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NUMERICAL SOLUTIONS OF INTEGRAL EQUATIONS USING SHIFTED FRACTIONAL VIETA-FIBONACCI POLYNOMIALS

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Abstract. In this paper, we propose a numerical technique to find approximate solutions of generalized Abel's integral equations, GAIEs, of the first and second kinds, based on the use of shifted fractional Vieta-Fibonacci polynomials. This possibility is created by establishing a relationship between the appearance of Abel's integral equations and the definition of fractional derivatives. The method reduces the numerical solutions of the Abel's integral equations to a system of algebraic equations. Convergence analysis and error bound of the proposed method are studied. The applicability and efficiency of the given methodology are demonstrated by a considerable number of examples. These examples show the remarkable superiority of our method.

Keywords: Singular Volterra integral equation, Generalized Abel's integral equatio, Fractional calculus, Vieta-Fibonacci polynomial, Collocation method.

AMS Subject Classification: 83-02, 99A00

1. Introduction

The importance of Abel integral equation, AIEs, appeared since Abel formulated it in general and presented its analytical solution [1]. The Abel integral equation provides an important tool in modeling of many complex systems in basic sciences and engineering from mathematical physics, biological processes, and mechanics [2–4].

In recent years, researchers have used many different methods, including analytical and numerical methods [5,6], to approximate the solution of the Volterra integral equations such as Block-Pulse functions [7,8], Laplace transform [9,10], spline collocation methods [4,11], and Taylor expansion [12]. There are also powerful methods that use polynomials and wavelets to solve Abel integral equation: shifted Legendre polynomials [13], Chebyshev polynomials [14], Jacobi polynomials [15, 16], Hermite wavelets [17, 18], Legendre wavelets [12, 13], Haar wavelets [19, 20], Boubaker wavelets [1], Bernstein wavelets [21].
Recently, Vieta-Fibonacci polynomials have been used frequently for function approximation because of their many special properties. In fact, the existence of these special features makes working with these polynomials easier and more effective. Such as recurrence relations defining these polynomials, analytically representation using the power series and also this Vieta-Fibonacci polynomial has a special property of being orthogonal in the entire domain $[-2, 2]$, equipped with a weight function. In this article, we use Vieta-Fibonacci polynomials for solving singular Volterra integral equations of the first and second kinds.

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Abel integral equations in the first and second kind [22] often appears as Eqs. (1) and (2), respectively:

$$\xi(x) = \int_0^x (x-t)^{-0.5}\eta(t)dt, \tag{1}$$

$$\eta(t) = \xi(x) + \int_0^x (x-t)^{-0.5}\eta(t)dt, \tag{2}$$

where $\xi(x)$ is a continuous function and $x \geq 0$.

Generalized Abel integral equations, GAIEs, in the first and second kind [9] with same α on the interval $0 < \alpha \leq 1$ are also given as Eqs. (3) and (4), respectively:

$$\xi(x) = \int_0^x (x-t)^{-\alpha}\eta(t)dt, \tag{3}$$

$$\eta(t) = \xi(x) + \int_0^x (x-t)^{-\alpha}\eta(t)dt, \tag{4}$$

where $x \geq 0$ and $\eta(x)$ is the unknown function and the expression $(x-t)^{-\alpha}$ is called Abel kernel [23].

The aim of this work is to present a new and easy-to-implement scheme for finding the approximate solution of first and second type GAIEs based on shifted Vita-Fibonacci polynomials. We show that fractional calculus plays an important role in the numerical solution of AIEs with the presented methodology. In fact the equation can be well described in terms of fractional integrals and the main motivation of this method is to use the properties of fractional integrals and Vita-Fibonacci polynomials for solving AIEs, numerically. The use of Vita-Fibonacci polynomials and their properties such as orthogonality helps to approximate the partial integral and thus, the original problem is reduced to a system of linear algebraic equations. Numerical examples and evaluated errors show the applicability and efficiency of the method.

The outline of this article is organized as follows. In section 2, we study the main preliminaries in fractional calculus. Moreover, in this section the shifted Vieta-Fibonacci polynomials and some of their properties are introduced. The function approximation by using Vieta-Fibonacci polynomials and their operational matrix of fractional integration are given in section 3. Also, description of proposed method to solve first and second kind GAIEs is given in section 3. section 4 is dedicated to studying convergence and error analysis of the given method. In section 5, we provide some of the numerical examples by applying the proposed approximation method.

