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CONVERGENCE OF CLIFFORD JACOBI WAVELETS IN L^p-NORM AND SHORT TIME FOURIER TRANSFORM

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Abstract. We have developed an approximate identity and studied the convergence of Clifford-Gegenbauer-Jacobi polynomials and their associated wavelets in L^p-sense. Additionally, the convergence of short-time Clifford Fourier transform in Feichtinger space will be studied. This study aims to extend prior findings in wavelet and Fourier analysis within the framework of Clifford analysis, addressing gaps in function reconstruction beyond L^2 spaces and utilizing approximate identities.

Keywords: Approximate identity; Clifford-Jacobi wavelets; Clifford-Jacobi continuous wavelet transform; Schwartz class; Clifford analysis; Short-time Clifford Fourier transform.

AMS Subject Classification: 68U10, 47G20

1. Introduction

It has been noted in [20] that wavelets are valuable tools in signal processing compared to Fourier analysis. Also, wavelets play a significant role in many fields, including electrical engineering, image processing, quantum physics, seismology, geology, and mathematics. Classical Fourier analysis employs a global approach to signal analysis. It replaces the analyzed function over the entire space, while in wavelet analysis the signal is decomposed in both time and frequency which describes it locally and globally as required.

The wavelets are generated by a single function $g \in L^2(\mathbb{R})$ by dilation and translation with dilation parameter $a > 0$ and translation parameter $b \in \mathbb{R}$ and defined as

$$g_{a,b}(x) = a^{-\frac{1}{2}} g(\frac{x-b}{a}).$$

The single function g is known as mother wavelet which satisfies the admissibility condition as

$$C_g = \int_{-\infty}^{\infty} \frac{|\hat{g}(u)|^2}{|u|} du < +\infty,$$

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where $\hat{g}$ denotes the Fourier transform of g.

The continuous wavelet transform is defined by

$$W_g(f) = < f, g_{a,b} > = \int_{\mathbb{R}} f(x)\overline{g_{a,b}(x)}dx.$$

If the admissibility condition satisfied, then the analyzed function f may be reconstructed in L^2-sense (see [23, 24])

$$f(x) = \frac{1}{C_g}\int_{\mathbb{R}}\int_0^{\infty} W_g(f)g_{a,b}(x)\frac{da}{a^2}db.$$

Clifford analysis deals with monogenic functions which are higher dimensional generalizations of holomorphic functions in complex plane. Wavelet analysis on the real line and in Euclidean spaces has been extended within the frame work of Clifford analysis (see [21, 8-13, 18, 22]).

Moreover, it is a generalization of the complex analysis and Hamiltonians and extends to some type of finite dimensional associative algebra known as Clifford algebra endowed with inner products and norms.

Although, the Clifford analysis generalizes the most important features of classical complex analysis, monogenic functions do not satisfies all properties of holomorphic functions of complex variables. Since Clifford algebras are non-commutative, the product of two monogenic functions is in general is not monogenic.There are several techniques to generate monogenic functions such as the Cauchy-Kowalevski extension (CK-extension) (see [30]).

Arfaoui et al. [3] developed new classes of Clifford wavelet functions. Such classes contain the well known Jacobi, Gegenbauer and Hermite ones. The constructed polynomials are applied to develop new Clifford wavelets. Additionally, reconstruction and Fourier-Plancherel formulas have also been proven.

Banouch and Ben Mabrouk [5] investigated the development of a new uncertainty principle based on the wavelet transform in the Clifford analysis/algebra framework. Moreover, they developed a sharp Heisenberg-type uncertainty principle for the continuous Clifford wavelet transform.

Recently, Arfaoui and Ben Mabrouk [2] introduced new classes of wavelet functions by extending some fractional calculus to the framework of Clifford analysis. Some classes of monogenic polynomials are provided which extend the classical Jacobi polynomials in the context of Clifford analysis. Additionally, they proved the reconstruction formula and the Fourier-Plancherel rule. In the present paper, our methods and results differ from all those discussed above.

Let Ω be an open subset of $\mathbb{R}^m$ or $\mathbb{R}^{m+1}$ and $f : \Omega \to A$, where A is the real Clifford algebra $\mathbb{R}_m$ (or $\mathbb{C}_m$), can be written in the form (see [1], pp. 2291)

$$f = \sum_A f_A e_A,$$

where e_A is a suitable basis of A and the functions f_A are $\mathbb{R}$ (or $\mathbb{C}$)-valued.

The real space $\mathbb{R}^m, (m \geq 2)$(or $\mathbb{C}^m$) endowed with an orthonormal basis $(e_1, \ldots, e_m)$. Throughout this paper, we consider $e_j^2 = -1$, $j = 1, \ldots, m$, $e_je_k + e_ke_j = 0$, $j \neq k$, j, $k = 1, \ldots, m$, and $e_\phi = 1$.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_m$ (or $\mathbb{C}_m$) by the vector $x = (x_1, \ldots, x_m)$ with the vector $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j.$$

The product of two vectors is defined as

$$\underline{x}\underline{y} = \underline{x}.\underline{y} + \underline{x} \wedge \underline{y},$$

where

$$\underline{x}.\underline{y} = -<\underline{x}, \underline{y}>= -\sum_{j=1}^{m} x_j y_j, \quad \underline{x} \wedge \underline{y} = \sum_{j=1}^{m} \sum_{k=j+1}^{m} e_j e_k (x_j y_k - x_k y_j).$$

We define

$$\underline{x}^2 = -<\underline{x}, \underline{x}>= -|\underline{x}|^2.$$

The Clifford Fourier transform of an analyzing function f is defined as (see [1], pp. 2290)

$$F(f(x))(y) = \int_{\mathbb{R}^m} e^{-i<\underline{x},\underline{y}>} f(\underline{x}) dv(\underline{x}),$$

where $dv(\underline{x})$ is the Lebesgue measure on $\mathbb{R}^m$.

