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SOME PROPERTIES OF ZERO-DIVISOR GRAPHS OF DIRECT PRODUCTS OF FINITE FIELDS

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ABSTRACT. This study investigates the zero-divisor graphs of the direct products of finite fields with vertex set consisting of non-zero zero divisors of the direct products of finite fields, and two distinct non-zero zero divisors are adjacent in the zero divisor graph if the product of the two distinct non-zero zero divisors is the additive identity of the direct products of finite fields. We prove that the metric chromatic number, clique number, vertex chromatic number are all equal to n, for the zero divisor graph of direct products of n finite fields, and also find the metric chromatic number, clique number, and vertex chromatic number of the complement graph of the zero-divisor graph of the direct product of n fields. The independence number, edge chromatic number, Eulerian and Hamiltonian properties of the zero-divisor graph and the complement graph are also determined.

Keywords: Zero-divisor graph, complement graph, finite field.

AMS Subject Classification: 05C15, 05C25, 05C45.

1. Introduction

In 1988, I. Beck [1] defined a graph on a commutative ring R, by considering all the elements of the ring R as the vertices of the graph and two distinct vertices x and y are adjacent if and only if $x \cdot y = 0$. In 1999, this definition of the graph given by Beck was modified by Anderson and Livingston [2] in which they considered only the non-zero zero divisors of R to be the vertex set of the zero divisor graph of R denoted by $\Gamma(R)$ and two distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $x \cdot y = 0$. In this paper we consider the zero divisor graph of the ring, $F_1 \times \cdots \times F_n,\ (n \geqslant 2)$, where $F_1, \cdots, F_n$ are finite fields.

Let $Z^*(F_1 \times \cdots \times F_n)$ be the set of non-zero zero-divisors of the ring $F_1 \times \cdots \times F_n,\ (n \geqslant 2)$ and $\Gamma(F_1 \times \cdots \times F_n)$ (defined by Anderson and Livingston [2]) denote the graph with vertex set $Z^*(F_1 \times \cdots \times F_n)$. Let $x = (x_1, \cdots, x_n),\ y = (y_1, \cdots, y_n) \in V(\Gamma(F_1 \times \cdots \times F_n)) = Z^*(F_1 \times \cdots \times F_n)$. Note that at least one coordinate of x and at least one coordinate of y

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contains 0 and at least one coordinate of x and at least one coordinate of y contains non-zero entry. Therefore, the product $x \cdot y = (x_1, \cdots, x_n) \cdot (y_1, \cdots, y_n) = (x_1 \cdot y_1, \cdots, x_n \cdot y_n)$, which is coordinate wise multiplication, is either equal to $(0, \cdots, 0)$ or the product $x \cdot y$ contains 0 in at least one coordinate and non-zero entry in at least one coordinate. In other words, $x \cdot y = (0, \cdots, 0)$ or $x \cdot y \in Z^*(F_1 \times \cdots \times F_n)$. Therefore, the edge set of $\Gamma(F_1 \times \cdots \times F_n)$ is

$$\Big\{xy \mid x \cdot y = (0, \cdots, 0), x,\ y \in Z^*(F_1 \times \cdots \times F_n)\Big\}$$

and the edge set of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is

$$\Big\{xy \mid x \cdot y \neq (0, \cdots, 0), x,\ y \in Z^*(F_1 \times \cdots \times F_n)\Big\}.$$

The complement graph [5] $\overline{G}$ of a simple graph G is defined by taking $V(\overline{G}) = V(G)$ and making two vertices u and v adjacent in $\overline{G}$ if and only if they are nonadjacent in G. Thus, if $x \cdot y = (0, \cdots, 0)$, then $xy \in E\big(\Gamma(F_1 \times \cdots \times F_n)\big)$, and if $x \cdot y \neq (0, \cdots, 0)$, then $xy \in E\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$ which implies $x \cdot y \in Z^*(F_1 \times \cdots \times F_n)$.

1.1. **Prerequisites.** For basic algebraic terminologies we refer to [3, 4]. For basic graph theoretical terminologies we adopt the definitions of [5, 6, 7].

(i) The degree of a vertex v in a graph G is the number of edges incident with v and is denoted by $deg_G(v)$ or simply by *deg* v. Also, *deg* v is the number of vertices adjacent to v. Two adjacent vertices are referred to as neighbors of each other. The set $N(v)$ of neighbors of a vertex v is called the neighborhood of v. Thus *deg* $v = |N(v)|$.[6] The minimum degree of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$, the maximum degree of G is the maximum degree among the vertices of G and is denoted by $\Delta(G)$.[6]

(ii) If every two distinct vertices in a subset $C \subseteq V(G)$ are adjacent in a graph G, the subset is said to be a clique in G. If there isn't another clique in the graph G with more vertices, then a clique C is said to be the maximum clique. The size of the maximal clique in a graph G is its clique number, which is represented by the symbol $\omega(G)$.[6] A set of vertices S in a graph G that has no two adjacent vertices is said to be an independent set. The highest possible size of an independent set in a graph is known as its independence number and is denoted by $\alpha(G)$.[6] A vertex cover of a graph is a collection of vertices such that each edge of the graph is incident to at least one vertex of the set. The cardinality of the smallest such set is known as the minimal vertex cover number and is represented by $\tau(G)$.[6, 7]

(iii) A graph's $k-$vertex(edge) coloring is the process of applying k different colors to its vertices(edges) while ensuring that no two adjacent vertices(edges) receive the same color. The least number of colors needed to color a graph's vertices(edges) so that no two adjacent vertices(edges) have the same color is the graph's vertex chromatic number(edge chromatic number), indicated by the symbol $\chi(G)(\chi_1(G))$.[6, 10] A simple graph G is Class-1, if $\chi_1(G) = \Delta(G)$ and Class-2, if $\chi_1(G) = \Delta(G) + 1$. [8]

(iv) An Eulerian trail is a path through a graph that includes every edge exactly once. It could be open or closed. If a graph has a closed Eulerian trail, it is referred to as an Eulerian graph.[10] A Hamiltonian cycle is a graph cycle that contains every vertex of the graph. A graph is referred to as a Hamiltonian graph if a Hamiltonian cycle exists in it.[10]

1.2. **Commonly used results.** The following results from graph theory are used to determine the properties of zero divisor graphs of direct product of finite fields.