2. Preliminaries

Definition 2.1. *For constant real number $r \in \mathbb{R}$, a real-valued function $\eta(x)$, $x \in \mathbb{R}^+$, is said to belong to the space $\mathbb{C}_r$ if there exists a real number $p > r$ such that $\eta(x) = x^p\tilde{\eta}(x)$ and $\tilde{\eta}(x) \in \mathbb{C}(0,\infty)$. Additionaly, $\mathbb{C}_r^n$, for all $n \in \mathbb{N} \cup \{0\}$, describes the set of real-valued functions $\eta \in \mathbb{C}_r$ with $\eta^{(n)}(x) \in \mathbb{C}_r$.*

Definition 2.2. *[19] Let $\alpha > 0$, $\eta(x) \in \mathbb{C}_r$, and $r \geq -1$ then, the Riemann-Liouville fractional integral operator of order α is defined by:*

$$\mathbb{I}_x^\alpha \eta(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\eta(\tau)d\tau. \tag{5}$$

Due to the definition of Riemann-Liouville fractional integral by Eq. (5), obviously

$$\mathbb{I}_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha}, \tag{6}$$

holds for $\beta > -1$.

Proposition 2.1. *For any constant $\lambda \in \mathbb{R}$, the following property is satisfied for Riemann-Liouville fractional integral operator:*

$$\mathbb{I}_x^\alpha \lambda\eta(x) = \lambda\mathbb{I}_x^\alpha \eta(x).$$

2.1. Vieta-Fibonacci Polynomials.

The Vieta-Fibonacci polynomials $\hat{VF}_n(x)$ are given in the interval $[-2,2]$ by [24]:

$$\hat{VF}_n(x) = \frac{sin(n\theta x)}{sin(\theta x)}, \tag{7}$$

where $x = 2cos(\theta)$ and $0 \leq \theta < \pi$. The nth term of the Vieta-Fibonacci polynomials can be obtained using the following recursive formula:

$$\hat{VF}_0(x) = 0, \quad \hat{VF}_1(x) = 1, \quad \hat{VF}_n(x) = x\hat{VF}_{n-1}(x) - \hat{VF}_{n-2}(x), \quad \forall n \geq 2. \tag{8}$$

The shifted Vieta-Fibonacci polynomials, denoted by $VF_n(x)$, are defined in the interval $[0,1]$ and given by recursive formula [25]:

$$VF_0(x) = 0, \quad VF_1(x) = 1, \quad VF_n(x) = (4x-2)VF_{n-1}(x) - VF_{n-2}(x), \quad \forall n \geq 2. \tag{9}$$

Theorem 2.1. *[26] The shifted Vieta-Fibonacci polynomials are orthogonal with respect to the weight function $\omega(x) = (x - x^2)^{\frac{1}{2}}$ on the interval $[0, 1]$, that is:*

$$\begin{aligned} < VF_n(x), VF_m(x) > &= \int_0^1 \omega(x) VF_n(x) VF_m(x) dx \\ &= \begin{cases} 0 & n \neq m, \\ \frac{\pi}{8} & n = m \neq 0. \end{cases} \end{aligned} \tag{10}$$

2.2. **Function Approxamation.** A function $\eta(x) \in L^2_w(0, 1)$ can be written as a linear combination of shifted Vieta-Fibonacci polynomials as follows:

$$\eta(x) = \sum_{i=1}^{\infty} c_i VF_i(x), \tag{11}$$

where $c_i \in \mathbb{R}$. By truncating the infinite series in Eq.(11) up to order N, $\eta(x)$ can be approximated as follows:

$$\begin{aligned} \eta(x) &\approx \sum_{i=1}^{N} c_i VF_i(x) \\ &\approx C^T \Theta_N(x), \end{aligned} \tag{12}$$

where

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}_{N\times 1}, \Theta_N(x) = \begin{pmatrix} VF_1(x) \\ VF_2(x) \\ \vdots \\ VF_N(x) \end{pmatrix}_{N\times 1}. \tag{13}$$

$\Theta_N(x)$ can be expressed as the following expression [27]:

$$\Theta_N(x) = \Lambda \rho(x), \tag{14}$$

where

$$\rho(x) = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{N-1} \end{pmatrix}_{N\times 1}, \Lambda = \begin{pmatrix} a_{1,1} & 0 & 0 & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 & \dots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & a_{N,32} & a_{N,4} & \ddots & a_{N,N} \end{pmatrix}_{N\times N}, \tag{15}$$

and $a_{i,j}$ are given by:

$$a_{i,j} = \begin{cases} \frac{(-1)^{i-j-1} 2^{2j} \Gamma(i+j+1)}{\Gamma(i-j)\Gamma(2j+2)} & i \geq j, \\ 0 & o.w. \end{cases} \tag{16}$$

3. Description of the method

In this section, we present a numerical method for obtaining the numerical solution of GAIEs given by Eqs.(3) and (4). By Eq.(5), the Riemann-Liouville fractional integral of order $1-\alpha$, $0 < \alpha \leq 1$, for $\eta(x)$ is:

$$\mathbb{I}_x^{1-\alpha} \eta(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\tau)^{-\alpha} \eta(\tau) d\tau.$$

