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COMPLETE SOFT SEMIGRAPHS: COMPREHENSIVE ANALYSIS OF $K_{m_1,m_2,\ldots,m_r}^{m_1-1,m_2-1,\ldots,m_r-1}$ STRUCTURES

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Abstract. Soft set theory provides a systematic approach for handling imprecision and uncertainty by categorizing elements of a set based on specific parameters. In semigraph theory, soft semigraphs utilize this approach, offering a parameterized perspective that has significantly advanced the field through effective parameter management. In this paper, we introduce and define complete and strongly complete soft semigraphs, focusing on their unique properties and structures. We then delve into an in-depth analysis of strongly complete soft semigraphs in the form $K_{m_1,m_2,\ldots,m_r}^{m_1-1,m_2-1,\ldots,m_r-1}$. Key properties such as the total number of f-edges, and various vertex degrees are examined through a series of theorems, providing valuable insights into the complex relationships and characteristics of these soft semigraphs.

Keywords: Soft Set, Semigraph, Soft Graph, Soft Semigraph.

AMS Subject Classification: 05C99

1. Introduction

Conventional methods in formal modelling, reasoning, and computation typically exhibit determinism, clarity, and precision. However, the complexities encountered in diverse fields like engineering, medicine, economics, and social sciences often involve data that lacks a clear definition. Various uncertainties present in these problem areas pose challenges for traditional methods. The fuzzy set theory addresses one form of uncertainty, termed "Fuzziness," arising from elements partially belonging to a set. While it effectively handles uncertainties related to vague or partially belonging elements, it doesn't encompass all uncertainties found in real-world problems. The emergence of soft set theory in 1999 by mathematician Molodtsov [31] offers a more practical approach compared to

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established theories like probability or fuzzy set theory, owing to its versatility. For example, fuzzy set theory lacks sufficient parameterization tools. Authors such as Maji, Biswas, and Roy [29, 30] have expanded on soft set theory, employing it to resolve decision-making problems.

The notion of soft graphs was introduced by Thumbakara and George [38]. In 2015, Akram and Nawas [1, 2] modified the definition of soft graphs. Further advancements in the field were made by Akram and Nawas [3, 4], who introduced fuzzy soft graphs, strong fuzzy soft graphs, complete fuzzy soft graphs, and regular fuzzy soft graphs, exploring their properties and potential applications. Akram and Zafar [6, 7] pioneered the concepts of soft trees and fuzzy soft trees. Fuzzy soft theory enables the handling of problems containing uncertain data by combining the characteristics of fuzzy sets and soft sets. Nawaz and Akram [32] explored the applications of fuzzy soft graphs, such as analyzing oligopolistic competition among wireless internet service providers in Malaysia. Additionally, Akram and Shahzadi [5] proposed a decision-making approach utilizing Pythagorean Dombi fuzzy soft graphs.

Contributions to the study of soft graphs have been made by Thenge, Jain, and Reddy [35, 36, 37]. Soft graphs, owing to their utility in handling parameterization, represent a growing domain within graph theory. George, Thumbakara, and Jose [28, 39, 40, 41] studied various concepts in soft graphs and introduced soft hypergraphs [8, 14], soft directed graphs [22, 24], soft directed hypergraphs [20] and soft disemigraphs [21], studying their properties. The operation of graph products, a method of combining two graphs, can be extended to soft graphs. They also explored various product operations in soft graphs [10, 11] and soft directed graphs [9, 23, 25, 26, 27] and investigated their properties. The concept of semigraphs, a broader version of graphs, was first introduced by Sampathkumar [33, 34]. Unlike hypergraphs, semigraphs maintain a specific order of vertices within their edges. When represented on a plane, semigraphs resemble conventional graphs. In 2022, George, Thumbakara, and Jose [12, 13] introduced soft semigraphs by applying soft set principles to semigraphs and defined some soft semigraph operations. Moreover, they introduced some product operations [17], connectedness [15] and various degrees, graphs, and matrices associated with soft semigraphs [13, 16]. George, Jose, and Thumbakara [19] also presented Eulerian and Hamiltonian soft semigraphs and the closure of a soft semigraph. In this paper, we introduce the complete and strongly complete soft semigraphs. We study in detail the strongly complete soft semigraphs in the form $K^{m_1-1,m_2-1,\ldots,m_r-1}_{m_1,m_2,\ldots,m_r}$.

2. Preliminaries

In this preliminary section, we lay the foundation for comprehending soft sets, semigraphs, and soft semigraphs. We define fundamental concepts such as partial edges and p-part, which are crucial to the structure of soft semigraphs. Finally, we provide a brief overview of topics including degrees and graphs associated with soft semigraphs.

2.1. **Semigraph.** The notion of semigraph was introduced by Sampathkumar [33, 34] as follows. "A *semigraph* G is a pair (V, X) where V is a nonempty set whose elements are called vertices of G, and X is a set of n-tuples, called edges of G, of distinct vertices, for various $n \geq 2$, satisfying the following conditions.

(1) Any two edges have at most one vertex in common
(2) Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_m)$ are considered to be equal if and only if
   (a) $m = n$ and
   (b) either $u_i = v_i$ for $1 \leq i \leq n$, or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Figure 1 shows an example for a semigraph $G^* = (V, X)$. The vertex set of this semigraph G^* is $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and the edge set is $X = \{(v_1, v_2, v_3), (v_3, v_4, v_5, v_6), (v_6, v_7, v_8), (v_5, v_8)\}$.