(i) For every graph $G, \chi(G) \geq \omega(G)$. [7]
(ii) For any graph G, $|V(G)| = \alpha(G) + \tau(G)$. [5, 7]
(iii) If graph G is bipartite, then $\chi_1(G) = \Delta(G)$. [7]
(iv) If G is simple graph, then $\Delta(G) \leqslant \chi_1(G) \leqslant \Delta(G) + 1$. [7]
(v) If G has a Hamiltonian cycle, then for each non-empty subset $S \subset V$, the sub graph $G - S$ has at most $|S|$ components. [7, 10]
(vi) If G is a simple graph with at least three vertices and $\delta(G) \geq \frac{n(G)}{2}$, then G is Hamiltonian.[7]
(vii) A connected graph G is Eulerian if and only if every vertex of G has even degree. [10]

1.3. **Known results of zero divisor graphs of direct product of finite fields.**

(i) $\big|Z^*(F_1 \times \cdots \times F_n)\big| = \big|V\big(\Gamma(F_1 \times \cdots \times F_n)\big)\big|$
$= \prod_{i=1}^{n} |F_i| - \prod_{i=1}^{n}(|F_i| - 1) - 1$. [11, Theorem 2.1]
(ii) If $|F_1| = \cdots = |F_n| = p$, then the number of vertices of degree $p^r - 1, (1 \leqslant r \leqslant n-1)$ in $\Gamma(F_1 \times \cdots \times F_n)$ is $C(n,r)(p-1)^{n-r}$.[11, Theorem 2.2]
(iii) If $|F_1| = \cdots = |F_n| = p$, then the number of vertices of degree
$p^n - (p-1)^n - p^r - 1,\ (1 \leqslant r \leqslant n-1)$ in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is $C(n,r)(p-1)^r$.[11, Corollary 2.3]
(iv) $\Gamma(F_1 \times F_2)$ is a complete bipartite graph, and hence $\Gamma(F_1 \times F_2)$ is disconnected. [11, Corollary 2.5]

1.4. **Observation.**

Observation 1. *The number of vertices in* $\Gamma(F_1 \times \cdots \times F_n)$ *with degree* $\prod_{i_k=1}^{r} |F_{i_k}| - 1$ *is* $\frac{\prod_{i=1}^{n}(|F_i|-1)}{\prod_{i_k=1}^{r}(|F_{i_k}|-1)}$ *where* $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$.

Observation 2. *Let* $S = \{v_i = (0, \cdots, 0, 1, 0, \cdots, 0) : 1 \leq i \leq n\}$. *In other words, if* $v_i \in S$, *then* v_i *contains* 1 *in* i^{th} *coordinate and* 0^s *in remaining coordinates. Clearly,* $v_i \cdot v_j = (0, \cdots, 0)$ *if* $i \neq j$, $(1 \leq i \neq j \leq n)$. *Therefore, the sub-graph spanned by* S *is complete graph* K_n *in* $\Gamma(F_1 \times \cdots \times F_n)$.

1.5. **Main Results.** Let $F_1, \cdots, F_n$ $(n \geqslant 2)$ be finite fields. In this paper, we consider the zero-divisor graph $\Gamma(F_1 \times \cdots \times F_n)$ defined by Anderson and Livingston [2], with vertex set $Z^*(F_1 \times \cdots \times F_n)$, and two distinct vertices $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in Z^*(F_1 \times \cdots \times F_n)$ are adjacent in $\Gamma(F_1 \times \cdots \times F_n)$ if and only if $x \cdot y = (0, \cdots, 0)$. We prove that the vertex chromatic number $\chi\big(\Gamma(F_1 \times \cdots \times F_n)\big)$, clique number $\omega\big(\Gamma(F_1 \times \cdots \times F_n)\big)$, metric chromatic number $\mu\big(\Gamma(F_1 \times \cdots \times F_n)\big)$ of the zero divisor graph $\Gamma(F_1 \times \cdots \times F_n)$ is n and also determine the vertex chromatic number, clique number, metric chromatic number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. We determine the Edge chromatic number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$, and also determine the independence number, Eulerian and Hamiltonian properties of the zero divisor graph $\Gamma(F_1 \times \cdots \times F_n)$ and the complement graph $\overline{\Gamma(F_1 \times \cdots \times F_n)}$.

2. Graph parameters of zero divisor graph

Example 2.1. *Consider the zero divisor graph* $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ *(as defined in* [2]*).*
$V\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big) = \Big\{v_1 = (1,0,0), v_2 = (0,1,0), v_3(0,0,1), v_4 = (0,0,2), v_5 = (1,1,0), v_6 = (1,0,1), v_7 = (1,0,2), v_8 = (0,1,1), v_9 = (0,1,2)\Big\}$.

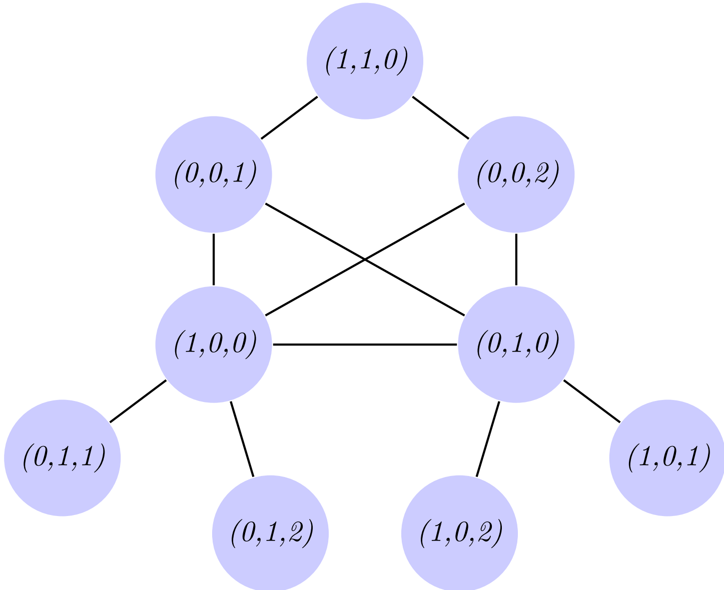

FIGURE 1. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$

(i) *Consider the subset* $S = \{(1,0,0),(0,1,0),(0,0,1)\} \subset V\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big)$. *The subgraph induced by* S *is complete graph* K_3.

(ii) *Consider the sub-partition* $V_1 = \big\{(1,0,0),(1,0,1),(1,0,2),(1,1,0)\big\}$, $V_2 = \{(0,1,0),(0,1,1),(0,1,2)\big\}, V_3 = \big\{(0,0,1),(0,0,2)\big\}$ *of* $V\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big)$. V_1, V_2, V_3 *are independent subsets and* $V_1 \cup V_2 \cup V_3 = V\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big)$. *We color the vertices in* V_i *with color* i $(i = 1,2,3)$. *Therefore, the vertex chromatic number* $\chi\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big) \leqslant 3$. *Since* K_3 *is subgraph, therefore,* $\chi\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big) \geqslant 3$. *Hence,* $\chi\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big) = 3$.

(iii) *Since* K_3 *is subgraph, therefore, the clique number* $\omega\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big) \geqslant 3$. *According to* [6, Theorem 10.5], $\omega(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) \leqslant \chi(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) = 3$. *Hence,* $\omega(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) = 3$.