The inner product of functions in the framework of Clifford analysis is defined by

$$<f, g>= \int_{\mathbb{R}^m} f(\underline{x}) \overline{g(\underline{x})} dv(\underline{x}).$$

2. Definitions and Auxiliary Results

We will use the following operators throughout the paper.
Translation: $T_{\underline{b}} f(\underline{x}) = f(\underline{x} - \underline{b})$ for $\underline{b} \in \mathbb{R}^m$.
Modulation: $E_{\underline{w}} f(\underline{x}) = e^{2\pi i <\underline{w}, \underline{x}>} f(\underline{x})$ for $\underline{w} \in \mathbb{R}^m$.
Dilation: $D_a f(\underline{x}) = |a|^{-\frac{1}{2}} f(\frac{\underline{x}}{a})$ for $a \in \mathbb{R} \setminus \{0\}$.

We denote the two parameters Clifford-Jacobi polynomials by $Z_{l,m}^{\alpha,\beta}(\underline{x})$. These polynomials are generated by the weight function (see [19], pp. 2295)

$$w_{\alpha,\beta}(\underline{x}) = (1 - |\underline{x}|^2)^{\alpha} (1 + |\underline{x}|^2)^{\beta}.$$

For more details about these polynomials (see [8-13]).

Definition 2.1 (Clifford-Jacobi Mother Wavelet). *The generalized 2-parameters Clifford-Jacobi mother wavelet (see [1], pp. 2299) is defined by*

$$\phi_{l,m}^{\alpha,\beta}(\underline{x}) = Z_{l,m}^{\alpha+l,\beta+l}(\underline{x}) w_{\alpha,\beta}(\underline{x}).$$

Definition 2.2 (Clifford-Jacobi Wavelet Transform). *The continuous Clifford-Jacobi wavelet transform of a function $f \in L^2(\mathbb{R}^m)$ (see [1], pp. 2301) is defined by*

$$W_{a,\underline{b}}(f) =< f, \phi_{l,m,a}^{\alpha,\beta,\underline{b}} >= \int_{\mathbb{R}^m} f(\underline{x}) \overline{\phi_{l,m,a}^{\alpha,\beta,\underline{b}}(\underline{x})} dv(\underline{x}) =< f, D_a T_{\underline{b}} \phi_{l,m}^{\alpha,\beta} > .$$

The mother wavelet $\phi_{l,m}^{\alpha,\beta}(\underline{x})$ satisfies the admissibility condition

$$C_{l,m}^{\alpha,\beta} = \frac{1}{w_m} \int_{\mathbb{R}^m} |\hat{\phi}_{l,m}^{\alpha,\beta}(\underline{x})|^2 \frac{dv(\underline{x})}{|\underline{x}|^m} < \infty,$$

where w_m *is the volume of the unit sphere* S^{m-1} *in* $\mathbb{R}^m$ *and* $\hat{\phi}^{\alpha,\beta}_{l,m}$ *denote the Fourier transform of* $\phi^{\alpha,\beta}_{l,m}$. *For more about admissibility and wavelet properties (see [1, 15, 16, 27, 30]).*

Following the approach in [9, pp. 402], it can be proven that if the admissibility condition satisfied, then the function f may be reconstructed in the L^2-sense

$$f(\underline{x}) = \frac{1}{C^{\alpha,\beta}_{l,m}} \int_{\mathbb{R}^m} \int_{a>0} W_{a,\underline{b}}(f)\phi(\frac{\underline{x}-\underline{b}}{a}) \frac{dadv(\underline{b})}{a^{m+1}}. \tag{2.1}$$

The formula (2.1) is true in L^2-sense but if $f \in L^1(\mathbb{R}^m)$ or $L^p(\mathbb{R}^m), p \neq 2, 1 \leq p < \infty$, then the function f may not be reconstructed by the formula (2.1). Therefore, in this paper, we address this problem by using an approximate identity. Finally, we discuss the convergence of the short-time Clifford Fourier transform in Feichtinger space.

Definition 2.3. *Let* $\psi^{\alpha,\beta}_{l,m} \in L^1(\mathbb{R}^m)$ *such that* $\int_{\mathbb{R}^m} \psi^{\alpha,\beta}_{l,m}(\underline{x})dv(\underline{x}) = 1$. *Then* $(\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{x}) = \varepsilon^{-m}\psi^{\alpha,\beta}_{l,m}(\frac{\underline{x}}{\varepsilon})$ *is an approximate identity if*

(1) $\int_{\mathbb{R}^m} (\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{x})dv(\underline{x}) = 1$,

(2) $\sup_{\varepsilon>0} \int_{\mathbb{R}^m} |(\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{x})|dv(\underline{x}) < \infty$,

(3) $\lim_{\varepsilon\to 0} \int_{|\underline{x}|>\delta} |(\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{x})|dv(\underline{x}) = 0$, *for every* $\delta > 0$.