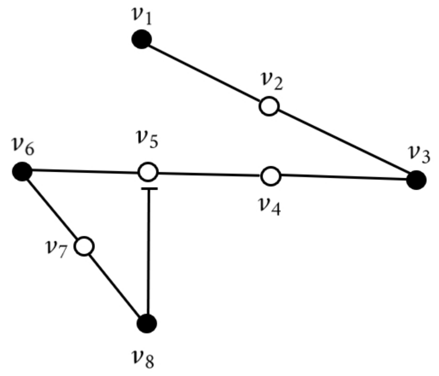

FIGURE 1. Semigraph $G^* = (V, X)$

Let $G = (V, X)$ be a semigraph and $E = (v_1, v_2, \ldots, v_n)$ be an edge of G. Then v_1 and v_n are the *end vertices* of E and $v_i, 2 \leq i \leq n-1$ are the *middle vertices(or m-vertices)* of E. If a vertex v of a semigraph G appears only as an end vertex then it is called an *end vertex.* If a vertex v is only a middle vertex then it is a *middle vertex* or *m-vertex* while a vertex v is called *middle-cum-end vertex or* (m, e)*-vertex* if it is a middle vertex of some edge and an end vertex of some other edge. A *subedge* of an edge $E = (v_1, v_2, \ldots, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ or $1 \leq i_k < i_{k-1} < \cdots < i_1 \leq n$. We say that the subedge E' is *induced* by the set of vertices $\{v_{i_1}, v_{i_2}, \ldots v_{i_k}\}$. A *partial edge* of $E = (v_1, v_2, \ldots, v_n)$ is a $(j-i+1)$-tuple $E(v_i, v_j) = (v_i, v_{i+1}, \ldots, v_j)$, where $1 \leq i < j \leq n$. $G' = (V', X')$ is a *partial semigraph* of a semigraph G if the edges of G' are partial edges of G. Two vertices u and v in a semigraph G are said to be *adjacent* if they belong to the same edge. If u and v are adjacent and consecutive in order then they are said to be *consecutively adjacent.* u and v are said to be *e-adjacent* if they are the end vertices of an edge and *1e-adjacent* if both the vertices u and v belong to the same edge and at least one of them, is an end vertex of that edge".

2.2. **Soft Set.** In 1999 Molodtsov [31] initiated the concept of soft sets. "Let U be an initial universe set and let A be a set of parameters. A pair (F, A) is called a soft set (over U) if and only if F is a mapping of A into the set of all subsets of the set U. That is, $F : A \to \mathcal{P}(U)$".

2.3. **Soft Semigraph.** George, Thumbakara and Jose [12, 13] introduced soft semigraph by applying the concept of soft set in semigraph as follows: "Let $G^* = (V, X)$ be a semigraph having vertex set V and edge set X. Consider a subset V_1 of V. Then a partial edge formed by some or all vertices of V_1 is said to be a *maximum partial edge* or *mp edge* if it is not a partial edge of any other partial edge formed by some or all vertices of V_1. Let X_p be the collection of all partial edges of the semigraph G and A be a nonempty set. Let a subset R of $A \times V$ be an arbitrary relation from A to V. We define a mapping Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy\}, \forall x \in A$, where $\mathcal{P}(V)$ denotes the power set of V. Then the pair (Q, A) is a soft set over V. Also define a mapping W from A to $\mathcal{P}(X_p)$ by $W(x) = \{\text{mp edges} < Q(x) >\}$, where $\{\text{mp edges} < Q(x) >\}$ denotes the set of all mp edges that can be formed by some or all vertices of $Q(x)$ and $\mathcal{P}(X_p)$ denotes the power set of X_p. The pair (W, A) is a soft set over X_p. Then we can define a soft semigraph as follows: The 4-tuple $G = (G^*, Q, W, A)$ is called a *soft semigraph* of G^* if the following conditions are satisfied:

(1) $G^* = (V, X)$ is a semigraph having vertex set V and edge set X,
(2) A is the nonempty set of parameters,
(3) (Q, A) is a soft set over V,
(4) (W, A) is a soft set over X_p,
(5) $H(a) = (Q(a), W(a))$ is a partial semigraph of $G^*, \forall a \in A$.

Let $G^* = (V, X)$ be a semigraph and $G = (G^*, Q, W, A)$ be a soft semigraph of G^* which is also given by $\{H(x) : x \in A\}$. Then the partial semigraph $H(x)$ corresponding to any parameter x in A is called a *p-part* of the soft semigraph G. An edge present in a soft semigraph G of G^* is called an *f-edge*. It may be a partial edge of some edge in G^* or an edge in G^*. A partial edge of any f-edge of a soft semigraph G is called a *p-edge* of G. An f-edge is a p-edge of itself. An f-edge or a p-edge of a soft semigraph G is called an *fp-edge* of G." An example of a soft semigraph is given below. Let $G^* = (V, X)$ be a semigraph as given in Figure 2 having the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ and the edge set $\{(v_1, v_2, v_3), (v_5, v_6, v_7), (v_{12}, v_{13}, v_{14}), (v_4, v_8, v_{10}, v_{13}), (v_9, v_{11}, v_{14}), (v_2, v_{10}), (v_5, v_{11})\}$.

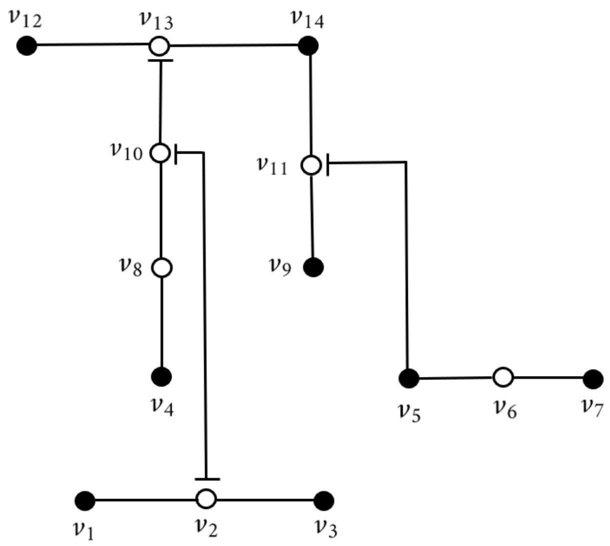

FIGURE 2. Semigraph $G^* = (V, X)$

Let $A = \{v_{13}, v_6\} \subseteq V$ be a set of parameters. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy \Leftrightarrow x = y \text{ or } x \text{ and } y \text{ are adjacent}\}, \forall x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp\ edges < Q(x) >\}, \forall x \in A$. That is, $Q(v_{13}) = \{v_4, v_8, v_{10}, v_{12}, v_{13}, v_{14}\}$ and $Q(v_6) = \{v_5, v_6, v_7\}$. Also $W(v_{13}) = \{(v_{12}, v_{13}, v_{14}), (v_4, v_8, v_{10}, v_{13})\}$ and $W(v_6) = \{(v_5, v_6, v_7)\}$. Then $H(v_{13}) = (Q(v_{13}), W(v_{13}))$ and $H(v_6) = (Q(v_6), W(v_6))$ are partial semigraphs of G^* as shown below in Figure 3. Also, (Q, A) is a soft set over V and (W, A) is a soft set over X_p. Hence $G = \{H(v_{13}), H(v_6)\}$ is a soft semigraph of G^*.