(iv) *Let* $S_j \subset V(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3))$ *be the set of elements with non-zero entry in* j^{th} *coordinate* $(1 \leqslant j \leqslant 3)$ *and* 0 *or non-zero entries in the remaining coordinates. Then,* $S_1 = \big\{(1,0,0),(1,1,0),(1,0,1),(1,0,2)\big\}$, $S_2 = \big\{(0,1,0),(1,1,0),(0,1,1),(0,1,2)\big\}$, $S_3 = \big\{(0,0,1),(1,0,1),(1,0,2),(0,1,1),(0,1,2)\big\}$ *and* $|S_1| = 4, |S_2| = 4$, $|S_3| = 6$. *Therefore,* S_3 *is maximal independent subset and thus, the independence number,* $\alpha\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big) = 6 = max\Big\{\frac{(|F_j|-1)\prod_{i=1}^{n}|F_i|}{|F_j|} - \prod_{i=1}^{n}(|F_i|-1) \mid 1 \leqslant j \leqslant 3\Big\}$.

(v) *Since,* $\chi\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big) = 3$, *therefore, according to* [12, Corollary 2.2], *the metric chromatic number,* $\mu\big(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)\big) = 3$.

(vi) *Except the vertex* $(1,1,0)$ *which is of degree 2, all the other remaining vertices in* $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ *are odd degree vertices. Therefore, according to* [10, Theorem 5.1], $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ *is not Eulerian.*

(vii) $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$ *contains four vertices of degree 1. Therefore, there does not exist any cycle in* $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$ *which contains all the vertices in* $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$, *since the degree 1 vertices can not belong to any cycle. In other words,* $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$ *does not contain a Hamiltonian cycle. Therefore,* $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$ *is not Hamiltonian.*

Now, consider the complement graph $\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}$.

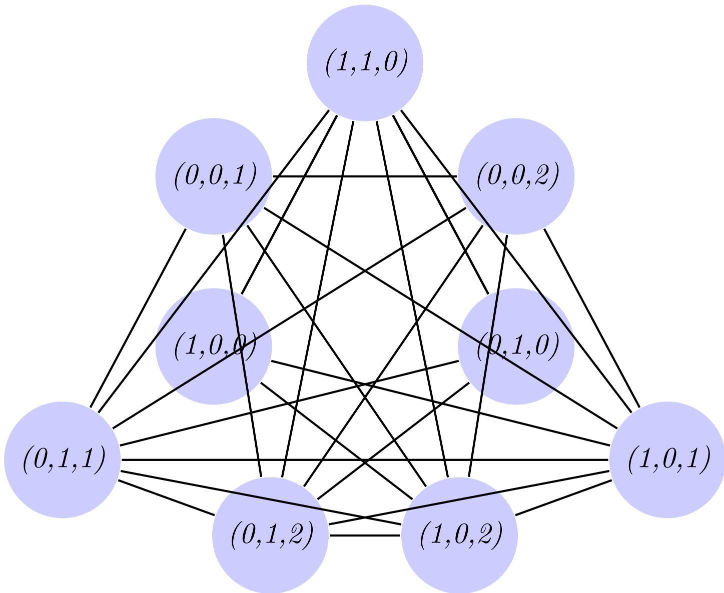

FIGURE 2. $\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}$

(i) *The clique number of* $\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}$ *is independence number of* $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$ *and independence number of* $\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}$ *is clique number of* $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$, *therefore,* $\omega\big(\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}\big) = 6 = max\Big\{\frac{(|F_j|-1)\prod_{i=1}^{n}|F_i|}{|F_j|} - \prod_{i=1}^{n}(|F_i|-1) \mid 1\leqslant j\leqslant 3\Big\}$ *and* $\alpha\big(\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}\big) = 3$.

(ii) *Consider the partition* $V_1=\{(1,0,0),(0,1,0),(0,0,1)\}, V_2=\{(0,0,2),(1,1,0)\}$, $V_3=\{(0,1,1)\}, V_4=\{(0,1,2)\}, V_5=\{(1,0,1)\}, V_6=\{(0,1,2)\}$. *Then* V_1,V_2,V_3, V_4,V_5,V_6 *are independent subsets and* $V_1\cup V_2\cup V_3\cup V_4\cup V_5\cup V_6 = V\big(\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}\big)$. *We color vertices in* V_i *with color* i $(1\leqslant i\leqslant 6)$. *Therefore,* $\chi\big(\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}\big)\leqslant 6$. *Also,* $\chi\big(\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}\big)\geqslant\omega\big(\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}\big)=6$. *Hence,* $\chi\big(\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}\big)=6$.

(iii) *In* [13, Remark 1], *Akbari and Mohammadian proved that if* G *is a graph such that for every vertex* u *of maximum degree there exists an edge* uv *such that* $\Delta(G)-d(v)+2$ *is more than the number of vertices with maximum degree in* G, *then* $\chi_1(G)=\Delta(G)$. *In* $\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}$, *there are four vertices of maximum degree, namely,* $(0,1,1),(0,1,2),(1,0,1),(1,0,2)$. *Consider* $(0,1,1)$. *We have* $deg(0,1,1)=7$. *The vertex* $(0,1,0)$ *is adjacent to* $(0,1,1)$ *and* $deg(0,1,0)=3$. *Therefore, if* $u=(0,1,1), v=(0,1,0)$, *then* $\Delta\big(\overline{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)}\big)-d(v)+2 = 7-3+2=6$. *Similarly for* $u=(0,1,2)$ *take* $v=(0,1,0)$, *for* $u=(1,0,1)$

take $v = (1,0,0)$*, for* $u = (1,0,2)$ *take* $v = (1,0,0)$*. Thus, according to* [13, Remark 1], $\chi_1\big(\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)}\big) = \Delta\big(\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)}\big) = 7$*. In other words,* $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)}$ *is Class-1.*

(iv) $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)}$ *contains exactly one even degree vertex and all other vertices are odd degree vertices. Therefore, according to* [7, Theorem 1.2.26], $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)}$ *is not Eulerian. The cycle* $(1,0,1),(1,0,0),(1,1,0),(0,1,0),(0,1,1),(0,1,2),(1,0,2),(0,0,1),(0,0,2),(1,0,1)$ *is a Hamiltonian cycle in* $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)}$*. Therefore,* $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)}$ *is Hamiltonian.*

2.1. **Independence number, Clique number, Vertex Chromatic number.** In Proposition 2.1, we determine the independence number of $\Gamma(F_1 \times \cdots \times F_n)$.

Proposition 2.1. *The independence number of* $\Gamma(F_1 \times \cdots \times F_n)$ *is* $\alpha\big(\Gamma(F_1 \times \cdots \times F_n)\big) = max\Big\{\frac{(|F_j|-1)\prod_{i=1}^n |F_i|}{|F_j|} - \prod_{i=1}^n(|F_i|-1) \mid 1 \leqslant j \leqslant n\Big\}$.