To prove (1) and (2), we see that

$$\int_{\mathbb{R}^m} (\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{x})dv(\underline{x}) = \int_{\mathbb{R}^m} \varepsilon^{-m}\psi^{\alpha,\beta}_{l,m}(\frac{\underline{x}}{\varepsilon})dv(\underline{x}) = \int_{\mathbb{R}^m} \psi^{\alpha,\beta}_{l,m}(\frac{\underline{x}}{\varepsilon})dv(\frac{\underline{x}}{\varepsilon}) = 1,$$

so

$$\sup_{\varepsilon>0} \int_{\mathbb{R}^m} |(\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{x})|dv(\underline{x}) < \infty.$$

In order to prove (3), we have

$$\int_{|\underline{x}|>\delta} |(\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{x})|dv(\underline{x}) = \int_{|\underline{x}|>\delta} \varepsilon^{-m}|\psi^{\alpha,\beta}_{l,m}(\frac{\underline{x}}{\varepsilon})|dv(\underline{x}) = \int_{\delta}^{\infty} \varepsilon^{-m}|\psi^{\alpha,\beta}_{l,m}(\frac{\underline{x}}{\varepsilon})|dv(\underline{x})$$
$$+ \int_{-\infty}^{-\delta} \varepsilon^{-m}|\psi^{\alpha,\beta}_{l,m}(\frac{\underline{x}}{\varepsilon})|dv(\underline{x}),$$

let $\frac{\underline{x}}{\varepsilon} = \underline{u}$, then

$$\lim_{\varepsilon\to 0} \int_{\frac{\delta}{\varepsilon}}^{\infty} |\psi^{\alpha,\beta}_{l,m}(\underline{u})|dv(\underline{u}) + \int_{-\infty}^{-\frac{\delta}{\varepsilon}} |\psi^{\alpha,\beta}_{l,m}(\underline{u})|dv(\underline{u}) = 0.$$

Definition 2.4. *Let* $\psi^{\alpha,\beta}_{l,m}(\underline{x}) \in L^1(\mathbb{R}^m)$ *with* $\psi^{\alpha,\beta}_{l,m}(\underline{0}) = 1$ *and* $(\psi^{\alpha,\beta}_{l,m})_n(\underline{x}) = n\psi^{\alpha,\beta}_{l,m}(n\underline{x})$, *where* $n = \frac{1}{\varepsilon}$ *as* $n \to \infty$, $\varepsilon \to 0$. *Then the sequence of functions* $\{(\psi^{\alpha,\beta}_{l,m})_n\}_{n=1}^{\infty}$ *is an approximate identity if*

(1′) $\int_{\mathbb{R}^m} (\psi^{\alpha,\beta}_{l,m})_n(\underline{x})dv(\underline{x}) = 1, \forall n$,

(2′) $\sup_n \int_{\mathbb{R}^m} |(\psi^{\alpha,\beta}_{l,m})_n(\underline{x})|dv(\underline{x}) < \infty$,

(3′) $\lim_{n\to\infty} \int_{|\underline{x}|>\delta} |(\psi^{\alpha,\beta}_{l,m})_n(\underline{x})|dv(\underline{x}) = 0$, *for every* $\delta > 0$.

Consider the Schwartz class $S(\mathbb{R}^m)$ such that

$$S(\mathbb{R}^m) = \{f : \mathbb{R}^m \to \mathbb{R}_m, \sup_{\underline{x}\in\mathbb{R}_m} (\underline{x}^n \frac{d^\eta}{d\underline{x}^\eta} f)(\underline{x}) < \infty\}; n, \eta \in \mathbb{N} \cup \{0\}.$$

Definition 2.5. *We define the Lebesgue space*

$$L^p(\mathbb{R}^m) = \{f :\| f \|_p = (\int_{\mathbb{R}^m} |f(\underline{x})|^p dv(\underline{x}))^{\frac{1}{p}} < \infty, 1 \leq p < \infty\}.$$

For $p = \infty$, we have

$$L^\infty(\mathbb{R}^m) = \{f :\| f \|_\infty = ess \sup_{\underline{x}\in\mathbb{R}_m} |f(\underline{x})| < \infty\}.$$

Now, we shall prove that $S(\mathbb{R}^m) \subseteq L^p(\mathbb{R}^m), 1 \leq p < \infty$. It holds that if $f \in S(\mathbb{R}^m)$, then $|f(\underline{x})| \leq \frac{c}{1+|\underline{x}|^n}$ for all $\underline{x} \in \mathbb{R}_m$. Let $n \in \mathbb{N}$ such that $np - (m-1) > 1$. Then we have

$$\int_{\mathbb{R}^m} |f(\underline{x})|^p dv(\underline{x}) \leq \int_{\mathbb{R}^m} (\frac{c}{1+|\underline{x}|^n})^p dv(\underline{x}).$$

Putting $\underline{x} = r\underline{w}, r = |\underline{x}|, \underline{w} \in S^{m-1}$, we get

$$\begin{aligned}\int_{\mathbb{R}^m} |f(\underline{x})|^p dv(\underline{x}) &\leq c^p \int_0^\infty \frac{1}{(1+r^n)^p} r^{m-1} dr \int_{S^{m-1}} d\sigma(\underline{w}) = c^p \gamma_m \int_0^\infty \frac{1}{(1+r^n)^p} r^{m-1} dr \\ &= c^p \gamma_m \int_0^\infty \frac{r^{m-1}}{r^{np} + P(r)} dr = c^p \gamma_m \frac{r^m}{m} 2F_1(\frac{m}{n}, p; 1+\frac{m}{n}; -r^n)|_0^\infty \\ &< \infty,\end{aligned}$$

for $\frac{m}{n} < p$, here $d\sigma(\underline{w})$ is the Lebesgue measure on unit sphere S^{m-1}, $\gamma_m = \frac{(2\pi)^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$ is the surface area of unit sphere in $\mathbb{R}^m$ and $2F_1$ is the hypergeometric function. Hence $f \in L^p(\mathbb{R}^m)$.