2.4. **Degrees Associated with Soft Semigraphs.** George, Jose and Thumbakara [13, 16] defined various types of degrees associated with soft semigraphs as follows: "Let $H(x)$ be any p-part of the soft semigraph G and let v be any vertex in $H(x)$. Then the *p-part degree* of v in $H(x)$ denoted by *deg* $v[H(x)]$ is defined as the number of f-edges having

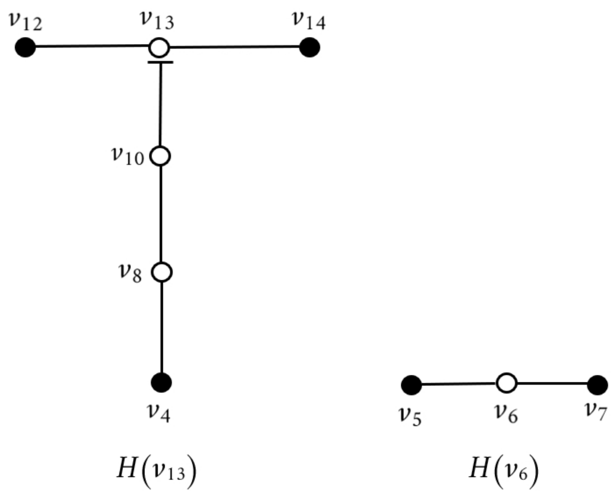

FIGURE 3. Soft Semigraph $G = \{H(v_{13}), H(v_6)\}$

v as an end vertex in $H(x)$. *Degree* of a vertex v in a soft semigraph G, denoted by $deg\ v$ is defined as $deg\ v = max\{deg\ v[H(x)] : x \in A\}$, where $deg\ v[H(x)]$ denotes the p-part degree of v in $H(x)$. The *p-part end degree* of v in $H(x)$ denoted by $deg_{ep}v[H(x)]$ is defined as the number of f-edges having v as an end vertex or partial end vertex in $H(x)$. *End degree* of a vertex v in a soft semigraph G, denoted by $deg_{ep}v$ is defined as $deg_{ep}v = max\{deg_{ep}v[H(x)] : x \in A\}$, where $deg_{ep}v[H(x)]$ denotes the p-part end degree of v in $H(x)$. The *p-part edge degree* of v in $H(x)$ denoted by $deg_e v[H(x)]$ is defined as the number of f-edges containing v in $H(x)$. *Edge degree* of a vertex v in a soft semigraph G, denoted by $deg_e v$ is defined as $deg_e v = max\{deg_e v[H(x)] : x \in A\}$, where $deg_e v[H(x)]$ denotes the p-part edge degree of v in $H(x)$. The *p-part adjacent degree* of v in $H(x)$ denoted by $deg_a v[H(x)]$ is defined as the number of vertices adjacent to v in $H(x)$. *Adjacent degree* of a vertex v in a soft semigraph G, denoted by $deg_a v$ is defined as $deg_a v = max\{deg_a v[H(x)] : x \in A\}$, where $deg_a v[H(x)]$ denotes the p-part adjacent degree of v in $H(x)$. The *p-part consecutive adjacent degree* of v in $H(x)$ denoted by $deg_{ca}v[H(x)]$ is defined as the number of vertices consecutively adjacent to v in $H(x)$. *Consecutive adjacent degree* of a vertex v in a soft semigraph G, denoted by $deg_{ca}v$ is defined as $deg_{ca}v = max\{deg_{ca}v[H(x)] : x \in A\}$, where $deg_{ca}v[H(x)]$ denotes the p-part consecutive adjacent degree of v in $H(x)$".

2.5. **Graphs Associated with Soft Semigraphs.** George, Jose and Thumbakara [13] defined various types of graphs associated with soft semigraphs as follows: "The *end vertex graph* G_e of the soft semigraph G is given by $G_e = \{H(x)_e : x \in A\}$ where $H(x)_e$ is a graph having vertex set $Q(x)$ and two vertices u and v in $H(x)_e$ are adjacent if they are the end vertices or a partial end vertices of an f-edge containing these vertices in the p-part $H(x)$. $H(x)_e$ is called *p-part end vertex graph* of $H(x)$. The *consecutive adjacency graph* G_{ca} of the soft semigraph G is given by $G_{ca} = \{H(x)_{ca} : x \in A\}$ where $H(x)_{ca}$ is a graph having vertex set $Q(x)$ and two vertices in $H(x)_{ca}$ are adjacent if they are consecutively adjacent in the p-part $H(x)$. $H(x)_{ca}$ is called *p-part consecutive adjacency graph* of $H(x)$. The *adjacency graph* G_a of the soft semigraph G is given by $G_a = \{H(x)_a : x \in A\}$ where $H(x)_a$ is a graph having vertex set $Q(x)$ and two vertices in $H(x)_a$ are adjacent if they are adjacent in the p-part $H(x)$. $H(x)_a$ is called *p-part adjacency graph* of $H(x)$".

3. Complete and Strongly Complete Soft Semigraphs

In this section, we explore the concepts of completeness within the framework of soft semigraphs. We introduce the definitions of complete and strongly complete soft semigraphs, essential for understanding the structural properties of soft semigraphs. These definitions help to distinguish between various levels of completeness and connectivity within the soft semigraph. We also establish the relationship between complete and strongly complete soft semigraphs, illustrating their differences with examples. Furthermore, we introduce specific notations for these soft semigraph structures to facilitate their identification and analysis in subsequent discussions.