Proof. First we observe that if two vertices are not adjacent in $\Gamma(F_1 \times \cdots \times F_n)$ they should have non-zero entries in the same coordinate. We fix the coordinate as j^{th} coordinate and count the number of vertices which are not adjacent to $x = (0, \cdots, t, \cdots, 0)$ where $t \in F_j, t \neq 0$, $(1 \leqslant j \leqslant n)$. Every vertex with non-zero entry in j^{th} coordinate is not adjacent to x and they are not mutually adjacent. We find the largest such set and its cardinality. Let S_j be the set of elements with non-zero entry in the j^{th} coordinate. Let $v = (a_1, \cdots a_n) \in S_j$. The choices for a_j is $\big(|F_j|-1\big)$ as $a_j \neq 0$ and choices for a_i is $|F_i|$ for $i \in \{1, \cdots, n\} \setminus \{j\}$ and S_j should contain elements of $V\big(\Gamma(F_1 \times \cdots \times F_n)\big)$.
Note that all coordinate having non-zero entries is not an element of $V\big(\Gamma(F_1 \times \cdots \times F_n)\big)$ and we should remove the elements with each coordinate entry non-zero. The total number of choices of a_i with each of $a_i \neq 0$ is $(|F_i|-1)$ and total such possible vertices not in $V\big(\Gamma(F_1 \times \cdots \times F_n)\big)$ are $\prod_{i=1}^n \big(|F_i|-1\big)$ and hence $|S_j| = \frac{\big(|F_j|-1\big)\prod_{i=1}^n |F_i|}{|F_j|} - \prod_{i=1}^n \big(|F_i|-1\big)$.
Let $|S_\Delta| = max\{|S_j| \mid 1 \leqslant j \leqslant n\}$ be the set with largest cardinality. Hence,
$\alpha\big(\Gamma(F_1 \times \cdots \times F_n)\big) = \big|S_\Delta\big| = max\big\{|S_j| \mid 1 \leqslant j \leqslant n\big\}$
$= max\Big\{\frac{\big(|F_j|-1\big)\prod_{i=1}^n |F_i|}{|F_j|} - \prod_{i=1}^n \big(|F_i|-1\big) \mid 1 \leqslant j \leqslant n\Big\}$. □

In [1], Beck conjectured that the vertex chromatic number $\chi(R)$ is equal to clique number $\Omega(R)$ for a commutative ring R. Beck proved this conjecture in case of reduced ring [1, Theorem 3.8]. Beck proved that if R is a reduced ring $\neq (0)$ and if $\chi(R) < \infty$, then R has only a finite number of minimal prime ideals. And if n is this number then $\chi(R) = \Omega(R) = n+1$. The result was proved using algebraic properties. Beck's result [1, Theorem 3.8] implies that chromatic number of the analogous zero divisor graph $\Gamma(R)$, as defined by Anderson and Livingston [2], is n.

In this paper, we provide graph theoretic approach for known results. We give the combinatorial proof for vertex chromatic number, clique number of $\Gamma(F_1 \times \cdots \times F_n)$ in Proposition 2.2. Since the vertex set of $\Gamma(F_1 \times \cdots \times F_n)$ does not contain identity element $(0, \cdots, 0)$ and units of $F_1 \times \cdots \times F_n$, we use graph theoretical properties to prove $\chi\big(\Gamma(F_1 \times \cdots \times F_n)\big) = \omega\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$.

Proposition 2.2. (i) *The vertex chromatic number* $\chi\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$*.*
(ii) *The clique number* $\omega\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$*.*

Proof.

(i) According to observation 2, $\Gamma(F_1 \times \cdots \times F_n)$ contains complete graph K_n as its subgraph. The vertex chromatic number $\chi(K_n)$ of the complete graph K_n is n. Therefore, the vertex chromatic number, $\chi\big(\Gamma(F_1 \times \cdots \times F_n)\big) \geqslant \chi(K_n) = n$.

Now we partition $V\big(\Gamma(F_1 \times \cdots \times F_n)\big)$ into n independent subsets. Let
$X_1 = \Big\{v = (a_1, \cdots, a_n) \in V(\Gamma(F_1 \times \cdots \times F_n)) \mid a_1 \in F_1 \setminus \{0\}, a_j \in F_j,\ 2 \leqslant j \leqslant n\Big\}$.
Then $|X_1| = (|F_1|-1)\prod_{i=2}^{n}|F_i| - \prod_{i=1}^{n}(|F_i|-1)$, since there are $(|F_1|-1)\prod_{i=2}^{n}|F_i|$ number of possible choices for the n-tuple and among these n-tuple, $\prod_{i=1}^{n}(|F_i|-1)$ are having non zero entries in each of n coordinate which are not elements of $V\big(\Gamma(F_1 \times \cdots \times F_n)\big)$. Also the vertices in X_1 are not mutually adjacent to each other in $\Gamma(F_1 \times \cdots \times F_n)$.
Let $X_2 = \Big\{v = (0, a_2, \cdots, a_n) \in V\big(\Gamma(F_1 \times \cdots \times F_n)\big) \mid a_2 \in F_2 \setminus \{0\}, a_j \in F_j, 3 \leqslant j \leqslant n\Big\}$.
Then $|X_2| = \big(|F_2|-1\big)\prod_{i=3}^{n}|F_i|$. Also, the vertices in X_2 are not mutually adjacent to each other in $\Gamma(F_1 \times \cdots \times F_n)$ and $X_1 \cap X_2 = \phi$.
Similarly, let $X_k = \Big\{v = (0, \cdots, 0, a_k, \cdots, a_n) \in V\big(\Gamma(F_1 \times \cdots \times F_n)\big) \mid a_k \in F_k \setminus \{0\}, a_j \in F_j,\ k+1 \leqslant j \leqslant n\Big\}$.
Then $|X_k| = \big(|F_k|-1\big)\prod_{i=K+1}^{n}|F_i|$ and the vertices in X_k are not mutually adjacent to each other in $\Gamma(F_1 \times \cdots \times F_n)$. Also, $X_i \cap X_j = \phi$ for $1 \leq i \neq j \leq k$.

Therefore, $|X_1| + |X_2| + \cdots + |X_k| + \cdots + |X_{n-1}| + |X_n|$

$= \big[(|F_1|-1)\prod_{i=2}^{n}|F_i| - \prod_{i=1}^{n}(|F_i|-1)\big] + (|F_2|-1)\prod_{i=3}^{n}|F_i| + \cdots + (|F_k|-1)\prod_{i=k+1}^{n}|F_i| + \cdots + (|F_{n-1}|-1)|F_n| + (|F_n|-1)$

$= \big[\prod_{i=1}^{n}|F_i| - \prod_{i=2}^{n}|F_i|\big] + \big[\prod_{i=2}^{n}|F_i| - \prod_{i=3}^{n}|F_i|\big] + \cdots + \big[\prod_{i=k}^{n}|F_i| - \prod_{i=k+1}^{n}|F_i|\big] + \cdots + \big[|F_{n-1}||F_n| - |F_n|\big] + (|F_n|-1) - \prod_{i=1}^{n}(|F_i|-1)$

$= \prod_{i=1}^{n}|F_i| - \prod_{i=1}^{n}(|F_i|-1) - 1$

$= \Big|V\big(\Gamma(F_1 \times F_2 \times \cdots \times F_n)\big)\Big|$. [11, Theorem 2.1]

Thus, $X_i, (1 \leqslant i \leqslant n)$ is a partition of $V\big(\Gamma(F_1 \times \cdots \times F_n)\big)$ such that, $\cup_{i=1}^{n} X_i = V\big(\Gamma(F_1 \times \cdots \times F_n)\big)$ and moreover each $X_i, (1 \leqslant i \leqslant n)$ is an independent set. If we color the vertices in set $X_i, (1 \leqslant i \leqslant n)$ with color $i, (1 \leqslant i \leqslant n)$, we get a $n-$coloring of $\Gamma(F_1 \times \cdots \times F_n)$. Therefore, $\chi\big(\Gamma(F_1 \times \cdots \times F_n)\big) \leqslant n$. Since, we have already proved that $\chi(\Gamma(F_1 \times \cdots \times F_n)) \geqslant \chi(K_n) = n$, we have that the vertex chromatic number $\chi\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$.

