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CHARACTERIZATIONS OF FUZZY STANDARD ELEMENT IN FUZZY LATTICES.

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Abstract. In this paper, we introduce the concepts of fuzzy neutral elements, fuzzy join-standard, and fuzzy meet-standard elements in fuzzy lattices. Also, we have proved the fundamental characterization theorem of fuzzy join-standard elements in fuzzy lattices. Moreover, we introduce the concepts of ∇_F-standard elements in fuzzy lattices. Further, we have studied the relationship among the fuzzy join-standard, ∇_F-standard, fuzzy meet-distributive, and fuzzy atom in fuzzy lattices. Also, we have introduced the notion and notation of fuzzy p-element in the fuzzy lattice.

Keywords: Fuzzy lattices, fuzzy distributive pair, fuzzy semi-distributive pair, fuzzy standard element, ∇_F-standard element, fuzzy p-element

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1. Introduction

Distributive lattice plays a major role in the theory of the lattices, as these lattices have important properties that lattices in general do not have. This fact gives the reason why some mathematicians tried to define types of elements that preserve some of the properties of distributive lattices. In 1936, Ore [13] introduced the concept of neutral elements in modular lattices and obtained their characterization in [14]. In 1940, Birkhoff [3] extended this concept to general lattices. In 1961, Grätzer and Schmidt [4] generalized the concept of neutral element to standard element. Thakare, Pawar and Waphare [17] introduced the concepts of neutral elements and standard elements in lattices.

In 1971, Zadeh [19] defined a fuzzy binary relation and a fuzzy partial order relation. Ajmal and Thomas [1] and Chon [2] defined fuzzy lattices. Recently, Wasadikar and Khubchandani [11] defined fuzzy distributive pairs, fuzzy semi-distributive pairs in fuzzy lattices. Also, Wasadikar and Khubchandani [10] defined the ∇_F relation in fuzzy lattices. Khubchandani and Khubchandani [15] defined fuzzy perspective and fuzzy subperspective in fuzzy lattices.

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In section 3, we have introduced the concepts of fuzzy join-standard (fuzzy meet-standard), fuzzy join-distributive (fuzzy meet-distributive) and fuzzy neutral element in fuzzy lattice and study relationship among them. Also, we have proved the fundamental characterization theorem of fuzzy join-standard element in fuzzy lattices.

In section 4, we introduce the concept of ∇_F-standard element in fuzzy lattices. We studied the relationship among the fuzzy join-standard, ∇_F-standard, fuzzy meet-distributive and fuzzy atom in fuzzy lattice. Moreover, we have introduced the notion and notation of fuzzy p-element in fuzzy lattice.

2. Preliminaries

Throughout in this paper, $(X,\ A)$ denotes a fuzzy lattice, where A is a fuzzy partial order relation on a non empty set X. We recall some concepts.

Definition 2.1. *(Chon* [2, Definition 2.1]*) A mapping $A : X \times X \to [0, 1]$ is called a fuzzy binary relation on X.*

For the definitions of a fuzzy partial order relation, fuzzy equivalence relation, fuzzy supremum, fuzzy infimum, fuzzy lattice etc. we refer to Chon [2]. We use the notations $a \vee_F b$ and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a, b \in X$ to distinguish the supremum and infimum of a, b in the lattice sense, if these exist in X.

We recall some known results from Chon [2] which we shall use in this paper.

Definition 2.2. *(Chon* [2, Definition 3.2]*) Let (X, A) be a fuzzy poset. Then, (X, A) is called a fuzzy lattice if and only if $a \vee_F b$ and $a \wedge_F b$ exist, for all $a, b \in X$.*

Definition 2.3. [6, Definition 3.4] *A fuzzy lattice $\mathcal{L} = (X, A)$ is bounded if there exist elements $\perp$ and $\top$ in X, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for all $a \in X$. In this case, $\perp$ and $\top$ are called bottom and top elements, respectively.*

We illustrate these concepts in the following example.

Example 2.1. *Let $X = \{\perp, a, b, c, x, y, s, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\perp,\perp) = A(a, a) = A(b, b) = A(c, c) = A(x, x) = A(y, y) = A(s, s) = A(\top,\top) = 1,$
$A(\perp, a) = 0.3,\ A(\perp, b) = 0.3,\ A(\perp, c) = 0.3,\ A(\perp, x) = 0.3,\ A(\perp, y) = 0.3,\ A(\perp, s) = 0.3,$
$A(\perp,\top) = 0.3,$
$A(a,\perp) = 0,\ A(a, b) = 0,\ A(a, c) = 0,\ A(a, x) = 0.5,\ A(a, y) = 0.5,\ A(a, s) = 0,$
$A(a,\top) = 0.03,$
$A(b,\perp) = 0,\ A(b, a) = 0,\ A(b, c) = 0,\ A(b, x) = 0.5,\ A(b, y) = 0,\ A(b, s) = 0,$
$A(b,\top) = 0.03,$
$A(c,\perp) = 0,\ A(c, a) = 0,\ A(c, b) = 0,\ A(c, x) = 0,\ A(c, y) = 0,\ A(c, s) = 0.5,$
$A(c,\top) = 0.03,$
$A(x,\perp) = 0,\ A(x, a) = 0,\ A(x, b) = 0,\ A(x, c) = 0,\ A(x, y) = 0,\ A(x, s) = 0,$
$A(x,\top) = 0.03,$
$A(y,\perp) = 0,\ A(y, a) = 0,\ A(y, b) = 0,\ A(y, c) = 0,\ A(y, x) = 0,\ A(y, s) = 0,$
$A(y,\top) = 0.03,$
$A(s,\perp) = 0,\ A(s, a) = 0,\ A(s, b) = 0,\ A(s, c) = 0,\ A(s, x) = 0,\ A(s, y) = 0,$
$A(s,\top) = 0.03,$
$A(\top,\perp) = 0,\ A(\top, a) = 0,\ A(\top, b) = 0,\ A(\top, c) = 0,\ A(\top, x) = 0,\ A(\top, y) = 0,$
$A(\top, s) = 0.$

Then A is a fuzzy partial order relation.
We note that (X, A) is a fuzzy lattice.

This fuzzy relation is summarized in the following table:

A	$\bot$	a	b	c	x	y	s	$\top$
$\bot$	1.0	0.3	0.3	0.3	0.3	0.3	0.3	0.3
a	0.0	1.0	0.0	0.0	0.5	0.5	0.0	0.03
b	0.0	0.0	1.0	0.0	0.5	0.0	0.0	0.03
c	0.0	0.0	0.0	1.0	0.0	0.0	0.5	0.03
x	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.03
y	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.03
s	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.03
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	x	y	s	$\top$
$\bot$	$\bot$	a	b	c	x	y	s	$\top$
a	a	a	x	$\top$	x	y	$\top$	$\top$
b	b	x	b	$\top$	x	$\top$	$\top$	$\top$
c	c	$\top$	$\top$	c	$\top$	$\top$	s	$\top$
x	x	x	x	$\top$	x	$\top$	$\top$	$\top$
y	y	y	$\top$	$\top$	$\top$	y	$\top$	$\top$
s	s	$\top$	$\top$	s	$\top$	$\top$	s	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	x	y	s	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a	$\bot$	$\bot$	a
b	$\bot$	$\bot$	b	$\bot$	b	$\bot$	$\bot$	b
c	$\bot$	$\bot$	$\bot$	c	$\bot$	$\bot$	c	c
x	$\bot$	a	b	$\bot$	x	a	$\bot$	x
y	$\bot$	$\bot$	$\bot$	$\bot$	a	y	$\bot$	y
s	$\bot$	$\bot$	$\bot$	c	$\bot$	$\bot$	s	s
$\top$	$\bot$	a	b	c	x	y	s	$\top$

Proposition 2.1. [2, Proposition 3.3] *and* [5, Proposition 2.4] *Let (X, A) be a fuzzy lattice. For $a, b, c \in X$. The following statements hold:*
i) $A(a, b) > 0$ iff $a \vee_F b = b$.
ii) $A(a, b) > 0$ iff $a \wedge_F b = a$.