Now, we have to prove that $S(\mathbb{R}^m)$ is dense in $L^p(\mathbb{R}^m)$. Let Ω be an open subset of $\mathbb{R}^m$, we define

$$D(\Omega) = \{\xi \in C^\infty(\Omega) : supp(\xi) \text{ is compact}\} = C_0^\infty(\Omega).$$

For $\infty > p \geq 1, D(\mathbb{R}^m)$ is dense in $L^p(\mathbb{R}^m)$. It also noted that $S(\mathbb{R}^m)$ is a vector space and $D(\mathbb{R}^m) \subseteq S(\mathbb{R}^m)$. Now, we get

$$D(\mathbb{R}^m) \subseteq S(\mathbb{R}^m) \subseteq L^p(\mathbb{R}^m), 1 \leq p < \infty.$$

The above arguments implies that $S(\mathbb{R}^m)$ is dense in $L^p(\mathbb{R}^m)$. It can be easily seen that if $f \in S(\mathbb{R}^m)$, then Fourier transform $\hat{f} \in S(\mathbb{R}^m)$.

Note. If $0 \leq \psi_{l,m}^{\alpha,\beta}(\underline{x}) \in S(\mathbb{R}^m)$, then $(\psi_{l,m}^{\alpha,\beta})_n(\underline{x}) = n\psi_{l,m}^{\alpha,\beta}(n\underline{x})$ is an approximate identity.

Theorem 2.1. *If $f \in L^1(\mathbb{R}^m)$ and $\psi_{l,m}^{\alpha,\beta}(\underline{x}) \in S(\mathbb{R}^m)$, then $(\psi_{l,m}^{\alpha,\beta} * f)(\underline{x}) \in S(\mathbb{R}^m)$.*

Proof. We see that

$$(\psi_{l,m}^{\alpha,\beta} * f)(\underline{x}) = \int_{\mathbb{R}^m} \psi_{l,m}^{\alpha,\beta}(\underline{y}) f(\underline{x}-\underline{y}) dv(\underline{y}),$$

$$\frac{d^\eta}{d\underline{x}^\eta}(\psi_{l,m}^{\alpha,\beta} * f)(\underline{x}) = \int_{\mathbb{R}^m} \psi_{l,m}^{\alpha,\beta}(\underline{y}) \frac{d^\eta}{d\underline{x}^\eta} f(\underline{x}-\underline{y}) dv(\underline{y}),$$

$$\begin{aligned}|\underline{x}|^n \frac{d^\eta}{d\underline{x}^\eta}(\psi_{l,m}^{\alpha,\beta} * f)(\underline{x}) &= |\underline{x}|^n \int_{\mathbb{R}^m} f(\underline{x}-\underline{y}) \frac{d^\eta}{d\underline{x}^\eta} \psi_{l,m}^{\alpha,\beta}(\underline{y}) dv(\underline{y}) \\ &= \int_{\mathbb{R}^m} f(\underline{y}) |\underline{x}|^n \frac{d^\eta}{d\underline{x}^\eta} \psi_{l,m}^{\alpha,\beta}(\underline{x}-\underline{y}) dv(\underline{y}).\end{aligned}$$

Since $|\underline{x}-\underline{y}| \leq |\underline{x}|+|\underline{y}| \leq \frac{3|\underline{x}|}{2}$, it gives

$$\int_{|\underline{y}|>\frac{|\underline{x}|}{2}} f(\underline{y})|\underline{x}|^n \frac{d^\eta}{d\underline{x}^\eta}\psi_{l,m}^{\alpha,\beta}(\underline{x}-\underline{y})dv(\underline{y}) + \int_{|\underline{y}|<\frac{|\underline{x}|}{2}} f(\underline{y})|\underline{x}|^n \frac{d^\eta}{d\underline{x}^\eta}\psi_{l,m}^{\alpha,\beta}(\underline{x}-\underline{y})dv(\underline{y}) \to 0.$$

Hence, the proof is complete. □

Theorem 2.2. *Let* $\phi_{l,m}^{\alpha,\beta}(\underline{x}) \in L^p(\mathbb{R}^m)$ *is admissible with* $C_{l,m}^{\alpha,\beta} = 1$ *and let* $\{(\psi_{l,m}^{\alpha,\beta})_n\}_{n=1}^{\infty}$ *be an approximate identity with* $(\psi_{l,m}^{\alpha,\beta})_n \in S(\mathbb{R}^m)$ *and* $(\psi_{l,m}^{\alpha,\beta})_n(\underline{x}) = (\psi_{l,m}^{\alpha,\beta})_n(-\underline{x})\forall(\psi_{l,m}^{\alpha,\beta})_n$. *Then* $\lim_{n\to\infty} \| f - f_n \|_p = 0$, *for all* $f \in L^p(\mathbb{R}^m), 1 \leq p < \infty$, *where*

$$f_n(\underline{x}) = \int_{\mathbb{R}^m}\int_{a>0} W_{a,\underline{b}}(f)((\psi_{l,m}^{\alpha,\beta})_n * D_a T_{\underline{b}}(\phi_{l,m}^{\alpha,\beta}))\frac{dadv(\underline{b})}{a^{m+1}}$$

and

$$f(\underline{x}) = \int_{\mathbb{R}^m}\int_{a>0} W_{a,\underline{b}}(f) D_a T_{\underline{b}}(\phi_{l,m}^{\alpha,\beta})(\underline{x})\frac{dadv(\underline{b})}{a^{m+1}}.$$