(ii) Since, K_n is a subgraph of $\Gamma(F_1 \times \cdots \times F_n))$, this implies, clique number $\omega\big(\Gamma(F_1 \times \cdots \times F_n)\big) \geq n$. According to [6, Theorem 10.5], $\omega\big(\Gamma(F_1 \times \cdots \times F_n)\big) \leqslant \chi\big(\Gamma(F_1 \times \cdots \times F_n)\big)$, therefore, we have, $\omega\big(\Gamma(F_1 \times \cdots \times F_n))\big) \leqslant n$. Hence, $\omega\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$. □

Corollary 2.1. (i) *The vertex cover number of* $\Gamma(F_1 \times \cdots \times F_n)$ *is* $\tau\big(\Gamma(F_1 \times \cdots \times F_n)\big)$ $= \Big[\prod_{i=1}^{n}|F_i| - \prod_{i=1}^{n}(|F_i|-1) - 1\Big] - max\Big\{\frac{(|F_k|-1)\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1) \;\Big|\; 1 \leqslant k \leqslant n\Big\}$.

(ii) *The clique number of* $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ *is* $\omega\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$
$= max\Big\{\frac{(|F_k|-1)\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1)|1 \leqslant k \leqslant n\Big\}$,

(iii) *The independence number of* $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ *is* $\alpha\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big) = n$

(iv) *The vertex cover number of* $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ *is*
$\tau(\overline{\Gamma(F_1 \times \cdots \times F_n)}) = \Big[\prod_{i=1}^{n}|F_i| - \prod_{i=1}^{n}(|F_i|-1) - 1\Big] - n.$

(v) *The vertex chromatic number of* $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ *is* $\chi\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$
$= max\Big\{\frac{(|F_k|-1)\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1) \mid 1 \leqslant k \leqslant n\Big\}.$

Proof. Using the fact, $\big|V\big(\Gamma(F_1\times\cdots\times F_n)\big)\big| = \alpha\big(\Gamma(F_1\times\cdots\times F_n)\big)+\tau\big(\Gamma(F_1\times\cdots\times F_n)\big)$[7], result (i) follows. Since the clique number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is independence number of $\Gamma(F_1 \times \cdots \times F_n)$ and independence number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is clique number of $\Gamma(F_1 \times \cdots \times F_n)$, and using the fact, $\big|V\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)\big| = \alpha\big(\overline{(\Gamma(F_1 \times \cdots \times F_n)}\big) + \tau\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$[7], the result (ii), (iii), (iv) follows.

(v) Consider $(0,\cdots,0,1,0,\cdots,0) \in X_k,\ 1 \leqslant k \leqslant n$ with a 1 in the k^{th} coordinate and 0 in the remaining coordinate. We collect those vertices in $\Gamma(F_1 \times \cdots \times F_n)$ which contains non-zero entry in the k^{th} coordinate in the $n-tuple$ and add it to the set X_k. The number of vertices of $\Gamma(F_1 \times \cdots \times F_n)$ which are not mutually adjacent to each other with a non zero entry in k^{th} coordinate is $(|F_k|-1)\frac{\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1)$. This implies $|X_k| = (|F_k|-1)\frac{\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1)$. Each X_k is an independent set in $\Gamma(F_1\times\cdots\times F_n)$. Therefore, the subgraph spanned by X_k in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is Complete graph $K_{|X_k|}$.
Let $|X_s| = max\{|X_k| \mid 1 \leqslant k \leqslant n\}$. Therefore, vertex chromatic number
$\chi\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big) \geq |X_s| = max\{|X_k|\}$
$= max\Big\{(|F_k|-1)\frac{\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1) \mid 1 \leqslant k \leqslant n\Big\}.$
Clearly, the vertex chromatic number of the subgraph spanned by X_s in $(\overline{\Gamma(F_1 \times \cdots \times F_n)})$ is $max\Big\{(|F_k|-1)\frac{\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1)\ :\ 1 \leqslant k \leqslant n\Big\}.$
Now if $y \in V\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big) \setminus X_s$, then y contains 0 in the k^{th} coordinate position. Therefore, y is adjacent to at least one vertex, say, $w \in X_s$ in $\Gamma(F_1 \times \cdots \times F_n)$. Thus, y is not adjacent to $w \in X_s$ in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. Thus, y can be colored with the same color as that of w in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. Since, subgraph spanned by X_k in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is complete graph $K_{|X_k|}$, the color of w is distinct from the color of vertices in $X_s \setminus \{w\}$. Now let $y_1, y_2 \in V\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big) \setminus X_s$. Let y_1 be adjacent to $w_1 \in X_s$ and y_2 be adjacent to $w_2 \in X_s$, such that $w_1 \neq w_2$. Then clearly the color of y_1 will be distinct from the color of y_2, as y_1 can be colored with the same color as that of w_1 and y_2 can be colored with the same color as that of w_2.
Now, suppose $N(y_1) \cap X_s = N(y_2) \cap X_s$ in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. If $y_1,\ y_2$ are not adjacent to each other then y_1 and y_2 can be colored with the same color of some vertex
$w \in X_s \setminus \{N(y_1) \cap X_s\}$, and this is possible since $|N(y_1) \cap X_s| < |X_s|$. Suppose $y_1,\ y_2$ are adjacent to each other then as $|X_s \setminus \{N(y_1) \cap X_s\}| \geq 2$, y_1 and y_2 can be colored with the same color of some vertices $w_1,\ w_2 \in X_s \setminus \{N(y_1) \cap X_s\}$ and the colors of $w_1,\ w_2$ are distinct from the colors of $N(y_1) \cap X_s$. Therefore, $\chi\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$

$\leq |X_s| = max\{|X_k|\} = max\Big\{(|F_k|-1)\frac{\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1) \;\Big|\; 1 \leqslant k \leqslant n\Big\}$.
Hence, $\chi\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big) = max\Big\{(|F_k|-1)\frac{\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1) \;\Big|\; 1 \leqslant k \leqslant n\Big\}$. □