Then, we obtain a key formula:

$$\Gamma(1-\alpha) \mathbb{I}_x^{1-\alpha} \eta(x) = \int_0^x (x-\tau)^{-\alpha} \eta(\tau) d\tau. \tag{17}$$

Comparing Eq.(3) with Eq.(17) gives:

$$\xi(x) = \Gamma(1-\alpha) \mathbb{I}_x^{1-\alpha} \eta(x). \tag{18}$$

Now, we use shifted Vieta-Fibonacci polynomials for approximating $\mathbb{I}_x^{1-\alpha}\eta(x)$ in (18):

$$\begin{aligned}\mathbb{I}_x^{1-\alpha}\eta(x) &= \mathbb{I}_x^{1-\alpha}C_N^T\Theta_N(x)\\ &= C^T\Lambda\mathbb{I}_x^{1-\alpha}\varphi(x).\end{aligned} \tag{19}$$

By Eq.(6), we have:

$$\mathbb{I}_x^{1-\alpha}\varphi(x) = \begin{pmatrix}\frac{\Gamma(1)}{\Gamma(2-\alpha)}x^{1-\alpha} & \frac{\Gamma(2)}{\Gamma(3-\alpha)}x^{2-\alpha} & \dots & \frac{\Gamma(N)}{\Gamma(N-\alpha+1)}x^{N-\alpha}\end{pmatrix}^T, \tag{20}$$

and by considering $\varphi(x) = \Lambda^{-1}\Theta_N(x)$, we can conclude:

$$\mathbb{I}_x^{1-\alpha}\eta(x) = x^{1-\alpha}C^T\Lambda G\Lambda^{-1}\Theta_N(x), \tag{21}$$

where

$$G = \begin{pmatrix}\frac{\Gamma(1)}{\Gamma(2-\alpha)} & 0 & \dots & 0\\ 0 & \frac{\Gamma(2)}{\Gamma(3-\alpha)} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \frac{\Gamma(N)}{\Gamma(N-\alpha)}\end{pmatrix}. \tag{22}$$

3.1. **Numerical Solution of the First Kind GAIEs using Shifted Vieta-Fibonacci Polynomials.** In order to solve Eq.(3), we use Eqs.(18) and (21) to obtain the following equivalent equation:

$$x^{1-\alpha}C^T\Lambda G\Lambda^{-1}\Theta_N(x) = \frac{1}{\Gamma(1-\alpha)}\xi(x). \tag{23}$$

Using collocation points $\{x_i\}_{i=1}^N$, such as uniform collocation points, in Eq.(23) we obtain a linear system of algebraic equations that gives unknown coefficient vector C:

$$x_i^{1-\alpha}C^T\Lambda G\Lambda^{-1}\Theta_N(x_i) = \frac{1}{\Gamma(1-\alpha)}\xi(x_i), \quad i = 1, \dots, N. \tag{24}$$

By specifying the constant coefficients, the approximate solution of the first kind GAIEs is obtained.

3.2. **Numerical Solution of the Second Kind GAIEs using Shifted Vieta-Fibonacci Polynomials.** Similar to the first kind GAIEs, Eq.(4) can be written for the second kind GAIEs in the following equivalent form:

$$\eta(x) - \Gamma(1-\alpha)\mathbb{I}_x^{1-\alpha}\eta(x) = \xi(x), \tag{25}$$

In order to solve Eq.(4), we use Eqs.(21) and (25) to obtain the following form of equivalent equation:

$$C^T(I - \Gamma(1-\alpha)x^{1-\alpha}\Lambda G\Lambda^{-1})\Theta_N(x) = \xi(x). \tag{26}$$

Using collocation points $\{x_i\}_{i=1}^N$ in Eq.(26), we obtain a linear system of algebraic equations that gives unknown coefficient vector C:

$$C^T(I - \Gamma(1-\alpha)x_i^{1-\alpha}\Lambda G\Lambda^{-1})\Theta_N(x_i) = \xi(x_i), \qquad i = 1, \dots, N. \tag{27}$$

By specifying the constant coefficients, the approximate solution of the second kind GAIEs is obtained.

4. Convergence and Error Analysis

In this section, we study and discuss the error estimation for the function approximation and error analysis of the proposed method. To this end, we consider the Sobolev space, $\mathbb{H}^m(a, b)$:

$$\mathbb{H}^m(a, b) = \{g : (a, b) \to \mathbb{R} | g^{(i)} \in L^2(a, b), \qquad i = 0, 1, \dots, m\}, \tag{28}$$

where $g^{(i)}$ denotes the i-th order derivative of g. Moreover, we consider the norm defined by:

$$\| g \|_{\mathbb{H}^m} = \sqrt{\sum_{k=0}^{m} \| g^{(k)} \|_{L^2}^2}. \tag{29}$$