Proposition 2.2. [2, Proposition 3.4] *Let (X, A) be a fuzzy lattice and let $a, b, c \in X$. Then following statements hold:*
i) $a \vee_F a = a$, $a \wedge_F a = a$.
ii) $a \vee_F b = b \vee_F a$, $a \wedge_F b = b \wedge_F a$.
iii) $(a \vee_F b) \vee_F c = a \vee_F (b \vee_F c)$, $(a \wedge_F b) \wedge_F c = a \wedge_F (b \wedge_F c)$.
iv) $(a \vee_F b) \wedge_F a = a$, $(a \wedge_F b) \vee_F a = a$.

Definition 2.4. *(Chon* [2]*) Let (X, A) be a fuzzy lattice. (X, A) is called a fuzzy distributive lattice, if $a \wedge_F (b \vee_F c) = (a \wedge_F b) \vee_F (a \wedge_F c)$ and*
$a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F (a \vee_F c)$ for all $a, b, c \in X$.

Definition 2.5. *(Wasadikar and Khubchandani* [8, Definition 4.4]*) Let $\mathcal{L} = (X, A)$ be a fuzzy lattice. Let $a, b \in X$. Then $b \prec_F a$ (a "fuzzy covers" b) if $0 < A(b, a) < 1$, $A(b, c) > 0$ and $A(c, a) > 0$ imply $c = b$ or $c = a$.*

Definition 2.6. *(Wasadikar and Khubchandani* [8, Definition 3.3]*) Let P denote the set of all $a \in X$ such that $\bot \prec_F a$. The elements of P are called fuzzy atoms.*

Definition 2.7. [9, Definition 3.2] *A fuzzy lattice (X, A) is called FM-symmetric if in (X, A), $(a, b)_F M_m$ implies $(b, a)_F M_m$.*

Definition 2.8. [10, Definition 3.1] *Let (X, A) be a fuzzy lattice. Let $a, b, c \in X$. We write $(a, b, c)_F D_j$ if $(a \vee_F b) \wedge_F c = (a \wedge_F c) \vee_F (b \wedge_F c)$ (I)*
and we write $(a, b, c)_F D_m$ if $(a \wedge_F b) \vee_F c = (a \vee_F c) \wedge_F (b \vee_F c)$. (II)
If (I) and (II) hold for all permutations of a, b and c, then we say that $\{a, b, c\}$ is a fuzzy distributive triplet and we write $(a, b, c)_F T$.

(X, A) is called fuzzy distributive when $(a, b, c)_F D_j$ and $(a, b, c)_F D_m$ hold for all elements $a, b, c \in X$.

Theorem 2.1. [11, Theorem 8] *Let $a, b \in X$. The following conditions are equivalent:*
(i) (a, b) is a fuzzy join-semidistributive pair.
(ii) $A(\{(a \vee_F b) \wedge_F c\} \vee_F b, (a \wedge_F c) \vee_F b) > 0$ for every $c \in X$.
(iii) $\{(a \vee_F b) \wedge_F c\} \vee_F b = (a \wedge_F c) \vee_F b$ for every $c \in X$.

Lemma 2.1. [11, Lemma 10] *For a pair (a, b) of elements in X, consider the following statements:*

(i) *(a, b) is a fuzzy join-distributive pair.*
(ii) *(a, b) is a fuzzy join-semidistributive pair.*
(iii) *(a, b) is a fuzzy join-modular pair.*

Then the following implications hold:
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Corollary 2.1. [11, Corollary 3] *Let (X, A) be a fuzzy lattice and $a, b, c, d \in X$. If $A(c, a) > 0$ and $A(d, b) > 0$, then $A(c \wedge_F d, a \wedge_F b) > 0$ and $A(c \vee_F d, a \vee_F b) > 0$.*

Definition 2.9. [12, Definition 5.2] *A fuzzy poset $\mathcal{L} = (X, A)$ with the least element $\perp$ is called fuzzy atomistic if every element $x \in X$ is the least upper bound of the set of fuzzy atoms less than or equal to x.*

Definition 2.10. [15, Definition 3.1]*) A fuzzy lattice (X, A) is called fuzzy section semi-complemented lattice ($FSSC$) if it satisfies the following condition:*
If $a \neq b$ in X, then there exists $c \in X$ such that $c \neq \perp$, $A(c, a) > 0$ and $c \wedge_F b = \perp$.

Definition 2.11. [15, Definition 3.3] *Let (X, A) be a fuzzy lattice with $\perp$ and $a, b \in X$. Then a and b are fuzzy perspective and written as $a \sim_x^F b$ (or simply $a \sim^F b$) if the following holds:*
$a \vee_F x = b \vee_F x$ and $a \wedge_F x = b \wedge_F x = \perp$ for some $x \in X$.

Lemma 2.2. [15, Lemma 3.8] *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$ and $a \in X$. An fuzzy atom $p \in X$ is fuzzy subperspective to a if and only if $a \nabla_F p$ does not hold.*

Definition 2.12. [16, Definition 3.3] *Let (X, A) be a fuzzy lattice. A pair of elements $a, b \in X$ is said to be fuzzy meet-modular pair, denoted by $(a, b)_F M_m$, if the following condition holds:*
$(a, b)_F M_m : \{a \vee_F (b \wedge_F c)\} \wedge_F b = (a \wedge_F b) \vee_F (b \wedge_F c)$ for every $c \in X$.

3. Fuzzy standard element in Fuzzy Lattices

In [11], Wasadikar and Khubchandani have proved results related to fuzzy join-distributive pairs and fuzzy meet-distributive pairs and this motivated us to introduce and study fuzzy join-standard, fuzzy meet-standard and fuzzy neutral elements in fuzzy lattices.

Definition 3.1. *An element $s \in X$ of fuzzy lattice (X, A) is said to be fuzzy join-standard if $(s, x, y)_F D_j$ for all $x, y \in X$, i.e., $\{(s \vee_F x) \wedge_F y\} = \{(s \wedge_F y) \vee_F (x \wedge_F y)\}$ for all $x, y \in X$.*

Dually, we can define the concept of a fuzzy meet-standard element.
It is also equivalent to say that $(s, x)_F D_j$ or $((x, s)_F D_j)$ for all $x \in X$.

Definition 3.2. *An element $s \in X$ of fuzzy lattice (X, A) is said to be fuzzy meet-distributive if $(x, y, s)_F D_m$ for all $x, y \in X$, i.e., $(x \wedge_F y) \vee_F s = (s \vee_F x) \wedge_F (s \vee_F y)$ for all $x, y \in X$.*

Dually, we can define the concept of a fuzzy join-distributive element.

Definition 3.3. *An element $s \in X$ of fuzzy lattice (X, A) is called fuzzy neutral if $(s, x, y)_F T$ for all $x, y \in X$.*

An element $s \in X$ is said to be fuzzy right meet-modular if $(x, s)_F M_m$ holds for every $x \in X$ and said to be fuzzy left meet-modular if $(s, x)_F M_m$ holds for every $x \in X$.

We define the concept of fuzzy left join-semidistributive element and fuzzy right join-semidistributive element in fuzzy lattice (X, A) as follows:
An element $s \in X$ is said to be fuzzy right join-semidistributive if $(x, s)_F SD_j$ for every $x \in X$ and is said to be fuzzy left join-semidistributive if $(s, x)_F SD_j$ for every $x \in X$.