Proof. Consider

$$\begin{aligned}
(f * (\psi_{l,m}^{\alpha,\beta})_n))(\underline{x}) &= \int_{\mathbb{R}^m} f(\underline{u})(\psi_{l,m}^{\alpha,\beta})_n(\underline{x}-\underline{u})d\underline{x} =< f, \overline{T_{\underline{x}}(\psi_{l,m}^{\alpha,\beta})_n} > \\
&=< W_{a,\underline{b}}f, W_{a,\underline{b}}\overline{(T_{\underline{x}}(\psi_{l,m}^{\alpha,\beta})_n)} > \\
&= \int_{\mathbb{R}^m}\int_{a>0} W_{a,\underline{b}}f(a,\underline{b}) < D_a T_{\underline{b}}(\phi_{l,m}^{\alpha,\beta}), \overline{T_{\underline{x}}(\psi_{l,m}^{\alpha,\beta})_n} > \frac{dadv(\underline{b})}{a^{m+1}} \\
&= \int_{\mathbb{R}^m}\int_{a>0} W_{a,\underline{b}}f(a,\underline{b})((\psi_{l,m}^{\alpha,\beta})_n * D_a T_{\underline{b}}(\phi_{l,m}^{\alpha,\beta}))(\underline{x})\frac{dadv(\underline{b})}{a^{m+1}}.
\end{aligned}$$

The last term of the second line above is followed by Plancherel theorem of Clifford Jacobi wavelet transform. We see that if $\phi_{l,m,a}^{\alpha,\beta,\underline{b}}(\underline{x}) \in L^p(\mathbb{R}^m)$, then $((\psi_{l,m}^{\alpha,\beta})_n * \phi_{l,m,a}^{\alpha,\beta,\underline{b}}) \in L^p(\mathbb{R}^m)$. Now, we consider

$$\begin{aligned}
&\left[\int_{\mathbb{R}^m} |((\psi_{l,m}^{\alpha,\beta})_n * f)(\underline{x}) - f(\underline{x})|^p dv(\underline{x})\right]^{\frac{1}{p}} \\
&= \left[\int_{\mathbb{R}^m} dv(\underline{x})|\int_{\mathbb{R}^m} (\psi_{l,m}^{\alpha,\beta})_n(\underline{x}-\underline{u})f(\underline{u})d\underline{u} - f(\underline{x})|^p\right]^{\frac{1}{p}}.
\end{aligned}$$

Putting $f(\underline{x}) = \int_{\mathbb{R}^m} f(\underline{x})(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})dv(\underline{u})$ in above equation, we get

$$\begin{aligned}
&\left[\int_{\mathbb{R}^m} dv(\underline{x})|\int_{\mathbb{R}^m}(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})(f(\underline{x}-\underline{u})-f(\underline{x}))dv(\underline{u})|^p\right]^{\frac{1}{p}}\\
&\leq \left[\int_{\mathbb{R}^m} dv(\underline{x})\int_{|\underline{u}|>\delta}|(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})|^p|f(\underline{x}-\underline{u})-f(\underline{x})|^p dv(\underline{u})|^p\right]^{\frac{1}{p}}\\
&+\left[\int_{\mathbb{R}^m} dv(\underline{x})\int_{|\underline{u}|\leq\delta}|(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})|^p|f(\underline{x}-\underline{u})-f(\underline{x})|^p dv(\underline{u})|^p\right]^{\frac{1}{p}}\\
&\leq \int_{|\underline{u}|>\delta} dv(\underline{u})|(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})|\left[\int_{\mathbb{R}^m} dv(\underline{x})|f(\underline{x}-\underline{u})-f(\underline{x})|^p|^p\right]^{\frac{1}{p}}\\
&+\int_{|\underline{u}|\leq\delta} dv(\underline{u})|(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})|\left[\int_{\mathbb{R}^m} dv(\underline{x})|f(\underline{x}-\underline{u})-f(\underline{x})|^p|^p\right]^{\frac{1}{p}}\\
&\leq \int_{|\underline{u}|>\delta} dv(\underline{u})|(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})|(2\parallel f\parallel_p)\\
&+\int_{|\underline{u}|\leq\delta} dv(\underline{u})|(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})|\sup_{|\underline{u}|<\delta}\left[\int_{\mathbb{R}^m} dv(\underline{x})|f(\underline{x}-\underline{u})-f(\underline{x})|^p|^p\right]^{\frac{1}{p}}.
\end{aligned}$$

The term $\int_{|\underline{u}|>\delta} dv(\underline{u})|(\psi^{\alpha,\beta}_{l,m})_n(\underline{u})|(2\parallel f\parallel_p)\to 0$ by (Def. 2.5(3′)) and

$$\sup_{|\underline{u}|<\delta}\left[\int_{\mathbb{R}^m} dv(\underline{x})|f(\underline{x}-\underline{u})-f(\underline{x})|^p|^p\right]^{\frac{1}{p}}\to 0.$$

This implies that both terms on the right-hand side of the above inequality tend to zero. Hence, we obtain

$$\lim_{n\to\infty}\parallel f*(\psi^{\alpha,\beta}_{l,m})_n - f\parallel_p = 0,$$

or

$$\lim_{n\to\infty}\parallel f_n - f\parallel_p = 0,$$

for $f\in L^p(\mathbb{R}^m)$. This completes the proof of Theorem 2.8. □

3. Feichtinger Space

The Feichtinger space originally was introduced in the technical report [17]. The popularity of the Feichtinger space increased with the turn of century due to [20,28]. Since then, the Feichtinger space and the more general modulation spaces have appeared in many books, e.g., [7,14]. For more recent applications (see [4, 6, 29]).