Definition 3.1. *Let $G^* = (V, X)$ be a semigraph and $G = (G^*, Q, W, A)$ be a soft semigraph of G^* which is also given by $\{H(x) : x \in A\}$. Let $H(x)$ be any p-part of G for some x in A. Then $H(x)$ is called a complete p-part if the p-part $H(x)$ is a complete partial semigraph of G^*. That is, any two vertices in $H(x)$ are adjacent.*

Definition 3.2. *A soft semigraph G given by $\{H(x) : x \in A\}$ is called a complete soft semigraph if $H(x)$ is a complete p-part for all x in A.*

Example 3.1. *Let $G^* = (V, X)$ be a semigraph as given in Figure 4 having the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and the edge set $\{(v_1, v_2, v_3), (v_2, v_4, v_6), (v_4, v_5), (v_6, v_7), (v_6, v_8), (v_6, v_9), (v_7, v_8, v_9)\}$.*

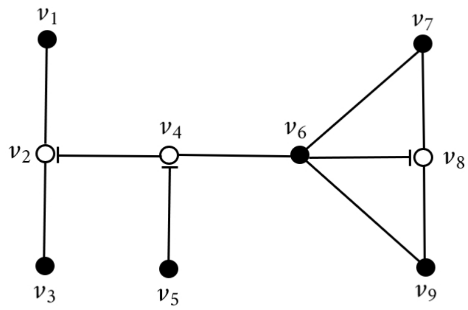

Figure 4. Semigraph $G^* = (V, X)$

Let $A = \{v_1, v_8\} \subseteq V$ be a set of parameters. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy \Leftrightarrow x = y$ or x and y are adjacent$\}, \forall x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp$ edges $< Q(x) >\}, \forall x \in A$. That is, $Q(v_1) = \{v_1, v_2, v_3\}$ and $Q(v_8) = \{v_6, v_7, v_8, v_9\}$. Also $W(v_1) = \{(v_1, v_2, v_3)\}$ and $W(v_8) = \{(v_6, v_7), (v_6, v_8), (v_6, v_9), (v_7, v_8, v_9)\}$. Then $H(v_1) = (Q(v_1), W(v_1))$ and $H(v_8) = (Q(v_8), W(v_8))$ are partial semigraphs of G^* as shown below in Figure 5. Also, (Q, A) is a soft set over V and (W, A) is a soft set over X_p. Hence $G = \{H(v_1), H(v_8)\}$ is a soft semigraph of G^*. Here, in $H(v_1)$ and $H(v_8)$, any two vertices are adjacent. Therefore, $H(v_1)$ and $H(v_8)$ are complete p-parts of G. So, G is a complete soft semigraph.

Definition 3.3. *Let $G^* = (V, X)$ be a semigraph and $G = (G^*, Q, W, A)$ be a soft semigraph of G^* which is also given by $\{H(x) : x \in A\}$. Let $H(x)$ be any p-part of G for some x in A. Then $H(x)$ is called a strongly complete p part if the p-part $H(x)$ is a strongly complete partial semigraph of G^*. That is, $H(x)$ is complete and every vertex in $H(x)$ is an end vertex of an f-edge in $H(x)$.*

Definition 3.4. *A soft semigraph G given by $\{H(x) : x \in A\}$ is called a strongly complete soft semigraph if $H(x)$ is a strongly complete p-part for all x in A.*

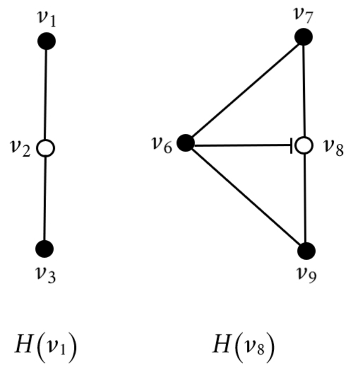

FIGURE 5. Soft Semigraph $G = \{H(v_1), H(v_8)\}$

Example 3.2. *Let $G^* = (V, X)$ be a semigraph as given in Figure 6 having the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and the edge set $\{(v_1, v_2), (v_2, v_3), (v_2, v_4), (v_1, v_4, v_3), (v_4, v_5, v_6), (v_4, v_7, v_8), (v_5, v_7), (v_5, v_8), (v_7, v_6), (v_6, v_8)\}$.*

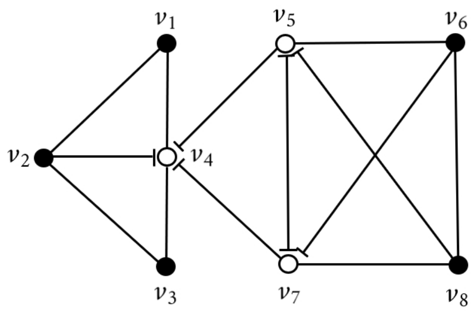

FIGURE 6. Semigraph $G^* = (V, X)$

Let $A = \{v_2, v_6\} \subseteq V$ be a set of parameters. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy \Leftrightarrow x = y$ or x and y are adjacent$\}, \forall x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp \text{ edges} < Q(x) >\}, \forall x \in A$. That is, $Q(v_2) = \{v_1, v_2, v_3, v_4\}$ and $Q(v_6) = \{v_4, v_5, v_6, v_7, v_8\}$. Also $W(v_2) = \{(v_1, v_2), (v_2, v_3), (v_2, v_4), (v_1, v_4, v_3)\}$ and $W(v_6) = \{(v_4, v_5, v_6), (v_4, v_7, v_8), (v_5, v_7), (v_5, v_8), (v_7, v_6), (v_6, v_8)\}$. Then $H(v_2) = (Q(v_2), W(v_2))$ and $H(v_6) = (Q(v_6), W(v_6))$ are partial semigraphs of G^* as shown below in Figure 9. Also, (Q, A) is a soft set over V and (W, A) is a soft set over X_p. Hence $G = \{H(v_2), H(v_6)\}$ is a soft semigraph of G^*. Here, in $H(v_2)$ and $H(v_6)$, any two vertices are adjacent (i.e., $H(v_2)$ and $H(v_6)$ are complete p-parts of G) and every vertex in $H(v_1)$ and $H(v_6)$ are end vertex of an f-edge in the corresponding p-part. That is, $H(v_1)$ and $H(v_6)$ are strongly complete p-parts of G. So, $G = \{H(v_2), H(v_6)\}$ is a strongly complete soft semigraph.