Remark 3.1. *From Lemma 2.1, it follows that if $s \in X$ is a fuzzy join-standard element then $(s, x)_F M_j$, $(x, s)_F M_j$ and $(s, x)_F M_m$ hold for every $x \in X$.*

Theorem 3.1. *An element $s \in X$ is fuzzy join-standard iff it is fuzzy left meet-modular and fuzzy right join-distributive.*

Proof. Suppose $s \in X$ is fuzzy join-standard element. From Lemma 2.1, it is clear that s is fuzzy right join-semidistributive and also $(x, s)_F M_j$ holds for every $x \in X$. As s is fuzzy right join-semidistributive, we have

$$\{(x \wedge_F y) \vee_F s\} = \{\{(x \vee_F s) \wedge_F y\} \vee_F s\} \tag{3.1}$$

for $x, y \in X$.
By $(y, s)_F M_j$ we get

$$\{\{y \wedge_F (s \vee_F x)\} \vee_F s\} = \{(x \vee_F s) \wedge_F (y \vee_F s)\}. \tag{3.2}$$

From (3.1) and (3.2) we get

$$\{(x \wedge_F y) \vee_F s\} = \{(x \vee_F s) \wedge_F (y \vee_F s)\}. \tag{3.3}$$

As s is fuzzy join-standard we have

$$(s \vee_F y) \wedge_F x = (s \wedge_F x) \vee_F (y \wedge_F x). \tag{3.4}$$

Taking meet x on both sides of equation (3.3) and then using fuzzy standardness of s, we have

$$\begin{aligned}\{\{(x \wedge_F y) \vee_F s\} \wedge_F x\} &= x \wedge_F (x \vee_F s) \wedge_F (y \wedge_F s) \\ &= \{x \wedge_F (y \vee_F s)\}, \text{ by absorption} \\ &= \{(s \wedge_F x) \vee_F (y \wedge_F x)\}, \text{by (3.4)}\end{aligned}$$

Therefore, $\{\{(x \wedge_F y) \vee_F s\} \wedge_F x\} = \{(s \wedge_F x) \vee_F (y \wedge_F x)\}$.
Thus, $(s, x)_F M_m$ holds i.e., (s, x) is fuzzy left meet-modular.
Conversely, suppose that s is fuzzy left meet-modular and fuzzy right join-semidistributive element. For $x \in X$, by using fuzzy right join-semidistributivity of s, we get

$$\{\{(x \vee_F s) \wedge_F y\} \vee_F s\} = \{(x \wedge_F y) \vee_F s\}. \tag{3.5}$$

Since

$$A(\{(s \vee_F x) \wedge_F y\}, \{\{(x \vee_F s) \wedge_F y\} \vee_F s\}) > 0. \tag{3.6}$$

Putting (3.5) in (3.6) we get

$$A(\{(s \vee_F x) \wedge_F y\}, (x \wedge_F y) \vee_F s) > 0.$$

Taking meet y on both sides we get $A(\{(s \vee_F x) \wedge_F y\} \wedge_F y, \{s \vee_F (x \wedge_F y)\} \wedge_F y) > 0$.
By using absorption property we get

$$A(\{(s \vee_F x) \wedge_F y\}, \{s \vee_F (x \wedge_F y)\} \wedge_F y) > 0. \tag{3.7}$$

As s is fuzzy left meet-modular, we get

$$\{\{s \vee_F (x \wedge_F y)\} \wedge_F y\} = \{(s \wedge_F y) \vee (x \wedge_F y)\}. \tag{3.8}$$

Using equation (3.8) in (3.7) we get

$$A((s \vee_F x) \wedge_F y, \{(s \wedge_F y) \vee_F (x \wedge_F y)\}) > 0. \tag{3.9}$$

As

$$A(\{(s \wedge_F y) \vee_F (x \wedge_F y)\}, (s \vee_F x) \wedge_F y) > 0 \tag{3.10}$$

always holds.
From (3.9) and (3.10) by fuzzy antisymmetry of A, we get

$$(s \wedge_F y) \vee_F (x \wedge_F y) = (s \vee_F x) \wedge_F y.$$

Hence s is fuzzy join-standard element. □

Definition 3.4. *An element $s \in X$ is called fuzzy separating if for $x, y \in X$, $s \wedge_F x = s \wedge_F y$ and $s \vee_F x = s \vee_F y$ imply $x = y$.*
An element s is called fuzzy strong separating if for $x, y \in X$, $A(s \wedge_F x, s \wedge_F y) > 0$ and $A(s \vee_F x, s \vee_F y) > 0$ imply $A(x, y) > 0$.

Theorem 3.2. *Let (X, A) be a fuzzy lattice and $s \in X$. Consider the following conditions:*
α) s is fuzzy join-standard.
β) The equality $v = (s \wedge_F v) \vee_F (v \wedge_F t)$ holds whenever $A(v, s \vee_F t) > 0$, $(v, t \in X)$.
γ) s is fuzzy meet-distributive and fuzzy separating.
Then the following implications hold:
$(\alpha) \Leftrightarrow (\beta) \Rightarrow (\gamma)$.

Proof. $(\alpha) \Leftrightarrow (\beta)$ Easy to prove.
$(\alpha) \Rightarrow (\gamma)$ Suppose s is a fuzzy join-standard element and $A(s \wedge_F y, s) > 0$.
By (i) of Proposition 2.1 we have

$$(s \wedge_F y) \vee_F s = s. \tag{3.11}$$

We have

$$\begin{aligned} s \vee_F (x \wedge_F y) &= (s \wedge_F y) \vee_F s \vee_F (x \wedge_F y), \text{ by putting (3.11)} \\ &= (s \wedge_F y) \vee_F (x \wedge_F y) \vee_F s, \\ &= \{(s \wedge_F y) \vee_F (x \wedge_F y)\} \vee_F s, \\ &= \{(s \vee_F x) \wedge_F y\} \vee_F s, \text{ as } s \text{ is fuzzy join-standard} \end{aligned}$$

From Remark (3.1), it is clear that $(y, s)_F M_j$ holds.
Therefore

$$\begin{aligned} s \vee_F (x \wedge_F y) &= \{\{y \wedge_F (s \vee_F x)\} \vee_F s\} \\ &= \{(s \vee_F x) \wedge_F (s \vee_F y)\}, \text{ by } (y, s)_F M_j \end{aligned}$$

i.e., $s \vee_F (x \wedge_F y) = (s \vee_F x) \wedge_F (s \vee_F y)$.
Hence s is fuzzy meet-distributive.
Now, to prove s is fuzzy separating.
Suppose

$$s \vee_F x = s \vee_F y \tag{3.12}$$

and

$$s \wedge_F x = s \wedge_F y \tag{3.13}$$

hold for $x, y \in X$.
As $A(x, s \vee_F x) > 0$ by (ii) of Proposition 2.1 we have $x = (s \vee_F x) \wedge_F x$.

$$\begin{aligned} x &= (s \vee_F x) \wedge_F x, \\ &= (s \vee_F y) \wedge_F x, \text{ by using (3.12)} \\ &= (s \wedge_F x) \vee_F (x \wedge_F y), \text{ as } s \text{ fuzzy meet-distributive} \\ &= (s \wedge_F y) \vee_F (x \wedge_F y), \text{ by using (3.13)} \\ &= (s \vee_F x) \wedge_F y, \text{ as } s \text{ fuzzy meet-distributive} \\ &= (s \vee_F y) \wedge_F y, \text{ by using (3.12)} \\ &= y, \text{ by absorption property} \end{aligned}$$

Therefore $x = y$.
Hence s is fuzzy meet distributive and fuzzy separating. □

We provide an example to show that the implication $(\alpha) \Rightarrow (\gamma)$ in Theorem 3.2 is not reversible i.e., an element may be fuzzy meet-distributive and fuzzy separating but not fuzzy join-standard.