Lemma 4.1. *[28] Assume that $g \in \mathbb{H}^m(-1,1)$ for $m \in \mathbb{N} \cup \{0\}$ and $P_N(g)(x) = \sum_{i=1}^{N} c_i VF_i(x)$, be the best approximation of g. Then, there is a constant $K > 0$ such that:*

$$\| g - P_N(g) \|_{L^2} \leq KN^{-m} |g|_{\mathbb{H}^{m;N}}, \tag{30}$$

where

$$|g|_{\mathbb{H}^{m;N}} = \sqrt{\sum_{l=n}^{m} \| g^{(l)} \|_{L^2}^2}. \tag{31}$$

Lemma 4.2. *Let $g \in \mathbb{H}^m(0,b)$ and $\bar{g} : (-1,1) \to \mathbb{R}$ be defined as $\bar{g}(x) = g\Big(\frac{b(x+1)}{2}\Big)$ on the interval $(-1,1)$. Then:*

$$\| \bar{g}^{(l)} \|_{L^2} = 2^{\frac{1}{2}-l} b^{l-\frac{1}{2}} \| g^{(l)} \|_{L^2}, \qquad l = 0, 1, \ldots, m. \tag{32}$$

Proof. Let $y = \frac{b(x+1)}{2}$. Then:

$$\begin{aligned} \| \bar{g}^{(l)} \|_{L^2(-1,1)}^2 &= \int_{-1}^{1} |\bar{g}^{(l)}(x)| dx \\ &= \int_{-1}^{1} \Big| g^{(l)}\Big(\frac{b(x+1)}{2}\Big)\Big|^2 dx \\ &= 2^{1-2l} b^{2l-1} \int_0^b \Big| g^{(l)}(y) \Big|^2 dy \\ &= 2^{1-2l} b^{2l-1} \| g^{(l)} \|_{L^2}^2 . \end{aligned} \tag{33}$$

This completes the proof. □

Theorem 4.1. *Let $g(x) \in \mathbb{H}^{m+1}(0,b)$ be the exact solution of a GAIE and $g_N(x)$ be the approximate solution computed using the proposed method. Then, there is a constant $K > 0$ such that:*

$$\| g - g_N \|_{L^2} \leq KN^{-m} |g|_{\mathbb{H}^{m;n}}, \tag{34}$$

Proof. Let $P_N(g)(x) = \sum_{i=1}^{N} c_i VF_i(x)$ be the best approximation of $g(x)$. Then:

$$\begin{aligned} \| g - g_N \|_{L^2}^2 &\leq \| g - P_N(g) \|_{L^2}^2 \\ &= \frac{b}{2} \| \bar{g} - P_N(\bar{g}) \|_{L^2}^2 . \end{aligned} \tag{35}$$

By Lemmas 4.1 and 4.2, we conclude that:

$$\begin{aligned} \| g - \bar{g}_N \|_{L^2}^2 &\leq \frac{b}{2} KN^{-2m} \sum_{l=n}^{m} \| \bar{g}^{(l)} \|_{L^2}^2 \\ &= KN^{-2m} \sum_{l=n}^{m} 2^{-2l} b^{2l} \| g^{(l)} \|_{L^2}^2 . \end{aligned} \tag{36}$$

This completes the proof. □

5. Numerical simulations

In this section, the efficiency and accuracy of the proposed method are evaluated using several numerical examples and compared with other methods presented in previous works. All computations are done using Matlab software and the error of the results is calculated with mean absolute error (MAE) in $\{x_i\}_{i=1}^{M}$:

$$||\eta(x) - \eta_N(x)||_{MAE} \approx \frac{\sum_{i=1}^{M} |\eta(x_i) - \eta_N(x_i)|}{M}. \tag{37}$$

Example 5.1. *Let us consider the following Abel integral equation of the first kind [29]:*

$$\frac{2\sqrt{x}(8x^2 + 10x + 15)}{15} = \int_0^x \frac{\eta(\tau)}{\sqrt{x - \tau}} d\tau. \tag{38}$$