Remark 3.1. *Every strongly complete soft semigraphs are complete. But the converse need not be true. For example, the complete soft semigraph G given in Figure 5 is not strongly complete since the p-part $H(v_1)$ of G is not a strongly complete p-part. Here, v_2 is not an end vertex of any f-edge in $H(v_1)$.*

Remark 3.2. *A complete p-part on m vertices can be denoted by C_m. If it is a strongly complete p-part on m vertices, it can be denoted by C^s_m. If G is a complete soft semigraph having r complete p-parts $C_{m1}, C_{m2}, \ldots, C_{mr}$, then G can be denoted by $K_{m_1, m_2, \ldots, m_r} =$*

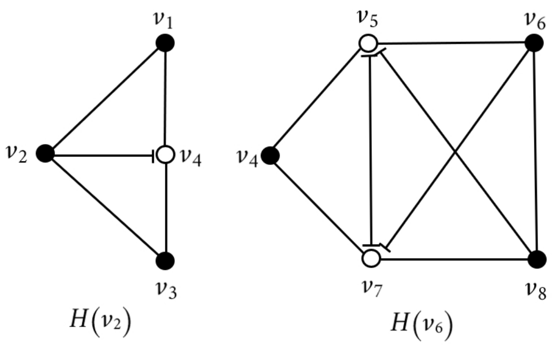

FIGURE 7. Soft Semigraph $G = \{H(v_2), H(v_6)\}$

$\{C_{m_1}, C_{m_2}, \ldots, C_{m_r}\}$. *If G is a strongly complete soft semigraph having r strongly complete p-parts* $C_{m_1}^s, C_{m_2}^s, \ldots, C_{m_r}^s$, *then G can be denoted by* $K^s_{m_1,m_2,\ldots,m_r} = \{C^s_{m_1}, C^s_{m_2}, \ldots, C^s_{m_r}\}$. *A strongly complete p-part having m vertices with one f-edge of cardinality* $m-1$ *and all other f-edges of cardinality two can be denoted by* C_m^{m-1} *where the cardinality of an f-edge means the number of vertices in an f-edge. A strongly complete soft semigraph G can be denoted as* $K^{m_1-1,m_2-1,\ldots,m_r-1}_{m_1,m_2,\ldots,m_r}$ *if its strongly complete p-parts are* $C_{m_1}^{m_1-1}$, $C_{m_2}^{m_2-1}$, $\ldots$, *and* $C_{m_r}^{m_r-1}$.

4. Some Properties of Strongly Complete Soft Semigraphs in the form $K^{m_1-1,m_2-1,\ldots,m_r-1}_{m_1,m_2,\ldots,m_r}$

In this section, we investigate several key properties of strongly complete soft semigraphs, specifically those denoted in the form $K^{m_1-1,m_2-1,\ldots,m_r-1}_{m_1,m_2,\ldots,m_r}$. Through a series of theorems, we quantify various aspects of these structures, such as the total number of f-edges, and various types of degrees associated with vertices. We examine the adjacent degree, consecutive adjacent degree, end degree, and edge degree within these semigraphs, providing precise formulations for the sum of these degrees. These results offer a deeper understanding of the intricate relationships and characteristics that define strongly complete soft semigraphs.

Theorem 4.1. *Total number of f-edges in the strongly complete soft semigraph* $K^{m_1-1,m_2-1,\ldots,m_r-1}_{m_1,m_2,\ldots,m_r}$ *is* $\sum_{i=1}^r m_i$.

Proof. The strongly connected complete p-parts of $K^{m_1-1,m_2-1,\ldots,m_r-1}_{m_1,m_2,\ldots,m_r}$ are $C_{m_1}^{m_1-1}$, $C_{m_2}^{m_2-1}$, $\ldots$, and $C_{m_r}^{m_r-1}$. Take an arbitrary p-part $C_{m_i}^{m_i-1}$. Here one f-edge E_1 of $C_{m_i}^{m_i-1}$ is of cardinality $m_i - 1$ and all other f-edges are of cardinality two. There is only one vertex v_i in $C_{m_i}^{m_i-1}$, which is different from the $m_i - 1$ vertices contained in E_1. Since $C_{m_i}^{m_i-1}$ is complete, v_i is adjacent to these $m_i - 1$ vertices through $m_i - 1$ f-edges of cardinality two. Since $C_{m_i}^{m_i-1}$ is a partial semigraph, no more edge is possible. So, total number of f-edges in $C_{m_i}^{m_i-1}$ is $m_i - 1 + 1 = m_i$, for $i = 1, 2, \ldots, r$. Therefore, the total number of edges in $K^{m_1-1,m_2-1,\ldots,m_r-1}_{m_1,m_2,\ldots,m_r}$ is $m_1 + m_2 + \ldots + m_r = \sum_{i=1}^r m_i = \sum_{i=1}^r m_i$. □

Theorem 4.2. *Consider a strongly complete soft semigraph* $K^{m_1-1,m_2-1,\ldots,m_r-1}_{m_1,m_2,\ldots,m_r}$. *Then*

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_a v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} m_i(m_i - 1),$$

where $V(C_{m_i}^{m_i-1})$ represents the vertex set of the strongly complete p-part $C_{m_i}^{m_i-1}$ and $deg_a v[C_{m_i}^{m_i-1}]$ represents the p-part adjacent degree of the vertex v in $C_{m_i}^{m_i-1}$.