Example 3.1. *Consider the fuzzy lattice in Example 2.1.*
As $s \vee_F (x \wedge_F y) = (s \vee_F x) \wedge_F (s \vee_F y) = \top$ *i.e.,* s *is fuzzy meet-distributive and fuzzy separating but not fuzzy join-standard as* $(s \vee_F x) \wedge_F y = y$ *and* $(s \wedge_F y) \vee_F (x \wedge_F y) = a \neq y$.

Theorem 3.3 (The fundamental characterization theorem of fuzzy join-standard elements)**.** *Let* (X, A) *be a fuzzy lattice and* $s \in X$*. Then the following conditions are equivalent:*
α) *s is fuzzy join-standard.*
β) *The equality* $v = (s \wedge_F v) \vee_F (v \wedge_F t)$ *holds whenever* $A(v, s \vee_F t) > 0$, $(v, t \in X)$.
γ) *s is fuzzy meet-distributive and fuzzy strong separating.*

Proof. $(\alpha) \Leftrightarrow (\beta)$ Follows from Theorem (3.2).
$(\beta) \Rightarrow (\gamma)$ Follows from Theorem (3.2).
We prove s is fuzzy strong separating element.
Suppose $x, y \in X$ such that $A(s \wedge_F x, s \wedge_F y) > 0$ and $A(s \vee_F x, s \vee_F y) > 0$.
As s is fuzzy join-standard. we have

$$\begin{aligned} x &= (s \vee_F x) \wedge_F x, \text{ by absorption property} \\ &= (s \vee_F y) \wedge_F x, \\ &= (s \wedge_F x) \vee_F (y \wedge_F x) \end{aligned}$$

i.e.,

$$x = (s \wedge_F x) \vee_F (y \wedge_F x). \tag{3.14}$$

As $A(s \wedge_F x, s \wedge_F y) > 0$ taking meet $(y \wedge_F x)$ on both sides we get

$$A((s \wedge_F x) \vee_F (y \wedge_F x), (s \wedge_F y) \vee (y \wedge_F x)) > 0. \tag{3.15}$$

By using fuzzy meet-distributivity of s, equation (3.15) reduces to

$$A((s \wedge_F x) \vee (y \wedge_F x), (s \vee_F x) \wedge_F y) > 0. \tag{3.16}$$

As $A(s \vee_F x, s \vee_F y) > 0$ taking meet y on both sides we get

$$A((s \vee_F x) \wedge_F y, (s \vee_F y) \wedge_F y) > 0. \tag{3.17}$$

From (3.16) and (3.17) by fuzzy transitivity of A, we get

$$A((s \wedge_F x) \vee_F (y \wedge_F x), (s \vee_F y) \wedge_F y) > 0. \tag{3.18}$$

Using equation (3.14) and fuzzy absorption property, equation (3.18) reduces to $A(x, y) > 0$.

$(\gamma) \Rightarrow (\alpha)$
Let $a, b \in X$, $x = (s \vee_F a) \wedge_F b$ and $y = (s \wedge_F b) \vee_F (a \wedge_F b)$.
By (i) and (ii) of Proposition 2.1 we have $A(x, s \vee_F a) > 0$, $A(x, b) > 0$, $A(s \wedge_F b, y) > 0$ and $A(a \wedge_F b, y) > 0$.
To prove s is fuzzy join-standard, it is sufficient to show that $A(x, y) > 0$.
As $A(x, b) > 0$ taking meet s on both sides we have

$$A(s \wedge_F x, s \wedge_F b) > 0. \tag{3.19}$$

Also, $A(s \wedge_F b, y) > 0$ taking meet s on both sides

$$A(s \wedge_F b, s \wedge_F y) > 0. \tag{3.20}$$

From (3.19) and (3.20) by fuzzy transitivity of A, we have $A(s \wedge_F x, s \wedge_F y) > 0$.
Consider

$$\begin{aligned} s \vee_F x &= s \vee_F \{(s \vee_F a) \wedge_F b\}, \text{ as } x = (s \vee_F a) \wedge_F b \\ &= s \vee_F \{(s \wedge_F b) \vee_F (a \wedge_F b)\}, \text{ as } s \text{ is fuzzy meet-distributive,} \\ &= s \vee_F (s \wedge_F b) \vee_F (a \wedge_F b), \\ &= s \vee_F (a \wedge_F b), \text{ by absorption property} \end{aligned}$$

Therefore we get

$$s \vee_F x = s \vee_F (a \wedge_F b). \tag{3.21}$$

As $A(a \wedge_F b, y) > 0$ taking join s on both sides we have

$$A(s \vee_F (a \wedge_F b), s \vee_F y) > 0. \tag{3.22}$$

Using (3.21) equation (3.22) reduces to

$$A(s \vee_F x, s \vee_F y) > 0. \tag{3.23}$$

Thus $A(s \vee_F x, s \vee_F y) > 0$.
As s is fuzzy strong separating element we have $A(x, y) > 0$. □

Corollary 3.1. *Let (X, A) be a fuzzy lattice and $s \in X$. Then following statements are equivalent:*
(α) *s is fuzzy neutral.*
(β) *s is fuzzy meet-distributive, fuzzy join-distributive and fuzzy strong separating.*
(γ) *s is fuzzy meet-standard and fuzzy join-standard (fuzzy meet-distributive).*

Lemma 3.1. *Every fuzzy strong separating element is fuzzy left meet-modular.*

Proof. Let s be a fuzzy strong separating element and $x \in X$ be an arbitrary element.
Let $v = \{s \vee_F (x \wedge_F y)\} \wedge x$ and $w = (s \wedge_F x) \vee (x \wedge_F y)$.
To show $(s, x)_F M_m$ it is enough to show that $A(v, w) > 0$.
The inequality $A(v, w) > 0$ follows by fuzzy strong separating of s,
if we prove $A(s \wedge_F v, s \wedge_F w) > 0$ and $A(s \vee_F w, s \vee_F v) > 0$ then we are through.
The inclusion $A(s \wedge_F v, s \wedge_F w) > 0$ follows from the fact that $A(v, x) > 0$ and $A(s \wedge_F x, w) > 0$. The other inclusion follows from the fact that $A(v, s \vee_F (x \wedge_F y)) > 0$ and $A(x \wedge_F y, w) > 0$. □

Lemma 3.2. *Every fuzzy join-distributive element is fuzzy right meet-modular.*

Proof. Let (X, A) be a fuzzy lattice. s be fuzzy join-distributive and $x \in X$.
Since for $y \in X$ we have $A(s \wedge_F y, y) > 0$.
Taking join x on both sides we get $A(x \vee_F (s \wedge_F y), x \vee_F y) > 0$.
Again taking meet s on both sides we get

$$A(\{\{x \vee_F (s \wedge_F y)\} \wedge_F s\}, (x \vee_F y) \wedge_F s) > 0. \tag{3.24}$$

Since s is fuzzy join-distributive we have

$$(x \vee_F y) \wedge_F s = (x \wedge_F s) \vee_F (s \wedge_F y). \tag{3.25}$$

Putting (3.25) in (3.24) we get

$$A(\{\{x \vee_F (s \wedge_F y)\} \wedge_F s\}, (x \wedge_F s) \vee (s \wedge_F y)) > 0. \tag{3.26}$$

As

$$A((x \wedge_F s) \vee_F (s \wedge_F y), \{x \vee_F (s \wedge_F y)\} \wedge_F s) > 0 \tag{3.27}$$

always holds.
From (3.26) and (3.27) fuzzy antisymmetry of A, we get
$\{x \vee_F (s \wedge_F y)\} \wedge_F s = (x \wedge_F s) \vee_F (s \wedge_F y)$.
Hence s is fuzzy right meet-modular. □

Lemma 3.3. *Every fuzzy meet-distributive, fuzzy left meet-modular element is fuzzy strong separating.*