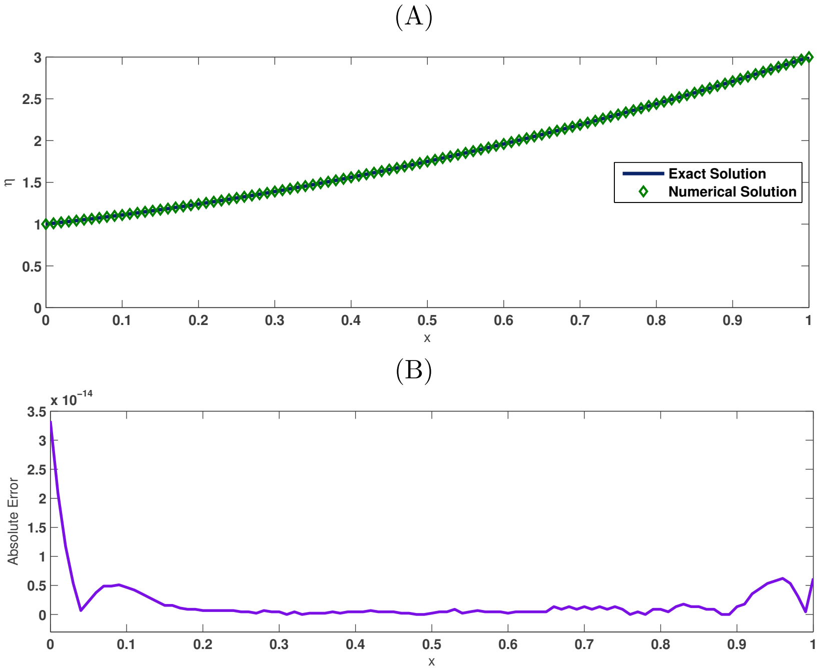

FIGURE 1. Numerical solution of Example 5.1 includes: (A) comparison our numerical results with the exact solution and (B) absolute error.

The exact solution is $\eta(x) = x^2 + x + 1$. We have solved the integral equation by applying the procedure introduced in Section 3 by taking $N = 10$ terms of Vieta-Fibonacci polynomailas; i.e. $\eta_{10}(x) = \sum_{i=1}^{10} c_i VF_i(x)$.
High ability of the method is obvious from Figure 1 and Table 1. For $N = 3$, the proposed method predicts the solution of the Eq.(38) as Eq.(39), which is consistent with the exact solution, $\eta(x)$:

$$\eta_3(x) = 1.000000000000000x^2 + 1.000000000000001x + 1.000000000000000. \tag{39}$$

TABLE 1. Comparing the numerical results for Example 5.1 obtained by the proposed method (based on Vieta-Fibonacci polynomailas) with the methods based on Legendre, 1^{st} Chebyshev and 2^{nd} Chebyshev polynomials proposed in [29].

		Methods in [29]			Proposed Method
x	Exact	Legendre wavelet	1^{st} Chebyshev wavelet	2^{nd}Chebyshev wavelet	Vieta Fibonacci
0.1	1.11	0.9295325636	1.0719999995	1.110000001	1.109999999999995
0.2	1.24	1.115745774	1.1546666661	1.240000002	1.239999999999999
0.3	1.39	1.309633163	1.2479999994	1.389999998	1.390000000000000
0.4	1.56	1.511194733	1.3519999994	1.560000001	1.560000000000001
0.5	1.75	1.720430483	1.466666666	1.750000001	1.750000000000000
0.6	1.96	1.937340412	1.5919999994	1.959999998	1.960000000000000
0.7	2.19	2.161924522	1.7279999994	2.190000001	2.190000000000001
0.8	2.44	2.394182812	1.8746666661	2.440000001	2.440000000000000
0.9	2.71	2.634115282	2.0319999994	2.709999998	2.710000000000001
MAE	–	1.804674364e-1	5.653333339e-1	2e-9	1.794989953100253e-15

Example 5.2. *Let us consider the Abel's integral equation of second kind with* $x \in [0, 1]$ *as the following [17, 22, 39]:*

$$4\eta(x) = \frac{4}{\sqrt{x+1}} - sin^{-1}(\frac{1-x}{1+x}) + \frac{\pi}{2} - \int_0^x \frac{\eta(t)}{\sqrt{x-t}} dt,$$

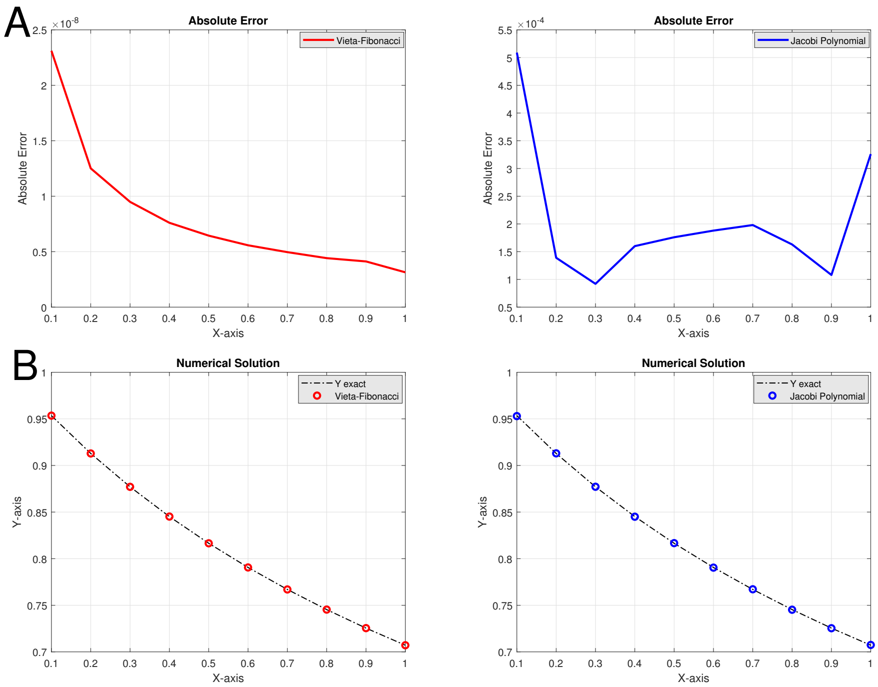

FIGURE 2. Comparison between our method and method in [22] for Example 5.2 (A) absolute error, (B) exact and approximate solutions.