2.2. **Metric Chromatic number.**

Definition 2.1. *For a set $S \subseteq V(G)$ and a vertex v of G, the distance $d(v, S)$ between v and S is defined as $d(v, S) = min\{d(v, x) \mid x \in S\}$.* [6, 10] *Suppose that $c : V(G) \to \{1, 2, \cdots, k\}$ is a $k-$coloring of G for some positive integer k where adjacent vertices may be colored the same and let $V_1, V_2, \cdots, V_k$ be the resulting color classes. With each vertex v, we can associate a $k-$vector*

$$code(v) = (a_1, a_2, \cdots, a_k) = a_1 a_2 \cdots a_k$$

called the metric color code of v, where for each i with $1 \leqslant i \leqslant k, a_i = d(v, V_i)$. If $code(u) \neq code(v)$ for every two adjacent vertices u and v of G, then c is called a metric coloring of G. The minimum k for which G has a metric $k-$coloring is called the metric chromatic number of G and is denoted by $\mu(G)$. Clearly, $\mu(G)$ is defined for every connected graph G and $\mu(G) \geqslant 2$ for every nontrivial connected graph G. [12]

Let c be a proper $k-$coloring of a nontrivial connected graph G with resulting color classes $V_1, V_2, \cdots, V_k$ and let u and v be two adjacent vertices of G. Then $u \in V_i$ and $v \in V_j$ for some $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$. Suppose that $code(u) = (a_1, a_2, \cdots, a_k)$ and $code(v) = (b_1, b_2, \cdots, b_k)$. Then $a_i = b_j = 0$ and $a_j = b_i = 1$. Thus $code(u) \neq code(v)$ and c is also a metric coloring of G. Thus $\mu(G) \leqslant \chi(G)$. [12]

Lemma 2.1. *If G be a connected graph on $n \geqslant 2$ vertices which contains complete graph $K_p; (2 \leqslant p \leqslant n)$ as an induced subgraph, then the metric chromatic number $\mu(G) \geqslant p$.*

Proof. Suppose $\mu(G) = t,\ 2 \leqslant t \leqslant p-1$. Then there exists a metric $t-$coloring $c : V(G) \to \{1, 2, \cdots, t\}$, such that, $code(u) \neq code(v)$ for every two adjacent vertices u and v of G. Then, $c|_{V(K_p)} : V(K_p) \to \{1, 2, \cdots, t\}$ is also a $t-$metric coloring of induced subgraph K_p. But this is a contradiction, as a connected graph G of order n has metric chromatic number n if and only if $G = K_n$ [12]. Hence, $\mu(G) \geqslant p$. □

Theorem 2.1. *The metric chromatic number of $\Gamma(F_1 \times \cdots \times F_n)$ is $\mu\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$.*

Proof. According to [12, Proposition 2.1], if $n = 2$, then $\mu\big(\Gamma(F_1 \times F_2)\big) = 2$, and if $n = 3$, then according to Proposition 2.2, $\chi\big(\Gamma(F_1 \times F_2 \times F_3)\big) = 3$, therefore, according to [12, Corollary 2.2], $\mu\big(\Gamma(F_1 \times F_2 \times F_3)\big) = 3$. Let $n \geqslant 4$. Claim: $\mu\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$. Suppose $\mu(\Gamma(F_1 \times \cdots \times F_n)) = t, (4 \leqslant t \leqslant n-1)$. Then there exists a metric $t-$coloring $c : V\big(\Gamma(F_1 \times \cdots \times F_n)\big) \to \{1, 2, \cdots, t\}$ where adjacent vertices may be colored the same and $code(u) \neq code(v)$ for every two adjacent vertices u and v of $\Gamma(F_1 \times \cdots \times F_n)$. According to observation 2, K_n is a subgraph of $\Gamma(F_1 \times \cdots \times F_n)$. Therefore, according to 2.1, $c|_{V(K_n)} : V(K_n) \to \{1, 2, \cdots, t\}$ is a metric $t-$coloring of the subgraph K_n, such that for every pair x, y of adjacent vertices in subgraph K_n, $code(x) \neq code(y)$. This is a contradiction to the fact that a connected graph G of order n has metric chromatic number n if and only if $G = K_n$ [12]. Thus, $\mu\big(\Gamma(F_1 \times \cdots \times F_n)\big) \geqslant n$. According to [12], since metric chromatic number $\leqslant$ vertex chromatic number, we have, $\mu\big(\Gamma(F_1 \times \cdots \times F_n)\big) \leqslant \chi\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$. Hence, $\mu\big(\Gamma(F_1 \times \cdots \times F_n)\big) = \chi\big(\Gamma(F_1 \times \cdots \times F_n)\big) = n$. □

Corollary 2.2. *The metric chromatic number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is $\mu\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$ $= \chi\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big) = max\Big\{(|F_k|-1)\frac{\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1) \;\Big|\; 1 \leqslant k \leqslant n\Big\}$.*

Proof. According to Corollary 2.1 (ii,iv), the clique number and vertex chromatic number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is $max\Big\{(|F_k|-1)\frac{\prod_{i=1}^{n}|F_i|}{|F_k|} - \prod_{i=1}^{n}(|F_i|-1) \;\Big|\; 1 \leqslant k \leqslant n\Big\}$. Hence, the result follows. □

2.3. **Edge Chromatic number.** In [13], Akbari and Mohammadian, have shown that for any finite commutative ring R, the edge chromatic number of $\Gamma(R)$ is equal to the maximum degree of $\Gamma(R)$, unless $\Gamma(R)$ is a complete graph of odd order. They have also proved in [13, Remark 1], that if G is a graph such that for every vertex u of maximum degree there exists an edge uv such that $\Delta(G)-d(v)+2$ is more than the number of vertices with maximum degree in G, then $\chi_1(G) = \Delta(G)$. We use this remark to determine the edge chromatic number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$.

Theorem 2.2. *If $n \geqslant 3$ then $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ belongs to Class-1.*

Proof. Let $|F_j| = min\Big\{|F_i| \mid i = 1, \cdots, n\Big\}$. Let u be a vertex with 0 in the j^{th} coordinate and non-zero entries in the remaining coordinate. Then

$$deg(u) = \prod_{i=1,\, i\neq j}^{n} |F_i| - 1 = \Delta\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big).$$