The Feichtinger space is a Banach space, which is dense in the space $L^2(\mathbb{R}^m)$. The Feichtinger space S_0 is much larger than Schwartz space $S(\mathbb{R}^m)$ and shares many properties such as both spaces are invariant under Fourier transform and time-frequency shifts. It was introduced by Feichtinger [18] and Reiter and Stegman [28] and proved that time-frequency shifts and Fourier transform are isometries on S_0. All members of S_0 are continuous and integrable functions. Also, the space S_0 does not depend on differentiability and it is the space of all functions on $\mathbb{R}^m$ which are represented in time-frequency domain by integrable functions [19]. In problem of sampling it has been observed that there are smooth functions $f\in L^2(\mathbb{R}^m)$ where the samples $\{f(n)\}_{n\in\mathbb{Z}}$ do not have any decay. For elements in

S_0 this degeneracy is no longer possible (see [26, Theorem 5.7 (ii)]). A generic function in $L^2(\mathbb{R}^m)$ does not even have a continuous representative. The Schwartz functions are individually so well-behaved that the pointwise product of $f, g \in S(\mathbb{R}^m)$ satisfies $f.g \in S(\mathbb{R}^m)$. Te individual niceness comes at the cost of weak collective properties. There is no way to make $S(\mathbb{R}^m)$ into a Banach space. Although, individual elements in $S(\mathbb{R}^m)$ are well-behaved, the space $S(\mathbb{R}^m)$ has poor collective properties. There is a trade-off between the collectively well-behaved $L^2(\mathbb{R}^m)$ and individually well-behaved $S(\mathbb{R}^m)$. The Feichtinger space provides a suitable spot between these two extremes.

For many purposes the short-time Fourier transform (STFT) of a signal is easier to handle, because it depends in a linear way on the analyzed signal. The (STFT) is, in language of applied time frequency analysis [20], a joint time-frequency depiction of f. As such, we should suspect that elements in S_0 are well-behaved both in time and frequency. Now, we define the norm of a function f in S_0 as the L^1-norm of short-time Clifford Fourier transform $\tilde{V}_g(f)$ with respect to Gaussian window g:

$$\| f \|_{S_0} = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |\tilde{V}_g f(\underline{b}, \underline{w})| dv(\underline{b}) dv(\underline{w}),$$

where $g(\underline{b}) = e^{-\pi <\underline{b},\underline{b}>}$ and the short-time Clifford Fourier transform $\tilde{V}_g f(\underline{b}, \underline{w})$ of analyzing function f is defined as

$$\tilde{V}_g f(\underline{b}, \underline{w}) = \int_{\mathbb{R}^m} f(\underline{t}) \overline{g(\underline{t} - \underline{b})} e^{-2\pi <\underline{w},\underline{t}>} dv(\underline{t}) = < f, E_{\underline{w}} T_{\underline{b}}(g) >,$$

where $\underline{b}, \underline{w} \in \mathbb{R}^m$. The Gaussian function g can be replaced by an arbitrary function from S_0, i.e, by the trapezoidal function, triangle function, or any Schwartz function [25]. The compactly supported function is in S_0 if and only if its Fourier transform is integrable. Due to the Fourier invariance of S_0, any integrable band limited function is in S_0. We know that if $(\psi^{\alpha,\beta}_{l,m})_\varepsilon \in S(\mathbb{R}^m)$ then $(\hat{\psi}^{\alpha,\beta}_{l,m})_\varepsilon \in S(\mathbb{R}^m)$ and $S(\mathbb{R}^m)$ is dense in $L^p(\mathbb{R}^m), p \geq 1$ this implies that $(\psi^{\alpha,\beta}_{l,m})_\varepsilon \in L^1(\mathbb{R}^m)$ and hence $(\psi^{\alpha,\beta}_{l,m})_\varepsilon \in S_0$.

Now, first we show that if $g, f \in S_0$, then the mapping $(\underline{b}, \underline{w}) \to E_{\underline{w}} T_{\underline{b}}(g) : \mathbb{R}^m \times \mathbb{R}^m \to L^1(\mathbb{R}^m)$ is continuous. $E_{\underline{w}} T_{\underline{b}}$ is the composition of two functions translation $T_{\underline{b}}$ and modulation $E_{\underline{w}}$. It is evident that $T_{\underline{b}}$ is continuous. To prove the continuity of $E_{\underline{w}}$, we need to show the following:

$$|E_{\underline{w}}(g) - E_{\underline{w'}}(g)| \to 0, \underline{w} > \underline{w'},$$

if $|\underline{w} - \underline{w'}| \to 0$.
We have

$$\begin{aligned} |E_{\underline{w}}(g) - E_{\underline{w'}}(g)| &= | \int_{\mathbb{R}^m} (e^{2\pi i <\underline{w},\underline{t}>} - e^{2\pi i <\underline{w'},\underline{t}>}) g(\underline{t}) dv(\underline{t})| \\ &\leq \int_{\mathbb{R}^m} |(e^{2\pi i <\underline{w},\underline{t}>} - e^{2\pi i <\underline{w'},\underline{t}>}) g(\underline{t})| dv(\underline{t}) \\ &\leq \int_{\mathbb{R}^m} (1 - e^{2\pi i <(\underline{w}-\underline{w'}),\underline{t}>}) g(\underline{t})| dv(\underline{t}) < \varepsilon. \end{aligned}$$

Since f and $E_{\underline{w}} T_{\underline{b}}(g)$ are continuous, therefore $\tilde{V}_g f(\underline{b}, \underline{w}) =< f, E_{\underline{w}} T_{\underline{b}}(g) >$ is also continuous.