TABLE 2. Comparison of approximate results and absolute errors reported in [22] of Example 5.2.

		Our Method		[22]	
x	Exact solution	Approximate solution	Absolute Error	Approximate solution	Absolute Error
0.1	0.953463	0.953462	2.311367e-08	0.952954	0.000509
0.2	0.912871	0.912870	1.251010e-08	0.913009	0.000139
0.3	0.877058	0.877058	9.490916e-09	0.877150	0.000092
0.4	0.845154	0.845154	7.604130e-09	0.844994	0.000160
0.5	0.816497	0.816496	6.444178e-09	0.816672	0.000176
0.6	0.790569	0.790569	5.575872e-09	0.790381	0.000188
0.7	0.766965	0.766964	4.958316e-09	0.767163	0.000198
0.8	0.745356	0.745355	4.417058e-09	0.745193	0.000163
0.9	0.725476	0.725476	4.119741e-09	0.725369	0.000108
1	0.707107	0.707106	3.139302e-09	0.707432	0.000326
$M_{Er}\lvert\eta(x)-\eta_n(x)\rvert$		0.953462	2.311×10^{-8}	0.952954	5.09×10^{-4}

where the exact solution of this example is $\eta(x) = \dfrac{1}{\sqrt{x+1}}$.

Table 2 compares the numerical solutions of Example 5.2 with the results of [22]. Also, in the results of [39] for this example the best error is 3.99925×10^{-5} while the absolute error of our proposed method does not exceed 2.31136×10^{-8}.

Example 5.3. *Let us consider the Abel's integral equation of first kind with* $x\in[0,1]$ *as the following [30, 31]:*

$$e^x - 1 = \int_0^x \frac{\eta(t)}{\sqrt{x-t}}dt,$$

with the exact solution $\eta(x) = \dfrac{e^x}{\sqrt{x}}erf(\sqrt{x})$, *where* $erf(x)$ *denotes the following error function*

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

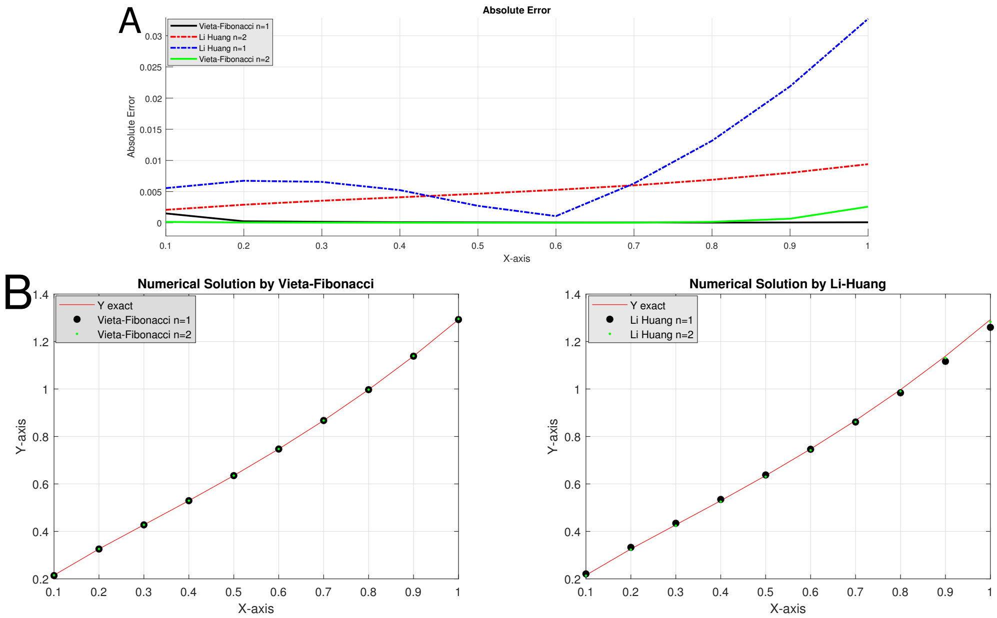

Figure 3. Comparison of numerical solutions of Example 5.3 between our method and method of [31]. (A): Absolute error, (B): Exact and approximate solutions.

Table 3. Comparison of numerical solutions of Example 5.3 between our method and method of [31].