Let H_j be the set of vertices with $'0'$ in the j^{th} coordinate and non-zero entries in the remaining $(n-1)$ coordinate. Then H_j contains maximum degree vertices and $|H_j| = \prod_{i=1,\, i\neq j}^{n} \big(|F_i|-1\big)$. Let k be such that $|F_j| \leqslant |F_k| \leqslant |F_i|$, $1 \leqslant i \leqslant n$, $i \neq j, k$. Let v be a vertex with non-zero entry in the k^{th} coordinate and $0's$ in the remaining coordinate. Since, the k^{th} co-ordinate of v and $u \in H_j$ is a non-zero entry, therefore, $uv \in E\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$ and $deg(v) = |F_k| - 1$. Therefore,
$\Delta\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big) - deg(v) + 2$
$= \prod_{i=1,\, i\neq j}^{n} |F_i| - 1 - (|F_k| - 1) + 2$
$= \prod_{i=1,\, i\neq j}^{n} |F_i| - 1 - |F_k| + 1 + 2$
$= \prod_{i=1,\, i\neq j}^{n} |F_i| - |F_k| + 2$
$> \prod_{i=1,\, i\neq j}^{n}(|F_i| - 1)$.
Thus, according to [13, Remark 1], $\overline{\Gamma(F_1 \times \cdots \times F_n)} = \Delta\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$. □

Remark 2.1. *If $n = 2$, then $\overline{\Gamma(F_1 \times F_2)}$ is disconnected graph with two copies of complete graph $K_{|F_1|-1}$ and $K_{|F_2|-1}$* [11, Corollary 2.5]. *Therefore, if $max\{|F_1|-1, |F_2|-1\}$ is an even number greater than or equal to 4, then $\overline{\Gamma(F_1 \times F_2)}$ is Class-1, and if $max\{|F_1|-1, |F_2|-1\}$ is an odd number greater than or equal to 3, then $\overline{\Gamma(F_1 \times F_2)}$ is Class-2.* [6, Theorem 10.15]

2.4. **Eulerian and Hamiltonian.** In [13], S. Akbari, A. Mohammadian, defined the zero-divisor graph of a ring R as the directed graph $\Gamma(R)$, such that its vertices are all non-zero zero-divisors of R in which for any two distinct vertices x and y, $x \to y$ is an edge if and only if $xy = 0$. They proved that [13, Proposition 1], if $n \geqslant 2$ and $R_1, \cdots, R_n$ are finite rings and $R = R_1 \times \cdots \times R_n$, then $\Gamma(R)$ is an Eulerian graph if and only if for $i = 1, \cdots, n$, either R_i is a field or $\Gamma(R_i)$ is an Eulerian graph. In this study, the zero divisor graph (defined by Anderson and Livingston [2]) we are considering is a simple undirected graph. In Proposition 2.3, we determine the conditions for which the zero divisor graph $\Gamma(F_1 \times \cdots \times F_n)$ is Eulerian or not Eulerian. The conditions are determined in terms of the order of the fields. Since the order of a finite field is a prime power, the following proposition fully characterizes the Eulerian property for the graphs under consideration in this study.

Proposition 2.3. *If $n \geqslant 2$ then $\Gamma(F_1 \times \cdots \times F_n)$ is*

(i) *not Eulerian if $|F_i| = 2^r$, for some i, $(1 \leqslant i \leqslant n)$, $r \in \mathbb{N}$.*

(ii) *Eulerian if $|F_i| \neq 2^r$ for all i, $(1 \leqslant i \leqslant n)$.*

Proof. Let $v \in V(\Gamma(F_1 \times \cdots \times F_n))$. Suppose v contains 0^s in coordinate $i_1, i_2, \cdots, i_r$ with $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$. Then $deg(v) = \prod_{k=1}^{r} |F_{i_k}| - 1$.

(i) If $|F_i| = 2^r$, for some i, $(1 \leqslant i \leqslant n)$, $r \in \mathbb{N}$, then clearly the degree of a vertex with 0 in i^{th} coordinate and non-zero entries in remaining coordinate is odd and thus $\Gamma(F_1 \times \cdots \times F_n)$ is not Eulerian.[10]

(ii) If $|F_i| \neq 2^r$ for all i, $(1 \leqslant i \leqslant n)$ then every vertex in $\Gamma(F_1 \times \cdots \times F_n)$ is of even degree. Hence according to [7, Theorem 1.2.26], $\Gamma(F_1 \times \cdots \times F_n)$ is Eulerian in this case. □

In Proposition 2.4, we determine the conditions for $\Gamma(F_1 \times \cdots \times F_n)$ to be Hamiltonian or non-Hamiltonian. We make use of the result [7, Proposition 7.2.3], "If a graph G has a Hamiltonian cycle then for each non-empty subset $S \subseteq V(G)$ the graph $G - S$ has atmost $|S|$ components. "

Proposition 2.4. (i) *$\Gamma(F_1 \times F_2)$ is Hamiltonian if $|F_1| = |F_2| \geqslant 3$, and $\Gamma(F_1 \times F_2)$ is not Hamiltonian if $|F_2| > |F_1|$ or $|F_1| > |F_2|$.*

(ii) *$\Gamma(F_1 \times \cdots \times F_n)$ is not Hamiltonian, if $n \geqslant 3$.*

Proof.

(i) $\Gamma(F_1 \times F_2)$ is complete bipartite graph $K_{|F_1|-1,|F_2|-1}$. A complete bipartite graph is hamiltonian if and only if the parts are of equal size and have 2 or more vertices[7]. Therefore, $K_{|F_1|-1,|F_2|-1}$ is Hamiltonian if and only if $|F_1| - 1 = |F_2| - 1 \geqslant 2$. Hence, the result follows.

(ii) Let $n \geqslant 3$. Without loss of generality, let $|F_1| = min\Big\{|F_1|, \cdots, |F_n|\Big\}$.
Consider the set $S = \big\{(t, 0, \cdots, 0) \;\big|\; t \in F_1 \setminus \{0\}\big\}$. We have $|S| = |F_1| - 1$.
Let $T = \Big\{(0, a_2, \cdots, a_n) : \; a_j \in \{1, 2, \cdots, |F_j| - 1\}, (2 \leqslant j \leqslant n)\Big\}$.
Then, $|T| = \prod_{j=2}^{n} \big(|F_j| - 1\big)$. Each vertex in S is adjacent with each and every vertex in T and the only vertices adjacent to a vertex of T are in S. That is, the vertices in T are not adjacent with any vertex in $\Gamma(F_1 \times \cdots \times F_n) - S \implies deg(x) = |F_1| - 1$ for $x \in T$. The number of components in $\Gamma(F_1 \times \cdots \times F_n) - S \geqslant \prod_{j=2}^{n} \big(|F_j| - 1\big) > |F_1| - 1 = |S|$, as $|F_1| = min\big\{|F_1|, \cdots, |F_n|\big\}$. Hence, if $n \geqslant 3$, then according to [7, Proposition 7.2.3], $\Gamma(F_1 \times \cdots \times F_n)$ is not Hamiltonian. □

Proposition 2.5. (i) *$\overline{\Gamma(F_1 \times F_2)}$ is neither Eulerian nor Hamiltonian.*

(ii) *$\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is Hamiltonian if $n \geqslant 3$.*

(iii) *$\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is Eulerian if $|F_i| = 2$ for all $1 \leq i \leq n$ and not Eulerian if at least one $|F_i| \geq 3$ for $1 \leq i \leq n$, $n \geqslant 3$.*

Proof. (i) $\overline{\Gamma(F_1 \times F_2)}$ is disconnected.