Proof. Let $s \in X$ be a fuzzy left meet-modular and fuzzy meet-distributive element of a fuzzy lattice (X, A). Let $x, y \in X$ such that
$A(s \wedge_F x, s \wedge_F y) > 0$ and $A(s \vee_F x, s \vee_F y) > 0$.
As we have $A(s \wedge_F x, s \wedge_F y) > 0$ taking join $(x \wedge_F y)$ on both sides we get
$A((s \wedge_F x) \vee_F (x \wedge_F y), (s \wedge_F y) \vee_F (x \wedge_F y)) > 0$.
As $(s, x)_F M_m$ holds we have $A(\{s \vee_F (x \wedge_F y)\} \wedge_F x), (s \wedge_F y) \vee_F (x \wedge_F y)) > 0$.
Taking join y on both sides we get
$A(y \vee_F \{\{s \vee_F (x \wedge_F y)\} \wedge_F x\}, y \vee_F (s \wedge_F y) \vee_F (x \wedge_F y)) > 0$.
By fuzzy absorption property we have $y \vee_F (x \wedge_F y) = y$ so, the above equation reduces to $A(y \vee_F \{\{s \vee_F (x \wedge_F y)\} \wedge_F x\}, y \vee_F (s \wedge_F y)) > 0$.
As s is fuzzy meet-distributive we get

$$A(y \vee_F \{\{(s \vee_F x) \wedge_F (s \vee_F y)\} \wedge_F x\}, y \vee_F (s \wedge_F y)) > 0. \tag{3.28}$$

As $A(s \vee_F x, s \vee_F y) > 0$ by (ii) of Proposition 2.1 we get $(s \vee_F x) \wedge_F (s \vee_F y) = (s \vee_F x)$.
Therefore equation (3.28) reduces to

$$A(y \vee_F \{(s \vee_F x) \wedge_F x\}, y \vee_F (s \wedge_F y)) > 0. \tag{3.29}$$

As $(s \vee_F x) \wedge_F x = x$ and $y \vee_F (s \wedge_F y) = y$, so equation (3.29) reduces to

$$A(y \vee_F x, y) > 0. \tag{3.30}$$

As

$$A(y, y \vee_F x) > 0 \tag{3.31}$$

always holds.
Therefore from equation (3.30) and (3.31) by fuzzy antisymmetry of A, we get $y \vee_F x = y$.
So by (i) of Proposition 2.1 we get $A(x, y) > 0$. □

Theorem 3.4. *Let (X, A) be a fuzzy lattice. For an element $s \in X$, the following conditions are equivalent:*
(α) s is fuzzy join-standard.

(β) s is fuzzy meet-distributive and fuzzy strong separating.
(γ) s is fuzzy meet-distributive and fuzzy left meet-modular.

Proof. (α) $\Leftrightarrow$ (β) Follows from Theorem 3.3.
(α) $\Rightarrow$ (γ) Follows from Theorem 3.1 and Theorem 3.3.
(γ) $\Rightarrow$ (β) Follows from Lemma 3.3. □

Corollary 3.2. *An element s of a fuzzy modular lattice (X, A) is fuzzy standard iff it is fuzzy meet-distributive.*

Theorem 3.5. *Let (X, A) be FM-symmetric lattice. An element $s \in X$ is fuzzy neutral iff it is both fuzzy meet-distributive and fuzzy join-distributive.*

Proof. Suppose $s \in X$ is fuzzy meet-distributive as well as fuzzy join-distributive. By Lemma 3.2, s is fuzzy right meet-modular using FM-symmetric, we get s is fuzzy left meet-modular. Hence it follows from Lemma 3.3 that s is fuzzy strong separating. By Theorem 3.3, s is fuzzy join-standard. Thus, by Corollary 3.1 s is fuzzy neutral.
The converse is trivial. □

4. ∇_F- Standard Elements

In this section, we have defined ∇_F-standard element in fuzzy lattice. Also, we have introduced the notion and notation of fuzzy p-element in fuzzy lattice.

Definition 4.1. *Let (X, A) be a fuzzy lattice with $\perp$. An element $s \in X$ is called ∇_F-standard element if, for $a \in X$, $s \wedge_F a = \perp_F$ implies $s\nabla_F a$ holds.*

(Here, by $s\nabla_F a$, we mean $(s \vee_F a) \wedge_F x = x \wedge_F a$ for every $x \in X$.)
More details about ∇_F relation can be found in P. Khubchandani and J. Khubchandani [15]

Lemma 4.1. *In a fuzzy lattice (X, A) with $\perp$, every fuzzy left join-semidistributive element is ∇_F-standard.*

Proof. Let $s \in X$ be a fuzzy left join-semidistributive element with $s \wedge_F a = \perp$, $a \in X$ and let $y = (s \vee_F x) \wedge_F a$ for $x \in X$.
To prove s is ∇_F-standard, it is sufficient to show $A(y, x) > 0$.
Clearly

$$A((s \vee_F x) \wedge_F a, \{(s \vee_F x) \wedge_F a\} \vee_F x) > 0. \tag{4.1}$$

By fuzzy left join-semidistributive of s, we get

$$\{(s \vee_F x) \wedge_F a\} \vee_F x = \{(s \wedge_F a) \vee_F x\}. \tag{4.2}$$

But $s \wedge_F a = \perp$. Therefore equation (4.2) reduces to $\{(s \vee_F x) \wedge_F a\} \vee x = x$.
Equation (4.1) reduces to

$$A((s \vee_F x) \wedge_F a, x) > 0. \tag{4.3}$$

As $y = (s \vee_F x) \wedge_F a$, equation (4.3) reduces to $A(y, x) > 0$.
Hence the proof. □

Remark 4.1. *From Lemma 2.1 and Lemma 4.1, we get that every fuzzy join-standard element is ∇_F-standard. However, the converse is not true.*

The following example shows that ∇_F-standard element need not be fuzzy join-standard.

Example 4.1. *Let $X = \{\bot, a, b, c, d, e, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\bot,\bot) = A(a,a) = A(b,b) = A(c,c) = A(d,d) = A(e,e) = A(\top,\top) = 1,$
$A(\bot,a) = 0.5,\ A(\bot,b) = 0.5,\ A(\bot,c) = 0.5,\ A(\bot,d) = 0.5,\ A(\bot,e) = 0.5,\ A(\bot,\top) = 0.5,$
$A(a,\bot) = 0,\ A(a,b) = 0,\ A(a,c) = 0,\ A(a,d) = 0.8,\ A(a,e) = 0.8,\ A(a,\top) = 0.05,$
$A(b,\bot) = 0,\ A(b,a) = 0,\ A(b,c) = 0.8,\ A(b,d) = 0.8,\ A(b,e) = 0.8,\ A(b,\top) = 0.05,$
$A(c,\bot) = 0,\ A(c,a) = 0,\ A(c,b) = 0,\ A(c,d) = 0,\ A(c,e) = 0,\ A(c,\top) = 0.05,$
$A(d,\bot) = 0,\ A(d,a) = 0,\ A(d,b) = 0,\ A(d,c) = 0,\ A(d,e) = 0.8,\ A(d,\top) = 0.05,$
$A(e,\bot) = 0,\ A(e,a) = 0,\ A(e,b) = 0,\ A(e,c) = 0,\ A(e,d) = 0,\ A(e,\top) = 0.05,$
$A(\top,\bot) = 0,\ A(\top,a) = 0,\ A(\top,b) = 0,\ A(\top,c) = 0,\ A(\top,d) = 0,\ A(\top,e) = 0.$

Then A is a fuzzy partial order relation.
We note that (X, A) is a fuzzy lattice.
This fuzzy relation is summarized in the following table:

A	$\bot$	a	b	c	d	e	$\top$
$\bot$	1.0	0.5	0.5	0.5	0.5	0.5	0.5
a	0.0	1.0	0.0	0.0	0.8	0.8	0.05
b	0.0	0.0	1.0	0.8	0.8	0.8	0.05
c	0.0	0.0	0.0	1.0	0.0	0.0	0.05
d	0.0	0.0	0.0	0.0	1.0	0.8	0.05
e	0.0	0.0	0.0	0.0	0.0	1.0	0.05
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	a	b	c	d	e	$\top$
a	a	a	e	$\top$	d	e	$\top$
b	b	e	b	c	d	e	$\top$
c	c	$\top$	c	c	$\top$	$\top$	$\top$
d	d	d	d	$\top$	d	d	$\top$
e	e	e	e	$\top$	d	e	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a	a	a
b	$\bot$	$\bot$	b	b	b	b	b
c	$\bot$	$\bot$	b	c	b	b	c
d	$\bot$	a	b	b	d	e	d
e	$\bot$	a	b	b	e	e	e
$\top$	$\bot$	a	b	c	d	e	$\top$

In the above fuzzy lattice (X, A), c is ∇_F-standard. There is only one element a with $a \wedge_F c = \bot$. Hence $c \nabla_F a$ must hold i.e., for every $x \in X$, $(x \vee_F c) \wedge_F a = x \wedge_F a$.
Now for $e \in X$ we have $(e \vee_F c) \wedge_F a = \top \wedge_F a = a = e \wedge_F a = a$.
For $d \in X$ we have $(d \vee_F c) \wedge_F a = 1 \wedge a = a = d \wedge_F a$.
For $b \in X$ we have $(b \vee_F c) \wedge_F a = c \wedge_F a = \bot = b \wedge_F a$.
Thus we have proved that c is ∇_F-standard but c is not fuzzy join-standard since $(e \vee_F c) \wedge_F d = \top \wedge_F d = d$ and $(e \wedge_F d) \vee_F (c \wedge_F d) = e \vee_F b = e \neq d$.

Theorem 4.1. *Let $s \in X$ be an element of an FSSC lattice. Then the following statements are equivalent.*
(α) *s is fuzzy join-standard element.*
(β) *s is a ∇_F-standard element.*
(γ) *s is fuzzy distributive element.*
(δ) *If $t \neq \bot$ and $A(t, s \vee x) > 0$ for $x \in X$, then there exists $t_1 \in X$ such that $0 < A(\bot, t_1) < 1$, $A(t_1, t) > 0$ and either $A(t_1, s) > 0$ or $A(t_1, x) > 0$.*

Proof. (α) $\Rightarrow$ (δ) Let $t \neq t_1$ and $A(t, s \vee_F x) > 0$ for $x \in X$.
Case i) If $A(t, s) > 0$, then $t = t_1$ will serve the purpose.
Case ii) Suppose $A(t, s) = 0$. Since (X, A) is a FSSC lattice, there exists $t_1 \in X$ such that

$0 < A(\bot, t_1) < 1$, $A(t_1, t) > 0$ and $t_1 \wedge_F s = \bot$.
As $A(t_1, t) > 0$ and $A(t, s \vee_F x) > 0$ by fuzzy transitivity of A, we get $A(t_1, s \vee_F x) > 0$.
By (ii) of Proposition 2.1 we get

$$\begin{aligned} t_1 &= (s \vee_F x) \wedge_F t_1 \\ &= (s \wedge_F t_1) \vee_F (x \wedge_F t_1), \text{ as } s \text{ is fuzzy join-standard element} \\ &= \bot \vee_F (x \wedge_F t_1), \text{ as } s \wedge_F t_1 = \bot \\ &= x \wedge_F t_1 \end{aligned}$$

i.e., $t_1 = x \wedge_F t_1$.
By (ii) of Proposition 2.1 we get $A(t_1, x) > 0$.
$(\delta) \Rightarrow (\gamma)$ Let $a = (s \vee_F x) \wedge_F (s \vee_F y)$ and $b = (x \wedge_F y) \vee_F x$ for $x, y \in X$.
To prove fuzzy meet-distributivity of s, it is sufficient to show that $A(a, b) > 0$.
Assume on the contrary that $A(a, b) = 0$.
Let (X, A) be FSSC lattice, there exists $c \in X$ such that $0 < A(\bot, c) < 1$, $A(c, a) > 0$ and $b \wedge_F c = \bot$.
As $a = (s \vee_F x) \wedge_F (s \vee_F y)$ by (ii) of Proposition 2.1 we have

$$A(a, s \vee_F x) > 0 \tag{4.4}$$

and

$$A(a, s \vee_F y) > 0. \tag{4.5}$$

As $A(a, s \vee_F x) > 0$ and $A(c, a) > 0$ by fuzzy transitivity of A, we get $A(c, s \vee_F x) > 0$.
Applying (δ) for the non zero element $A(c, s \vee_F x) > 0$, there exists $c_1 \in X$ such that $0 < A(\bot, c_1) < 1$, $A(c_1, c) > 0$ and either $A(c_1, s) > 0$ or $A(c_1, x) > 0$.
As $A(c_1, c) > 0$ and $A(c, a) > 0$ by fuzzy transitivity of A, we get

$$A(c_1, a) > 0. \tag{4.6}$$

From (4.5) and (4.6) by fuzzy transitivity of A, we get $A(c_1, s \vee_F y) > 0$.
Again by (δ), there exists $c_2 \in X$ such that $0 < A(\bot, c_2) < 1$,

$$A(c_2, c_1) > 0 \tag{4.7}$$

and either $A(c_2, s) > 0$ or $A(c_2, y) > 0$.
Note that $A(s, b) > 0$, $A(c_1, c) > 0$,

$$A(c_2, c) > 0. \tag{4.8}$$

Taking meet b on both sides we get $A(b \wedge_F c_1, b \wedge_F c) > 0$ and $A(b \wedge_F c_2, b \wedge_F c) > 0$.
But $b \wedge_F c = \bot$. Therefore $A(b \wedge_F c_1, \bot) > 0$ and $A(b \wedge_F c_2, \bot) > 0$.
As $A(\bot, b \wedge_F c_1) > 0$ and $A(\bot, b \wedge_F c_2) > 0$ always holds.
Therefore by fuzzy antisymmetry of A, we have $b \wedge_F c_1 = \bot$ and $b \wedge_F c_2 = \bot$.
Therefore $A(c_1, s) = 0$ and $A(c_2, s) = 0$.
Hence the only possibility is that $A(c_1, c) > 0$ and $A(c_2, y) > 0$.
Therefore we have $A(c_1 \wedge_F c_2, x \wedge_F y) > 0$.
By (4.7) and (ii) Proposition 2.1 we have

$$A(c_2, x \wedge_F y) > 0. \tag{4.9}$$

As $b = (x \wedge_F y) \vee_F x$ by (i) of Proposition 2.1 we have

$$A(x \wedge_F y, b) > 0. \tag{4.10}$$

From (4.9) and (4.10) by fuzzy transitivity of A, we get $A(c_2, b) > 0$.
Taking meet c on both sides we get $A(c \wedge_F c_2, b \wedge_F c) > 0$.
From (4.8) we have $A(c_2, b \wedge_F c) > 0$, so we get $A(c_2, \bot) > 0$ (as $b \wedge_F c = \bot$).