Proof. Consider the strongly complete soft semigraph $K_{m_1,m_2,\ldots,m_r}^{m_1-1,m_2-1,\ldots,m_r-1}$ given by $\{C_{m_1}^{m_1-1}, C_{m_2}^{m_2-1}, \ldots, C_{m_i}^{m_i-1}\}$. Take any strongly complete p-part $C_{m_i}^{m_i-1}$ for some $i = 1, 2, \ldots, r$. In $C_{m_i}^{m_i-1}$ there is one f-edge E of cardinality $m_i - 1$ and $m_i - 1$ f-edges of cardinality two. There is only one vertex v_i in $C_{m_i}^{m_i-1}$ other than the $m_i - 1$ vertices contained in the f-edge E. Each vertex u_i in E is adjacent to $m_i - 2$ other vertices in E and to the vertex v_i. So, the p-part adjacent degree of each vertex contained in E is $m_i - 2 + 1 = m_i - 1$. Therefore, these $m_i - 1$ vertices give $(m_i - 1)(m_i - 1)$ to the sum of p-part adjacent degrees. Also, the vertex v_i is adjacent to the $m_i - 1$ vertices in E. So, its p-part adjacent degree is also $m_i - 1$. Therefore,

$$\sum_{v \in V(C_{m_i}^{m_i-1})} deg_a v[C_{m_i}^{m_i-1}] = (m_i-1)(m_i-1)+(m_i-1) = (m_i-1)[(m_i-1)+1] = m_i(m_i-1).$$

This is true for each strongly complete p-part $C_{m_i}^{m_i-1}, i = 1, 2, \ldots, r$. Hence,

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_a v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} m_i(m_i - 1).$$

□

Theorem 4.3. *Let v be any vertex of the strongly complete soft semigraph $K_{m_1,m_2,\ldots,m_r}^{m_1-1,m_2-1,\ldots,m_r-1}$. Then $deg_a v = max\{(m_i - 1) : v \in V(C_{m_i}^{m_i-1}), i = 1, 2, \ldots, r\}$.*

Proof. From the proof of Theorem 2, it is clear that all vertices in the strongly complete p-part $C_{m_i}^{m_i-1}$ have p-part adjacent degree $m_i - 1$. So, to find the adjacent degree of a vertex v in the strongly complete soft semigraph $K_{m_1,m_2,\ldots,m_r}^{m_1-1,m_2-1,\ldots,m_r-1}$, we have to find the maximum of p-part adjacent degrees of v among all strongly complete p-parts which contains the vertex v. Therefore, $deg_a v = max\{(m_i - 1) : v \in V(C_{m_i}^{m_i-1}), i = 1, 2, \ldots, r\}$. □

Theorem 4.4. *In a strongly complete soft semigraph $K_{m_1,m_2,\ldots,m_r}^{m_1-1,m_2-1,\ldots,m_r-1}$,*

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ca} v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} (4m_i - 6),$$

where $deg_{ca} v[C_{m_i}^{m_i-1}]$ represents the p-part consecutive adjacent degree of the vertex v in $C_{m_i}^{m_i-1}$.

Proof. Take any strongly complete p-part $C_{m_i}^{m_i-1}$ of $K_{m_1,m_2,\ldots,m_r}^{m_1-1,m_2-1,\ldots,m_r-1}$ for some $i = 1, 2, \ldots, r$. We know that $C_{m_i}^{m_i-1}$ has totally m_i vertices, say $v_1, v_2, \ldots, v_{m_i}$, an f-edge of cardinality $m_i - 1$, say $E = (v_1, v_2, \ldots, v_{m_i-1})$ and $m_i - 1$ f-edges of cardinality two which has an end v_{m_i} (if we take the vertex which is not part of E as v_{m_i}). Then v_1 and v_{m_i-1} are consecutively adjacent to two vertices (v_1 is adjacent to v_2 and v_{m_i} and v_{m_i-1} are consecutively adjacent to v_{m_i-2} and v_{m_i}) and v_{m_i} is consecutively adjacent to $m_i - 1$ vertices of E. Also, all the vertices of E other than v_1 and v_{m_i-1} are consecutively adjacent to two vertices of E and v_{m_i}. That is, the $(m_i - 3)$ vertices of the f-edge E are consecutively adjacent to three vertices each. Therefore,

$$\sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ca} v[C_{m_i}^{m_i-1}] = 2 \times 2 + (m_i - 1) + (m_i - 3) \times 3 = 4 + m_i - 1 + 3m_i - 9 = 4m_i - 6.$$

This is true for all $C_{m_i}^{m_i-1}, i = 1, 2, \ldots, r.$. Therefore,

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ca} v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} (4m_i - 6).$$

□

Theorem 4.5. *In a strongly complete soft semigraph* $K_{m_1, m_2, \ldots, m_r}^{m_1-1, m_2-1, \ldots, m_r-1}$,

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ep} v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} 2m_i,$$

where $deg_{ep} v[C_{m_i}^{m_i-1}]$ *represents the p-part end degree of the vertex v in* $C_{m_i}^{m_i-1}$.

Proof. Consider the strongly complete soft semigraph $K_{m_1, m_2, \ldots, m_r}^{m_1-1, m_2-1, \ldots, m_r-1}$ given by $\{C_{m_1}^{m_1-1}, C_{m_2}^{m_2-1}, \ldots, C_{m_i}^{m_i-1}\}$. Take any strongly complete p-part $C_{m_i}^{m_i-1}$ for some $i = 1, 2, \ldots, r$. We know that $C_{m_i}^{m_i-1}$ has an f-edge E of cardinality $m_i - 1$ and $m_i - 1$ edges of cardinality two. There is only one vertex v_{m_i} in $C_{m_i}^{m_i-1}$ which is not contained in the f-edge E. Then, the two end vertices of the edge E say, v_1 and v_{m_i-1} are the end vertices or partial end vertices of 2 f-edges each, one is the f-edge E and the other is the f-edge from v_{m_i} to them. Other $m_i - 3$ vertices in the edge are the end vertices or partial end vertices of one f-edge each which is the f-edge from the vertex v_{m_i} to these $m_i - 3$ vertices. Also v_{m_i} is the end vertex or partial end vertex of $m_i - 1$ f-edges which are the f-edges from v_{m_i} to the $m_i - 1$ vertices of E. Therefore,

$$\sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ep} v[C_{m_i}^{m_i-1}] = 2 \times 2 + (m_i - 3) + (m_i - 1) = 4 + m_i - 3 + m_i - 1 = 2m_i.$$

This is true for all $C_{m_i}^{m_i-1}, i = 1, 2, \ldots, r.$. Hence,

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ep} v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} (2m_i).$$

□

Theorem 4.6. *Consider a strongly complete soft semigraph* $K_{m_1, m_2, \ldots, m_r}^{m_1-1, m_2-1, \ldots, m_r-1}$. *Then*

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_{e} v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} (3m_i - 3),$$

where $deg_{e} v[C_{m_i}^{m_i-1}]$ *represents the p-part edge degree of the vertex v in* $C_{m_i}^{m_i-1}$.