(ii) Let $n \geqslant 3$. We prove that $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is Hamiltonian. We demonstrate a Hamiltonian cycle in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. First we partition $V\Big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\Big)$ into n subsets such that each subset is a clique in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$.
Let $X_1 = \Big\{(a_1, a_2, \cdots, a_n) \in V(\overline{\Gamma(F_1 \times \cdots \times F_n)}) \;\Big|\; a_1 \in F_1 \setminus \{0\}, a_j \in F_j, \; 2 \leqslant j \leqslant n\Big\}$.
Then $|X_1| = \big(|F_1| - 1\big) \prod_{i=2}^{n} |F_i| - \prod_{i=1}^{n} \big(|F_i| - 1\big)$, since there are $(|F_1| - 1) \prod_{i=2}^{n} |F_i|$ number of possible choices for the n-tuple and among these n-tuple, $\prod_{i=1}^{n} \big(|F_i| - 1\big)$ are having

non zero entries in each of n coordinate which are not elements of $V\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$. Also the vertices in X_1 are all mutually adjacent to each other in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$.
Let $X_2 = \Big\{(0, a_2, \cdots, a_n) \in V(\overline{\Gamma(F_1 \times \cdots \times F_n)}) \;\Big|\; a_2 \in F_2 \setminus \{0\}, a_j \in F_j,\ 3 \leqslant j \leqslant n\Big\}$.
Then $|X_2| = (|F_2| - 1)\prod_{i=3}^{n} |F_i|$. And the vertices in X_2 are all mutually adjacent to each other in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. Also, $X_1 \cap X_2 = \phi$. Similarly, let
$X_k = \Big\{(0, \cdots, 0, a_k, \cdots, a_n) \in V\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big) \;\Big|\; a_k \in F_k \setminus \{0\}, a_j \in F_j,$
$k + 1 \leqslant j \leqslant n\Big\}$.
Then $|X_k| = (|F_k| - 1)\prod_{i=K+1}^{n} |F_i|$. And the vertices in X_k are all mutually adjacent to each other in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. Also, $X_i \cap X_j = \phi$ for $1 \leq i \neq j \leq k$.
Therefore, $|X_1| + |X_2| + \cdots + |X_k| + \cdots + |X_{n-1}| + |X_n|$
$= \Big[(|F_1|-1)\prod_{i=2}^{n} |F_i| - \prod_{i=1}^{n}(|F_i|-1)\Big] + (|F_2|-1)\prod_{i=3}^{n} |F_i| + \cdots + (|F_k|-1)\prod_{i=k+1}^{n} |F_i|$
$+ \cdots + (|F_{n-1}| - 1)|F_n| + (|F_n| - 1)$
$= \Big[\prod_{i=1}^{n} |F_i| - \prod_{i=2}^{n} |F_i|\Big] + \Big[\prod_{i=2}^{n} |F_i| - \prod_{i=3}^{n} |F_i|\Big] + \cdots + \Big[\prod_{i=k}^{n} |F_i| - \prod_{i=k+1}^{n} |F_i|\Big] + \cdots$
$+ \Big[|F_{n-1}||F_n| - |F_n|\Big] + (|F_n| - 1) - \prod_{i=1}^{n}(|F_i| - 1)$
$= \prod_{i=1}^{n} |F_i| - \prod_{i=1}^{n}(|F_i| - 1) - 1$
$= |V(\overline{\Gamma(F_1 \times \cdots \times F_n)})|$.
Thus, $X_i, (1 \leqslant i \leqslant n)$ is a partition of $V\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$ such that, $\cup_{i=1}^{n} X_i = V\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$ and moreover each $X_i, (1 \leqslant i \leqslant n)$ is a clique set in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$.
Let P_1 be a Hamiltonian path of vertices in X_1 such that the starting vertex of path P_1 contains 1 in the first coordinate and last n^{th} coordinate and the last vertex of path P_1 contains 1 in the second coordinate. Let P_2 be a Hamiltonian path of vertices in X_2 such that the last vertex of path P_2 contains 1 in the third coordinate. Clearly, the last vertex of path P_1 is adjacent to first vertex of path P_2. Therefore, $P_1 \cup P_2$ is a Hamiltonian path of vertices in $X_1 \cup X_2$. Similarly, let P_k be a Hamiltonian path of vertices in X_k $(3 \leqslant k \leqslant n - 1)$ such that the last vertex of path P_k contains 1 in the $k + 1$ coordinate. Clearly, the last vertex of path P_k is adjacent to first vertex of path P_{k+1} and $P_1 \cup P_2 \cup \cdots \cup P_k$ is a Hamiltonian path of vertices in $X_1 \cup X_2 \cup \cdots \cup X_k$. Let P_n be a Hamiltonian path of vertices in X_n. The vertices of X_n contain non-zero entry in the last n^{th} coordinate. Therefore, the last vertex of path P_n is adjacent to the first vertex of path P_1. Therefore, $P_1 \cup P_2 \cup \cdots \cup P_n$ is a Hamiltonian path of vertices in $V\big(\overline{\Gamma(F_1 \times \cdots \times F_n)}\big)$ such that the last vertex of path P_n is adjacent to first vertex of path P_1. Thus, we get a Hamiltonian cycle in $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. Hence, $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is Hamiltonian.

(iii) Let $n \geqslant 3$. The degree of an arbitrary vertex $x \in \Gamma(F_1 \times \cdots \times F_n)$ is $\prod_{i_k=1}^{r} |F_{i_k}| - 1$ and $|V(\Gamma(F_1 \times \cdots \times F_n))| = |V(\overline{\Gamma(F_1 \times \cdots \times F_n)})| = \prod_{i=1}^{n} |F_i| - \prod_{i=1}^{n}(|F_i| - 1) - 1$. Hence degree of $x \in \overline{\Gamma(F_1 \times \cdots \times F_n)}$ is $\prod_{i=1}^{n} |F_i| - \prod_{i=1}^{n}(|F_i| - 1) - \prod_{k=1}^{r} |F_{i_k}| - 1$ where $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$, $(1 \leqslant r \leqslant n - 1)$. If $|F_1| = \cdots = |F_n| = 2$, then clearly the degree of every vertex is even and hence $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ is Eulerian in this case. If at least one $|F_i| \geq 3$ for $1 \leq i \leq n$ then $|F_i|$ is odd and thus $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ contains at least one odd degree vertex and hence is not Eulerian. □

Conclusion

We calculated the independence number and determined the Eulerian and Hamiltonian properties of the zero-divisor graph $\Gamma(F_1 \times \cdots \times F_n)$ and the complement graph $\overline{\Gamma(F_1 \times \cdots \times F_n)}$. We determined the Edge chromatic number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$ and proved that the metric chromatic number clique number and vertex chromatic number of $\Gamma(F_1 \times \cdots \times F_n)$ is n and also determined clique number, vertex chromatic number, metric chromatic number of $\overline{\Gamma(F_1 \times \cdots \times F_n)}$.

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