As $A(\perp, c_2) > 0$ always holds.
So by fuzzy antisymmetry of A, we have $c_2 = \perp$, a contradiction to $0 < A(\perp, c_2) < 1$. Therefore $A(a, b) > 0$, as required.
$(\gamma) \Rightarrow (\beta)$ Let $a \in X$ such that $s \wedge_F a = \perp$ and let $x, y \in X$ such that $y = (s \vee_F x) \wedge_F a$. To show s is ∇_F-standard, it is sufficient to show that $A(y, x) > 0$.
Suppose $A(y, x) = 0$. Since (X, A) is FSSC lattice, there exists $c \in X$ such that $0 < A(\perp, c) < 1$, $A(c, y) > 0$ and $c \wedge_F x = \perp$. As $A(c, x \vee_F s) > 0$.
Taking join s on both sides we get $A(c \vee_F s, x \vee_F s) > 0$. By (ii) of Proposition 2.1 we get

$$\begin{aligned} c \vee_F s &= (c \vee_F s) \wedge_F (x \vee_F s) \\ &= (c \wedge_F x) \vee_F s, \quad \text{as } s \text{ is fuzzy meet-distributive} \\ &= \perp \vee_F s, \\ &= s \end{aligned}$$

i.e., $c \vee_F s = s$, by (i) of Proposition 2.1 we get $A(c, s) > 0$.
But $A(c, y) > 0$ and $A(y, a) > 0$ gives $c = s \wedge_F a = \perp$, a contradiction to the fact that $0 < A(\perp, c) < 1$.
$(\beta) \Rightarrow (\alpha)$ Let s be ∇_F-standard. Suppose s is not fuzzy join-standard. Then for some $x, y \in X$, there exists $w = (s \vee_F x) \wedge_F y$ and $v = (s \wedge_F y) \vee_F (x \wedge_F y)$ such that $w \neq v$. Since (X, A) is a FSSC lattice, then there exists $c \in X$ such that $c \wedge_F v = \perp$, $0 < A(c, w) < 1$ and

$$A(c, w) > 0. \tag{4.11}$$

As $v = (s \wedge_F y) \vee (x \wedge_F y)$ then by (i) of Proposition 2.1 we have

$$A(s \wedge_F y, v) > 0. \tag{4.12}$$

Also, $w = (s \vee_F x) \wedge_F y$ so by (ii) of Proposition 2.1 we have

$$A(w, y) > 0. \tag{4.13}$$

From (4.11) and (4.13) by fuzzy transitivity of A, we have $A(c, y) > 0$.
Therefore by (ii) of Proposition 2.1 we have

$$c \wedge_F y = c. \tag{4.14}$$

As $A(s \wedge_F y, v) > 0$. Taking meet c on both sides we get

$$A(c \wedge_F s \wedge_F y, c \wedge_F v) > 0. \tag{4.15}$$

By using (4.14) equation (4.15) reduces to

$$A(s \wedge_F c, c \wedge_F v) > 0. \tag{4.16}$$

But $c \wedge_F v = \perp$. Therefore equation (4.16) reduces to

$$A(s \wedge_F c, \perp) > 0. \tag{4.17}$$

As

$$A(\perp, s \wedge_F c) > 0 \tag{4.18}$$

always holds.
Therefore from (4.17) and (4.18) by fuzzy antisymmetry of A, we get $s \wedge_F c = \perp$.
Similarly using $A(x \wedge_F y, v) > 0$, we get $x \wedge_F c = \perp$.
Since s is ∇_F-standard and $s \wedge_F c = \perp$ therefore $s \nabla_F c$ holds.
Hence using $s \nabla_F c$ and the fact $x \wedge_F c = \perp$ we get

$$\perp = x \wedge_F c = \{(s \vee x) \wedge_F c\} = c,$$

a contradiction to $0 < A(\perp, c) < 1$. □

Combining Theorem 4.1 with Corollary 3.1 we have

Corollary 4.1. *An element s of a FSSC lattice (X, A) is fuzzy neutral iff it is both fuzzy meet-distributive and fuzzy join-distributive.*

Theorem 4.2. *In a fuzzy atomistic lattice (X, A) the following statements are equivalent for an element $s \in X$:*
(α) s is fuzzy join-standard element.
(β) s is a ∇_F-standard element.
(γ) s is fuzzy meet-distributive element.
(δ) If p is a fuzzy atom and $A(p, s) = 0$, then $s\nabla_F p$.
(ϵ) If a fuzzy atom p is fuzzy subperspective to s, then $A(p, s) > 0$.
(ξ) If a fuzzy atom $A(p, s \vee_F x) > 0$ for $x \in X$, then $A(p, s) > 0$ or $A(p, x) > 0$.

Proof. Equivalence of (α), (β), (γ) and (ξ) follows from Theorem 4.1 as every fuzzy atomistic lattice is FSSC lattice.
$(\beta) \Rightarrow (\delta)$ Suppose s is a ∇-standard element and let p be a fuzzy atom with $p \wedge_F s = \perp$. Hence by (β), $s\nabla_F p$ holds.
$(\delta) \Rightarrow (\epsilon)$ Let p be a fuzzy atom which is fuzzy subperspective to s, then by Lemma 2.2, $s\nabla_F p$ does not hold. Hence by (δ), $A(p, s) > 0$.
$(\epsilon) \Rightarrow (\xi)$ Suppose $p \in X$ be a fuzzy atom with $A(p, s \vee_F x) > 0$ for an element $x \in X$.
If $A(p, x) = 0$, then $p \wedge_F x = \perp$. Hence p is fuzzy subperspective to s.
Therefore by (ϵ), we get $A(p, s) > 0$. □

Definition 4.2. *An element s of a fuzzy lattice (X, A) with $\perp$ is called a fuzzy p-element if $A(a, s) > 0$ and $a \sim^F b$ together imply $A(b, s) > 0$.*

Lemma 4.2. *let (X, A) be a fuzzy lattice with $\perp$. If an element $s \in X$ is fuzzy join-standard, then s is a fuzzy p-element.*

Proof. Let $A(a, s) > 0$, $a \sim^F b$ i.e., for some $x \in X$, $a \wedge_F x = b \wedge_F \perp$ and $a \vee_F x = b \vee_F x$. Then

$$\begin{aligned} b &= b \wedge_F (b \vee_F x), \\ &= b \wedge_F (a \vee_F x), \text{ as } a \vee_F x = b \vee_F x \end{aligned}$$

i.e., we have

$$b = b \wedge_F (a \vee_F x). \tag{4.19}$$

As $A(a, s) > 0$ taking join x on both sides we get $A(a \vee_F x, s \vee_F x) > 0$.
Taking meet b on both sides we get $A(b \wedge_F (a \vee_F x), b \wedge_F (s \vee_F x)) > 0$.
As s is fuzzy join-standard we get

$$A(b \wedge_F (a \vee_F x), (b \wedge_F s) \vee (b \wedge_F x)) > 0. \tag{4.20}$$

As $b \wedge_F x = \perp$.
Equation (4.20) reduces to

$$A(b \wedge_F (a \vee_F x), b \wedge_F s) > 0. \tag{4.21}$$

Using equation (4.19) in (4.21) we get

$$A(b, b \wedge_F s) > 0. \tag{4.22}$$

As

$$A(b \wedge_F s, b) > 0 \tag{4.23}$$

always holds.
Hence from equation (4.22) and (4.23) by antisymmetry of A we get $b \wedge_F s = b$.

By (ii) of Proposition 2.1 we get $A(b, s) > 0$.
Thus s is a fuzzy p-element. □

Lemma 4.3. *Let (X, A) be a fuzzy atomistic lattice. Then every ∇_F-standard element is fuzzy p-element i.e., ∇_F-standard $\Rightarrow$ fuzzy p-element.*

Proof. Suppose $s \in X$ is ∇_F-standard. Let $A(a, s) > 0$, $a \wedge_F x = b \wedge_F x = \perp$, $a \vee_F x = b \vee_F x$ but $A(b, s) = 0$. Since (X, A) is fuzzy atomistic, there exists an atom q such that

$$A(q, b) > 0 \tag{4.24}$$

with $A(q, s) = 0$. Hence $q \wedge_F s = \perp$. Since s is ∇_F-standard, we have $s \nabla_F q$. Therefore

$$((a \vee_F x) \vee_F s) \wedge_F q = (a \vee_F x) \wedge_F q = (b \vee_F x) \wedge_F q = q.$$

i.e., $(x \vee_F s) \wedge_F q = q$ which implies $x \wedge_F q = q$ by $s \nabla_F q$. Thus

$$A(q, x) > 0. \tag{4.25}$$

From (4.24), (4.25) and by Corollary 2.1 we have $A(q, b \wedge_F x) > 0$. But $b \wedge_F x = \perp$. Therefore $A(q, \perp) > 0$, which is not possible, since q is a fuzzy atom. Hence s is a fuzzy p-element. □

In the following example we show that the implication in Lemma 4.3 is not reversible. To prove this, we shall give counter example. Thus, we emphasize that the above implication is strict. We show that every fuzzy p-element need not be ∇_F-standard.