		Our Method		[31]	
x	Exact	n=1	n=2	n=1	n=2
0.1	0.21529	0.21379	0.21515	0.22085	0.21321
0.2	0.32588	0.32564	0.32583	0.33262	0.32298
0.3	0.42757	0.42741	0.42754	0.43413	0.42403
0.4	0.52933	0.52924	0.52931	0.53457	0.52524
0.5	0.63503	0.63496	0.63502	0.63775	0.63037
0.6	0.74704	0.74699	0.74703	0.74597	0.74176
0.7	0.86719	0.86714	0.86721	0.86088	0.86118
0.8	0.99709	0.99706	0.99724	0.98393	0.99019
0.9	1.13830	1.13825	1.13895	1.11642	1.13029
1	1.29239	1.29247	1.29497	1.25968	1.28299
$M_{Er}\lvert\eta(x)-\eta_n(x)\rvert$		0.00149	2.584×10^{-3}	0.03271	9.398×10^{-3}

Example 5.4. *[31, 36]. Let us consider the Generalized Abel's integral equation of first kind with* $\alpha=\frac{1}{3}$, $0\le x\le 1$ *and* $\xi(x)=x^{\frac{5}{3}}$ *as the following:*

$$\xi(x)=\int_0^x (x-t)^{-\frac{1}{3}}\eta(t)dt,$$

where the exact solution is $\eta(x)=\frac{10}{9}x$.

Comparison of approximate results and maximum absolute errors obtained by our proposed method with [36] are provided in Table 4.

Example 5.5. *[32, 33]. Let us consider the Generlazed Abel's integral equation of second kind with* $\alpha=\frac{1}{4}$ *as the following :*

$$\eta(t)=2x-\frac{32}{21}x^{\frac{7}{4}}+\frac{4}{3}x^{\frac{3}{4}}-\int_0^x (x-t)^{-\frac{1}{4}}dt$$

where the exact solution is $\eta(x)=1-2x$.

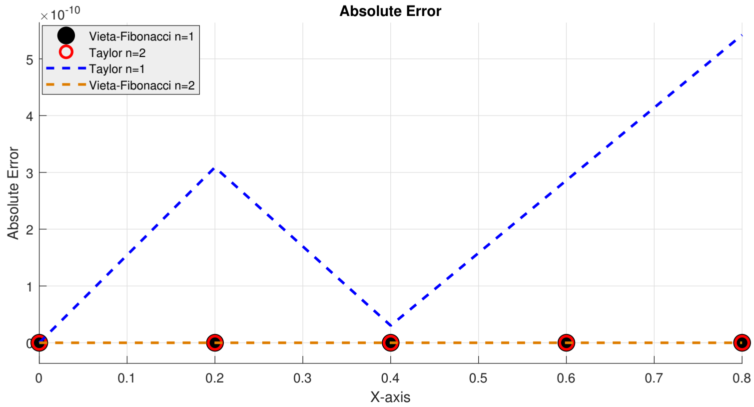

FIGURE 4. Comparison of absolute error between the results of our method and method of [36] for Example 5.4

TABLE 4. Comparison of numerical solutions of Example 5.4 between our method and method of [36].

		Method in [36]		Our Method	
x	Exact	N=4	N =6	N =4	N =6
0	0.00000	0.000E-0	0.000E-0	0.000E-0	0.000E-0
0.2	0.32588	0.309E-9	0.125E-14	0.138E-15	3.053E-16
0.4	0.52933	0.303E-10	0.392E-15	1.665E-16	6.661E-16
0.6	0.74704	0.286E-9	0.842E-16	0.000E-0	1.221E-16
0.8	0.99709	0.542E-9	0.684E-15	1.110E-16	2.331E-16
$M_{Er}\lvert\eta(x)-\eta_n(x)\rvert$		0.542×10^{-9}	0.125×10^{-14}	0.138×10^{-15}	6.661×10^{-16}

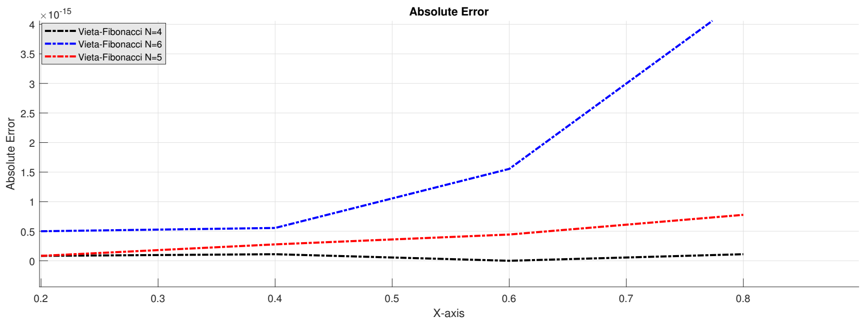

FIGURE 5. Comparison of absolute errors between the results of our method and method [32] for Example 5.5

The results of applying the method on this example can be seen in Table 5 and Figure 5.