Proof. Consider any strongly complete p-part $C_{m_i}^{m_i-1}$ of $K_{m_1, m_2, \ldots, m_r}^{m_1-1, m_2-1, \ldots, m_r-1}$ for some $i = 1, 2, \ldots, r$. In $C_{m_i}^{m_i-1}$, we have an f-edge E containing $m_i - 1$ vertices, say $v_1, v_2, \ldots, v_{m_i-1}$ and another vertex v_{m_i} which is not contained in E. There is also $m_i - 1$ f-edges of cardinality two from v_{m_i} to the $m_i - 1$ vertices in E. So, each vertex v_i in E, $i = 1, 2, \ldots, m_i - 1$, is the part of two f-edges, one is E and the other is the f-edge to v_{m_i}. Also, there are $m_i - 1$ edges which contains v_{m_i}. Therefore,

$$\sum_{v \in V(C_{m_i}^{m_i-1})} deg_{e} v[C_{m_i}^{m_i-1}] = 2 \times (m_i - 1) + (m_i - 1) = 3(m_i - 1) = 3m_i - 3.$$

This is true for all $C_{m_i}^{m_i-1}, i = 1, 2, \ldots, r.$. Hence,

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_e v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} (3m_i - 3).$$

□

Example 4.1. *Let $G^* = (V, X)$ be a semigraph as given in Figure 8 having the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ and the edge set $\{(v_1, v_2, v_3), (v_1, v_4), (v_2, v_4), (v_3, v_4), (v_4, v_6, v_{10}), (v_5, v_6, v_8), (v_7, v_8, v_9), (v_{10}, v_{11}), (v_{10}, v_{12}), (v_{10}, v_{13}), (v_{10}, v_{14}), (v_{11}, v_{12}, v_{13}, v_{14})\}$.*

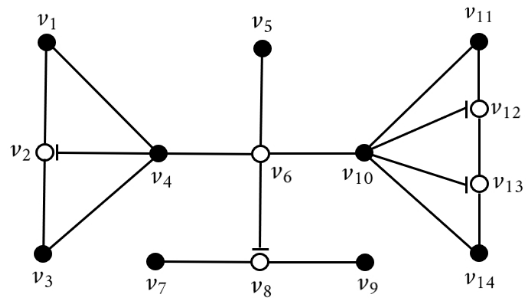

FIGURE 8. Semigraph $G^* = (V, X)$

Let $A = \{v_2, v_{12}\} \subseteq V$ be a set of parameters. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V | xRy \Leftrightarrow x = y$ or x and y are adjacent$\}, \forall x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp$ edges $< Q(x) >\}, \forall x \in A$. That is, $Q(v_2) = \{v_1, v_2, v_3, v_4\}$ and $Q(v_{12}) = \{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$. Also $W(v_2) = \{(v_1, v_2, v_3), (v_1, v_4), (v_2, v_4), (v_3, v_4)\}$ and $W(v_{12}) = \{(v_{10}, v_{11}), (v_{10}, v_{12}), (v_{10}, v_{13}), (v_{10}, v_{14}), (v_{11}, v_{12}, v_{13}, v_{14})\}$. Then $H(v_2) = (Q(v_2), W(v_2))$ and $H(v_{12}) = (Q(v_{12}), W(v_{12}))$ are partial semigraphs of G^* as shown below in Figure 9. Also, (Q, A) is a soft set over V and (W, A) is a soft set over X_p. Hence $G = \{H(v_2), H(v_{12})\}$ is a soft semigraph of G^*.

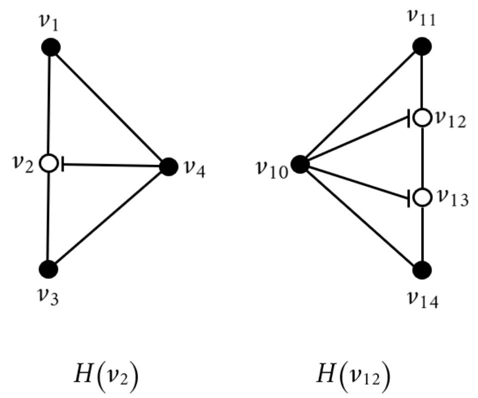

FIGURE 9. Soft Semigraph $G = \{H(v_2), H(v_{12})\}$

Here, in $H(v_2)$ and $H(v_{12})$, any two vertices are adjacent (i.e., $H(v_2)$ and $H(v_{12})$ are complete p-parts of G) and every vertex in $H(v_1)$ and $H(v_{12})$ are end vertex of an f-edge in the corresponding p-part. That is, $H(v_1)$ and $H(v_{12})$ are strongly complete p-parts of

G. So, $G = \{H(v_2), H(v_{12})\}$ is a strongly complete soft semigraph. Also, in $H(v_2)$ there is exactly one f-edge of cardinality 3 and all other f-edges are of cardinality 2 and $H(v_2)$ has totally 4 vertices. Therefore, $H(v_2)$ is C_4^3. Here, $H(v_{12})$ has totally 5 vertices, exactly one f-edge of cardinality 4, and all other f-edges are of cardinality 2. So, $H(v_{12})$ is C_5^4. Hence, $G = \{C_4^3, C_5^4\}$. Therefore, G can be denoted as $K_{4,5}^{3,4}$. Let us verify the above results in this strongly complete soft semigraph.

Here, $m_1 = 4, m_2 = 5, r = 2$. We have the total number of f-edges in $K_{4,5}^{3,4} = \sum_{i=1}^{r} m_i = \sum_{i=1}^{2} m_i = m_1 + m_2 = 4 + 5 = 9$.