Example 4.2. *Let $X = \{\perp, a, b, c, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\perp, \perp) = A(a, a) = A(b, b) = A(c, c) = A(\top, \top) = 1$,
$A(\perp, a) = 0.6$, $A(\perp, b) = 0.6$, $A(\perp, c) = 0.6$, $A(\perp, \top) = 0.6$,
$A(a, \perp) = 0$, $A(a, b) = 0.9$, $A(a, c) = 0$, $A(a, \top) = 0.06$,
$A(b, \perp) = 0$, $A(b, a) = 0$, $A(b, c) = 0$, $A(b, \top) = 0.06$,
$A(c, \perp) = 0$, $A(c, a) = 0$, $A(c, b) = 0$, $A(c, \top) = 0.06$,
$A(\top, \perp) = 0$, $A(\top, a) = 0$, $A(\top, b) = 0$, $A(\top, c) = 0$.

Then A is a fuzzy partial order relation.
We note that (X, A) is a fuzzy lattice.
This fuzzy relation is summarized in the following table:

A	$\perp$	a	b	c	$\top$
$\perp$	1.0	0.6	0.6	0.6	0.6
a	0.0	1.0	0.9	0.0	0.06
b	0.0	0.0	1.0	0.0	0.06
c	0.0	0.0	0.0	1.0	0.06
$\top$	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\perp$	a	b	c	$\top$
$\perp$	$\perp$	a	b	c	$\top$
a	a	a	b	$\top$	$\top$
b	b	b	b	$\top$	$\top$
c	c	$\top$	$\top$	c	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\perp$	a	b	c	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	a	a	$\perp$	a
b	$\perp$	a	b	$\perp$	b
c	$\perp$	$\perp$	$\perp$	c	c
$\top$	$\perp$	a	b	c	$\top$

Here c is a fuzzy p-element, since there is no element which is fuzzy perspective to c. Here c is not ∇_F-standard, because $c \wedge_F b = \perp$, but $c \nabla_F a$ fails, since

$$(a \vee_F c) \wedge_F b = \top \wedge_F b = b \neq a = a \wedge_F b.$$

Thus, c is a fuzzy p-element $\nRightarrow$ c is ∇_F-standard.

The next question is whether the meet of two fuzzy p-elements is a fuzzy p-element. This is answered in the following Lemma.

Lemma 4.4. *Let (X, A) be a fuzzy lattice. If s_1, s_2 are fuzzy p-elements, then so $s_1 \wedge_F s_2$.*

Proof. Suppose $A(a, s_1 \wedge_F s_2) > 0$, $a \wedge_F x = \bot = b \wedge_F x$ and $a \vee_F x = b \vee_F x$. Then $A(a, s_1) > 0$ and $A(a, s_2) > 0$. Since s_1, s_2 are fuzzy p-elements we have $A(b, s_1) > 0$, $A(b, s_2) > 0$. Thus $A(b, s_1 \wedge_F s_2) > 0$. Therefore $s_1 \wedge_F s_2$ is a fuzzy p-element. □

However, join of two fuzzy p-elements need not be a fuzzy p-element. The counter example given below brings this fact explicitly.

Example 4.3. *Let $X = \{\bot, a, b, s, s_1, s_2, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\bot,\bot) = A(a,a) = A(b,b) = A(s,s) = A(s_1,s_1) = A(s_2,s_2) = A(\top,\top) = 1$,
$A(\bot,a) = 0.1$, $A(\bot,b) = 0.1$, $A(\bot,s) = 0.1$, $A(\bot,s_1) = 0.1$, $A(\bot,s_2) = 0.1$,
$A(\bot,\top) = 0.1$,
$A(a,\bot) = 0$, $A(a,b) = 0$, $A(a,s) = 0$, $A(a,s_1) = 0.5$, $A(a,s_2) = 0$, $A(a,\top) = 0.01$,
$A(b,\bot) = 0$, $A(b,a) = 0$, $A(b,s) = 0$, $A(b,s_1) = 0$, $A(b,s_2) = 0.5$, $A(b,\top) = 0.01$,
$A(s,\bot) = 0$, $A(s,a) = 0$, $A(s,b) = 0$, $A(s,s_1) = 0.5$, $A(s,s_2) = 0.5$, $A(s,\top) = 0.01$,
$A(s_1,\bot) = 0$, $A(s_1,a) = 0$, $A(s_1,b) = 0$, $A(s_1,s) = 0$, $A(s_1,s_2) = 0$, $A(s_1,\top) = 0.01$,
$A(s_2,\bot) = 0$, $A(s_2,a) = 0$, $A(s_2,b) = 0$, $A(s_2,s) = 0$, $A(s_2,s_1) = 0$, $A(s_2,\top) = 0.01$,
$A(\top,\bot) = 0$, $A(\top,a) = 0$, $A(\top,b) = 0$, $A(\top,s) = 0$, $A(\top,s_1) = 0$, $A(\top,s_2) = 0$.

Then A is a fuzzy partial order relation.
We note that (X, A) is a fuzzy lattice.
This fuzzy relation is summarized in the following table:

A	$\bot$	a	b	s	s_1	s_2	$\top$
$\bot$	1.0	0.1	0.1	0.1	0.1	0.1	0.1
a	0.0	1.0	0.0	0.0	0.5	0.0	0.01
b	0.0	0.0	1.0	0.0	0.0	0.5	0.01
s	0.0	0.0	0.0	1.0	0.5	0.5	0.01
s_1	0.0	0.0	0.0	0.0	1.0	0.0	0.01
s_2	0.0	0.0	0.0	0.0	0.0	1.0	0.01
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	s	s_1	s_2	$\top$
$\bot$	$\bot$	a	b	s	s_1	s_2	$\top$
a	a	a	$\top$	$\top$	a	$\top$	$\top$
b	b	$\top$	b	$\top$	$\top$	b	$\top$
s	s	$\top$	$\top$	s	s	s	$\top$
s_1	s_1	a	$\top$	s	s_1	s	$\top$
s_2	s_2	$\top$	b	s	s	s_2	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	s	s_1	s_2	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	s_1	s_1	$\bot$	a
b	$\bot$	$\bot$	b	s_2	$\bot$	s_2	b
s	$\bot$	s_1	s_2	s	s_1	s_2	s
s_2	$\bot$	s_1	$\bot$	s_1	s_1	$\bot$	s_1
s_1	$\bot$	$\bot$	s_2	s_2	$\bot$	s_2	s_2
$\top$	$\bot$	a	b	s	s_1	s_2	$\top$

Here s_1 and s_2 are fuzzy p-elements. But $s_1 \vee_F s_2 = s$ is not fuzzy p-element, since $A(s_2, s) > 0$, $s_2 \vee_F a = b \vee_F a = \top$ and $s_2 \wedge_F a = b \wedge_F a = \bot$ (i.e., $s_2 \sim b$) do not imply $A(b, s) > 0$.

5. Conclusion

In this paper, we have introduced the notion of fuzzy join-standard, fuzzy neutral element, ∇_F-standard element and p-element in fuzzy lattice and studied relationship among them.

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