Now, we prove

Theorem 3.1. *Let* $g \in S_0$ *and* $(\psi^{\alpha,\beta}_{l,m})_\varepsilon$ *be an approximate identity such that* $(\psi^{\alpha,\beta}_{l,m})_\varepsilon \in S_0$ *and* $(\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{b}) = (\psi^{\alpha,\beta}_{l,m})_\varepsilon(-\underline{b})$ *for all* $\underline{b}$. *Then*

$$\| f - (f^{\alpha,\beta}_{l,m})_\varepsilon \|_2 \to 0 \quad as \quad \varepsilon \to 0,$$

where

$$(f^{\alpha,\beta}_{l,m})_\varepsilon(\underline{t}) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \tilde{V}_g f(\underline{b},\underline{w}) < (\psi^{\alpha,\beta}_{l,m})_\varepsilon * E_{\underline{w}} T_{\underline{b}}(g) > \underline{t} dv(\underline{b}) dv(\underline{w}),$$

and

$$f(\underline{t}) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \tilde{V}_g f(\underline{b},\underline{w}) E_{\underline{w}} T_{\underline{b}}(g) \underline{t} dv(\underline{b}) dv(\underline{w}).$$

Proof. We see that

$$\begin{aligned} f * (\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{b}) &= \int_{\mathbb{R}^m} f(\underline{t})(\psi^{\alpha,\beta}_{l,m})_\varepsilon(\underline{b}-\underline{t}) dv(\underline{t}) \\ &= < f, \overline{T_{\underline{b}}(\psi^{\alpha,\beta}_{l,m})_\varepsilon} > = < \tilde{V}_g f, \overline{\tilde{V}_g(T_{\underline{b}}(\psi^{\alpha,\beta}_{l,m})_\varepsilon)} > \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \tilde{V}_g f(\underline{b},\underline{w}) < E_{\underline{w}} T_{\underline{b}}(g), \overline{T_{\underline{b}}(\psi^{\alpha,\beta}_{l,m})_\varepsilon} > dv(\underline{b}) dv(\underline{w}) \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \tilde{V}_g f(\underline{b},\underline{w}) < ((\psi^{\alpha,\beta}_{l,m})_\varepsilon * E_{\underline{w}} T_{\underline{b}}(g)) \underline{t} dv(\underline{b}) dv(\underline{w}). \end{aligned}$$

Since $f * (\psi^{\alpha,\beta}_{l,m})_\varepsilon \to f \in S_0$ and $((\psi^{\alpha,\beta}_{l,m})_\varepsilon * E_{\underline{w}} T_{\underline{b}}(g)) \to E_{\underline{w}} T_{\underline{b}}(g)) \in S_0$. Since S_0 is dense in $L^2(\mathbb{R}^m)$, we obtain

$$\| f * (\psi^{\alpha,\beta}_{l,m})_\varepsilon - f \|_2 \to 0 \quad as \quad \varepsilon \to 0.$$

This completes the proof of Theorem 3.1. □

4. Practical Applications of the Work.

The Fourier transform of a time-domain signal gives the frequency-amplitude representation of the signal. The frequency spectrum of a real valued signal is always symmetric. The information which can not be seen in the time domain can be seen in the frequency domain. For example, the typical shape of a *ECG* signal (Electrocardiography, Graphical recording of heart's electrical activity) is well known to Cardiologists and they use the time-domain *ECG* signals which are recorded on strip-charts to analyze these signals. Now, computerized *ECG* recorders utilize the frequency information to check the pathological condition. Some times the frequency content of the signal is more useful to diagnose the existence of pathological condition. There is no frequency information is available in the time-domain signal and no time information is available in Fourier transformed signal. The Fourier transform gives the frequency information of the signal at all time. The Fourier transform can be used for non-stationary signals if we need only spectral components but not interested where these occur. If we need both information then Fourier transform is not suitable. All biological signals are non-stationary such as *ECG*, *EEG* (Electro activity of the brain, electroencephalograph) and *EMG* (Electrical activity of the muscles, electromyogram). The wavelet transform gives the time-frequency representation of the non-stationary signals. Short-time Fourier transform (STFT) also give this information. The wavelet transform was developed as an alternative to the (STFT). In

(STFT) the signal is divided into small enough segments where the signal can be assumed stationary.

5. Conclusions

Arfaoui and Ben Mabrouk [2] introduced new classes of wavelet functions to the framework of Clifford analysis. Some classes of monogenic polynomials are provided, which extend the classical Jacobi polynomials in the context of Clifford analysis. They proved the reconstruction formula and Fourier-Plancherel rule in L^2 space but if $f \in L^1(\mathbb{R}^m)$ or $L^p(\mathbb{R}^m), p \neq 2, 1 \leq p < \infty$, then the function f may not be reconstructed by the formula given by Arfaoui and Ben Mabrouk. Therefore, in this paper we have tried to solve this problem by using approximate identity. Also, we have discussed the convergence of short-ime Clifford Fourier transform in Feichtinger space.

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References

[1] Arfaoui, S., Ben Mabrouk, A., (2017), Some ultraspheroidal monogenic Clifford Gegenbauer Jacobi polynomials and associated wavelets, Adv. Appl. Clifford Algebras, 27, pp. 2287-2306.