Let us find the p-part adjacent degrees. Here, $deg_a v_1[C_4^3] = 3$, $deg_a v_2[C_4^3] = 3$, $deg_a v_3[C_4^3] = 3$, $deg_a v_4[C_4^3] = 3$, $deg_a v_{10}[C_5^4] = 4$, $deg_a v_{11}[C_5^4] = 4$, $deg_a v_{12}[C_5^4] = 4$, $deg_a v_{13}[C_5^4] = 4$, and $deg_a v_{14}[C_5^4] = 4$. Then

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_a v[C_{m_i}^{m_i-1}] = (3+3+3+3) + (4+4+4+4+4) = 12 + 20 = 32.$$

Also,

$$\sum_{i=1}^{r} m_i(m_i - 1) = \sum_{i=1}^{2} m_i(m_i - 1) = 4 \times 3 + 5 \times 4 = 12 + 20 = 32.$$

That is,

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_a v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} m_i(m_i - 1).$$

Also, we can see that $deg_a v = max\{(m_i - 1) : v \in V(C_{m_i}^{m_i-1}), i = 1, 2, \dots, r\}$.

The p-part consecutive adjacent degrees are $deg_{ca} v_1[C_4^3] = 2$, $deg_{ca} v_2[C_4^3] = 3$, $deg_{ca} v_3[C_4^3] = 2$, $deg_{ca} v_4[C_4^3] = 3$, $deg_{ca} v_{10}[C_5^4] = 4$, $deg_{ca} v_{11}[C_5^4] = 2$, $deg_{ca} v_{12}[C_5^4] = 3$, $deg_{ca} v_{13}[C_5^4] = 3$, and $deg_{ca} v_{14}[C_5^4] = 2$. Then

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ca} v[C_{m_i}^{m_i-1}] = (2+3+2+3) + (4+2+3+3+2) = 10 + 14 = 24.$$

Also,

$$\sum_{i=1}^{r} (4m_i - 6) = \sum_{i=1}^{2} (4m_i - 6) = (4m_1 - 6) + (4m_2 - 6) = 10 + 14 = 24.$$

That is,

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ca} v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r} (4m_i - 6).$$

The p-part end degrees are $deg_{ep} v_1[C_4^3] = 2$, $deg_{ep} v_2[C_4^3] = 1$, $deg_{ep} v_3[C_4^3] = 2$, $deg_{ep} v_4[C_4^3] = 3$, $deg_{ep} v_{10}[C_5^4] = 4$, $deg_{ep} v_{11}[C_5^4] = 2$, $deg_{ep} v_{12}[C_5^4] = 1$, $deg_{ep} v_{13}[C_5^4] = 1$, and $deg_{ep} v_{14}[C_5^4] = 2$. Then

$$\sum_{i=1}^{r} \sum_{v \in V(C_{m_i}^{m_i-1})} deg_{ep} v[C_{m_i}^{m_i-1}] = (2+1+2+3) + (4+2+1+1+2) = 8 + 10 = 18.$$

Also,

$$\sum_{i=1}^{r} (2m_i) = \sum_{i=1}^{2} (2m_i) = 2m_1 + 2m_2 = 8 + 10 = 18.$$

That is,

$$\sum_{i=1}^{r}\sum_{v\in V(C_{m_i}^{m_i-1})} deg_{ep}v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r}(2m_i).$$

The p-part edge degrees are $deg_e v_1[C_4^3] = 2$, $deg_e v_2[C_4^3] = 2$, $deg_e v_3[C_4^3] = 2$, $deg_e v_4[C_4^3] = 3$, $deg_e v_{10}[C_5^4] = 4$, $deg_e v_{11}[C_5^4] = 2$, $deg_e v_{12}[C_5^4] = 2$, $deg_e v_{13}[C_5^4] = 2$, *and* $deg_e v_{14}[C_5^4] = 2$. *Then*

$$\sum_{i=1}^{r}\sum_{v\in V(C_{m_i}^{m_i-1})} deg_e v[C_{m_i}^{m_i-1}] = (2+2+2+3)+(4+2+2+2+2) = 9+12 = 21.$$

Also,

$$\sum_{i=1}^{r}(3m_i-3) = \sum_{i=1}^{2}(3m_i-3) = (3m_1-3)+(3m_2-3) = 9+12 = 21.$$

That is,

$$\sum_{i=1}^{r}\sum_{v\in V(C_{m_i}^{m_i-1})} deg_e v[C_{m_i}^{m_i-1}] = \sum_{i=1}^{r}(3m_i-3).$$

Theorem 4.7. *Let* $G^* = (V, X)$ *be a semigraph and* $G = (G^*, Q, W, A)$ *be a soft semigraph of* G^* *given by* $\{H(x) : x \in A\}$. *Then* G *is a complete soft semigraph if and only if its p-part adjacency graph* $H(x)_{ca}$ *is complete for all* $x \in A$.

Proof. Suppose that $G = \{H(x) : x \in A\}$ is a complete soft semigraph. Then $H(x)$ will a complete p-part of G for all x in A. That is, any two vertices in $H(x)$ are adjacent for all x in A. Then, if we draw $H(x)_a$, there will be an edge connecting any two vertices in $H(x)$ for all x in A. That is, $H(x)_a$ is a complete graph for all x in A.

Conversely, assume that p-part adjacency graph $H(x)_a$ is complete for all x in A. That is, any two vertices in $H(x)_a$ are connected by an edge for all x in A. Therefore, every vertex in $H(x)$ is adjacent, for all x in A. That is $H(x)$ is a complete p-part for all x in A. Hence, $G = \{H(x) : x \in A\}$ is a complete soft semigraph. ☐

5. Conclusion

This paper has introduced and explored the concepts of complete and strongly complete soft semigraphs, contributing significantly to the field of soft set theory and semigraph theory. By defining these structures and analyzing their properties, particularly in the form $K_{m_1,m_2,\ldots,m_r}^{m_1-1,m_2-1,\ldots,m_r-1}$ we have provided a deeper understanding of their mathematical underpinnings. The detailed examination of various degrees and the total number of f-edges has elucidated the intricate nature of strongly complete soft semigraphs. These findings enhance the theoretical framework of soft semigraphs, offering new avenues for research and potential applications in areas requiring parameter management and uncertainty handling. The results presented in this paper lay a robust foundation for further investigations into the complexities of soft semigraphs.

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