions

In the present work, we consider a nonlinear Langevin fractional differential equation with nonlocal mixed (multi-point, fractional integral and fractional derivative) boundary conditions involving a Caputo-exponential. We have proved two theorems with an example to illustrate the following results:

i) The existence and uniqueness of solutions: A technique of fixed point theorem is used to prove the results. Prior to the main theorem, the form of solution is derived for nonlinear problem.
ii) The existence of at least one solution: A technique of Schaefer's theorem is used to prove the results.
iii) Applications: A particular example is addressed at the end of the paper to show the consistency of the theoretical results.

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