Example 5.6. *[34]. Let us consider the Generalized Abel's integral equation of second kind with* $\alpha = 0.8$ *,* $0 \le x \le 1$ *and* $\xi(x) = x + 1$ *as the following:*

$$\xi(x) = \int_0^x \frac{\eta(t)}{(x-t)^{0.8}} dt,$$

where the exact solution take the form $\eta(x) = \dfrac{(1+1.25x)}{\pi x^{0.2}} sin(0.8\pi)x^{0.8}$.

The results of applying the method on this example can be seen in Table 5 and Figure 6.

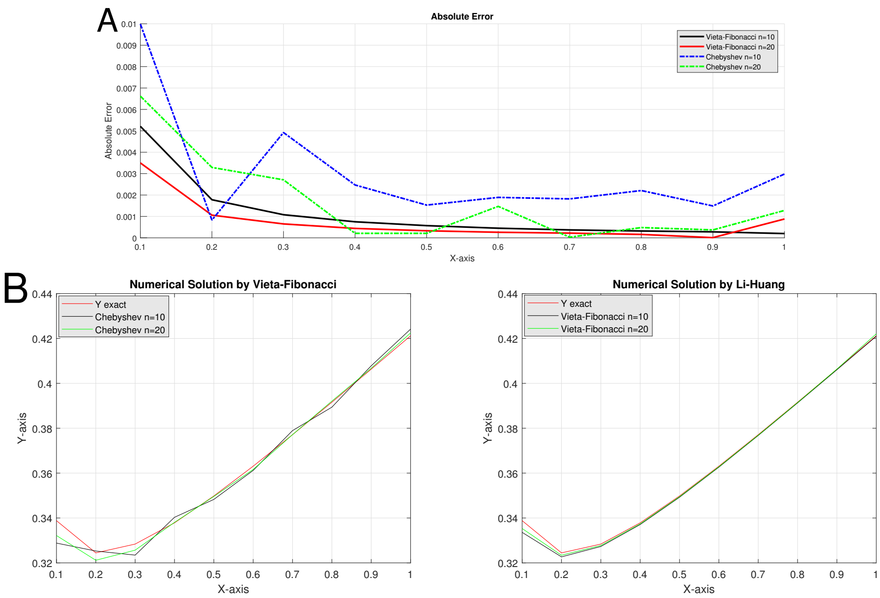

FIGURE 6. Comparison of numerical solutions of Example 5.6 between our method and method of [34]. (A): Absolute error, (B): Exact and approximate solutions.

TABLE 5. Absolute errors for approximate results of Example 5.5 obtained by proposed method for different values of N.

x	Exact	$Er_N = 4$	$Er_N = 5$	$Er_N = 6$
0.2	0.22222	8.3267e-17	8.3267e-17	4.996e-16
0.4	0.44444	1.1102e-16	2.7756e-16	5.5511e-16
0.6	0.66667	0.0000e-0	4.4409e-16	1.5543e-15
0.8	0.88889	1.1102e-16	7.7716e-16	4.4409e-15
$Max_{Er}\lvert\eta(x) - \eta_n(x)\rvert$		1.1102×10^{-16}	7.7716×10^{-16}	4.4409×10^{-15}

TABLE 6. Comparison of numerical solutions of Example 5.6 between our method and method of [34].

		Avazzadeh Method in [34]		Our Method	
x	Exact	$Chebyshev_{n=10}$	$Chebyshev_{n=20}$	$VFPOLY_{n=10}$	$VFPOLY_{n=20}$
0.1	0.33881	0.32882	0.33219	0.33360	0.33531
0.2	0.32446	0.32529	0.32117	0.32268	0.32340
0.3	0.32838	0.32346	0.32567	0.32730	0.32773
0.4	0.33784	0.34031	0.33805	0.33709	0.33740
0.5	0.34981	0.34828	0.34960	0.34924	0.34948
0.6	0.36309	0.36120	0.36162	0.36264	0.36283
0.7	0.37712	0.37894	0.37715	0.37675	0.37690
0.8	0.39159	0.38938	0.39207	0.39127	0.39143
0.9	0.40633	0.40782	0.40670	0.40605	0.40634
1	0.42117	0.42415	0.42245	0.42097	0.42205
$M_{Er}\lvert\eta(x) - \eta_n(x)\rvert$		2.81×10^{-3}	1.11×10^{-3}	1.17×10^{-3}	1.07×10^{-3}

6. CONCLUSIONS

The numerical scheme used in this article was very accurate in determining approximate results of GAIEs. The proposed method used simplicity the Vieta Fibonacci polynomial to create an efficient and accurate method for solving [GAIEs]. The equation has been converted into a system of linear algebraic equations. The illustrate examples showed that the obtained results were very better compared with the results of many other methods.

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