[2] Arfaoui, S., Ben Mabrouk, A., (2021), Some generalized Clifford-Jacobi polynomials and associated spheroidal wavelets, Anal. Theory Appl., 37 (10), pp. 1-24.

[3] Arfaoui, S., Ben Mabrouk, A., Cattani, C., (2020), New type of Gegenbauer-Jacobi-Hermite Monogenic polynomials and associated continuous Clifford wavelet transform: Some monogenic Clifford polynomials and associated wavelets, Acta Applicandae Math.,170 (3), Doi;10,1007/s 10440-020-00322-0.

[4] Austad, A., Enstad, U., (2020), Heisenberg modules as function spaces, J. Fourier Anal. Appl., 26, 24, https://doi.org/10.1007/s00041-020-09729-7.

[5] Banouch, H., Ben Mabrouk, A., (2020), A sharp Clifford wavelet Heisenberg-type uncertainty principle, J. Math. Phys., 61 (9), https://doi.org/10.1063/5.0015989.

[6] Bayer, D., Cordero, E., Gröchenig, K., Trapasso, S.I., (2020), Linear perturbations of the Wigner transform and the weyl quantization. In Advances in Microlocal and Time Frequency Analysis, Springer, pp.79-120.

[7] Bényi, Á., Okoudjou, K., (2020), Modulation Spaces, Birkhauser, Besel.

[8] Brackx, F., Delanghe, R., Sommen, F., (1982), Clifford Analysis, Pitman Publication, London.

[9] Brackx, F., De Schepper, N., Sommen, F., (2004), The Clifford- Gegenbauer polynomials and the associated continuous wavelet transform, Integral Transforms Special Funct., 15 (5), pp. 387-404.

[10] Brackx, F., De Schepper, N., Sommen, F., (2006), The two dimensional Clifford Fourier transform, J. Math. Imaging Vision, 26 (1-2), pp. 5-18.

[11] Brackx, F., De Schepper, N., Sommen, F., (2009), The Fourier transform in Clifford analysis, Adv. Imaging Electron Phys.,156, pp. 55-201, http://hdl.handle.net/1854/LU-681383.

[12] Brackx, F., De Schepper, N., Sommen, F., (2006), Clifford-Jacobi polynomials and the associated continuous wavelet transform in Eucllidean space, in: Qian T., Vai, M.I., Xu, Y. (eds) Wavelet Analysis and Applications, pp. 185-198.

[13] Brackx, F., De Schepper, N., Sommen, F., (2003), The Clifford-Laguerre Continuous Wavelet Transform, pp. 201-215.

[14] Cordero, E., Rodino, L., (2020), Time Frequency Analysis of Operators, De Gruyter.

[15] Craddock, M.J., Hogan, J. A., (2013), The fractional Clifford-Fourier kernel, vol. 2411. The Erwin Schrodinger International Institute for Mathematical Physics ESI Vienna, 27, pp. 1-28.

[16] Delanghe, R., (2001), Clifford analysis: history and perspectives, Comput. Methods Funct. Theory, 1 (1), pp. 107-153.

[17] Feichtinger, H. G., (1983), Modulation space on locally compact abelian groups, Technical Report, University of Vienna.

[18] Feichtinger, H. G., (1981), On a new Segal algebra, Monatsh. Math., 92 (4), pp. 269-289.

[19] Feichtinger, H. G., Gröchenig, K., (1992), Gabor wavelets and Heisenberg group, Gabor expansions and short time Fourier transform from the group theoretical point of view, In Wavelets: A Tutorial in Theory and Applications (Ed. C. Chui), Academic Press, Boston, pp. 359-397.
[20] Gröchenig, K., (2001), Foundations of Time-Frequency Analysis, Springer Science and Business Media.
[21] Hemmata, A., Rahbani, Z., (2010), Clifford wavelets and Clifford-valued MRAs, Iran. J. Math. Sci. Inf., 5 (1), pp. 7-18.
[22] Hitzer, E., Sangwine, S. J., (2013), Quaternion and Clifford Fourier transforms and wavelets. In: Hitzer, E., Sangwine, S.J. (eds) Trends in Mathematics, Birkhauser, Springer, Basel.
[23] Jaffard, S., (1989), Exposants de Hölder en des points donnés et coefficients d'ondelettes, C. R. Acad. Sci. Paris Sér. I. Math., 308, pp. 79-81.
[24] Jaffard, S., (1991), Pointwise smoothness, two microlocalization and wavelet coefficients, Pub. Mat., 35, pp. 155-168.
[25] Jakobsen, M. S., (2018), On a (no longer) new Segal algebra: A review of the Feichtinger algebra, J. Fourier Anal. Appl., 24, pp. 1579-1660.
[26] Jakobsen, M. S., Luef, F., (2019), Sampling and periodization of generators of Heisenberg modules, International Journal of Mathematics, 30 (10), https://doi.org/10.1142/S0129167X19500514.
[27] Mitrea, M., (1994), Clifford Wavelets, Singular Integrals and Hardy Spaces, Springer, New York.
[28] Reiter, H., Stegeman, J. D., (2000), Classical Harmonic Analysis and Locally Compact Groups, Second Edition, Clarendon Press, Oxford.
[29] Sugimoto, M., Wang, B., Zhang, R., (2015), Local well-posedness for the Davey-Stewartson equation in a generalized Feichtinger algebra, J. Fourier Anal. Appl., 21, pp. 1105-1129.
[30] Vieira, N., (2015), Cauchy-Kovalevskaya extension theorem in fractional Clifford analysis, Complex Anal. Oper. Theory, 9 (5), pp. 1089-1109.

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