TWMS J. App. and Eng. Math. V.15, N.8, 2025, pp. 2031-2040

DIFFERENT ENERGIES OF THE ONE-POINT UNION OF COMPLETE BIPARTITE GRAPH

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Abstract. The concept of energy of graphs originated with chemical applications and was first introduced by Gutman in 1978. The energy of a graph $\mathcal{E}(G)$ was defined as the sum of the absolute eigenvalues of the adjacency matrix. In the present work, the energy of graph, general extended adjacency energy of graph in the context of graph operation, namely one point union of k-copies of complete bipartite graph $K_{p,q}$ have been discussed. Also, MATLAB coding has been generated for all the results. In this paper, we have investigated the bounds on energy in terms of order of disjoint union of graphs and one point union of graphs.

Keywords: Energy of graph; General extended adjacency energy; Complete bipartite graph.

AMS Subject Classification: 05C50, 05C76.

1. Introduction

In this research work, the graphs under consideration are finite, connected, undirected and simple graph $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, ..., v_n\}$ the set of vertices and $E(G) = \{e_1, e_2, ..., e_m\}$ the set of edges of graph G. The energy of graph is introduced and described as a frontier between chemistry and mathematics by Gutman [7]. The adjacency matrix $A(G)$ of G is defined as $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i is adjacent to v_j, and 0 otherwise. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $A(G)$ with multiplicities $\{m_1, m_2, ..., m_n\}$, respectively. Hence, $spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_n \\ m_1 & m_2 & ... & m_n \end{pmatrix}$.
The energy of a graph G is the sum of the absolute values of the eigenvalues of the adjacency matrix of graph G and denoted by $\mathcal{E}(G)$. That is, $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$.
Subsequently, several degree-based energies were identified for simple graphs. In general

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§ Manuscript received: June 24, 2024; accepted: October 26, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.8;

terms, $TI(G) = \sum_{v_i \sim v_j} F(d_i, d_j)$ represents the degree-based topological indices for an undirected graph, where F is a symmetric function of two variables.

The authors of the article [4] introduced general extended adjacency matrix with each degree-based topological index. $ATI(G) = (a_{ij})$, which is defined as $a_{ij} = F(d_i, d_j)$, if v_i and v_j are adjacent and 0 otherwise. Let $f_1, f_2, f_3, \ldots, f_n$ be the eigenvalues of the extended adjacency matrix $ATI(G)$. The energy of the general extended adjacency matrix is defined as $ETI(G) = \sum_{i=1}^{n} |fi|$.

Definition 1.1. [8] *A graph $G = (V(G), E(G))$ is said to be* Bipartite *if its vertices can be partitioned into two disjoint subsets in such a way, that no edge joins two vertices in the same set. And a* Complete bipartite graph *is a simple bipartite graph in which each vertex in one partite set is adjecent to all other vertices in other partition set. If the two partite sets have cardinality p and q, then this graph is denoted by $K_{p,q}$.*

Definition 1.2. [3] *A graph G in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called the root of G. Let G be a rooted graph. The graph $G^{(n)}$ obtained by identifying the roots of n copies of G is called a* One-point union *of the n copies of G.*

Let $K_{p,q}^{(k)}$ be the one-point union of k-copies of $K_{p,q}$. $\{v_{i1}, v_{i2}, ..., v_{i(p+q)}\}$ be the vertices of i^{th} copy of $K_{p,q}$, where $i = 1, 2, ..., k$. Let $K_{p,q}^{(k)}$ be the graph obtained by identifying the vertices $v_{11}, v_{21}, v_{31} ... v_{k1}$ of degree q which is denoted by v_{11}. Here the total number of vertices in $K_{p,q}^{(k)}$ is $k(p+q) - k + 1$. $K_{p,q}^{(k)}$ be the rooted graph as $K_{p,q}$ with root v_{i1}, where $i = 1, 2, ..., k$.

Definition 1.3. [1] *Let $A \in R^{m \times n}$, $B \in R^{p \times q}$. Then the* Kronecker product (or tensor product) *of A and B is defined as the matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

Definition 1.4. [2] *The* Randić matrix *of the graph $G = (V(G), E(G))$ is the square matrix of order n, whose $(i,j)-$element is equal to $\frac{1}{\sqrt{deg(v_i)deg(v_j)}}$ if v_i and v_j are adjacent vertices, and is zero otherwise. The* Randić energy *is the sum of the absolute values of the eigenvalues of the Randić matrix.*

Definition 1.5. [11] *The* Extended adjacency matrix *of the graph $G = (V(G), E(G))$ is the square matrix of order n, whose $(i,j)-$element is equal to $\frac{1}{2}(\frac{deg(v_i)}{deg(v_j)} + \frac{deg(v_j)}{deg(v_i)})$ if v_i and v_j are adjacent vertices, and zero otherwise. The* Extended energy *is the sum of the absolute values of the eigenvalues of the extended adjacency matrix.*

Lemma 1.1. [10] *If G and H are two rooted graphs with roots r and s, then the charecteristic polynomial of the coalescence $G \cdot H$ is*
$\phi(G \cdot H) = \phi(G)\phi(H - s) + \phi(G - r)\phi(H) - x\phi(G - r)\phi(H - s).$

Lemma 1.2. [7] *The energy of complete bipartite graph $K_{p,q}$ is $2\sqrt{pq}$.*

Proposition 1.1. [12] *Let M, N, P, Q be the matrices with the order $p \times p, p \times q, q \times p, q \times q$ respectively, and let Q be invertible and*

$$S=\begin{bmatrix} M & N \\ P & Q \end{bmatrix}$$

Then $detS = detQ \cdot det[M - NQ^{-1}P]$.

Proposition 1.2. [1] *If* $A \in R^{n\times n}$ *and* $B \in R^{n\times n}$ *be invertible matrices then* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Proposition 1.3. [9] *The matrix* $A_{n\times n}=\begin{bmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}_{n\times n}$

then their determinant will be $(a+(n-1)b)(a-b)^{n-1}$ *and their inverse will be*

$$A^{-1}=\frac{1}{(a-b)(a+(n-1)b)}\begin{bmatrix} a+(n-2)b & -b & -b & \cdots & -b \\ -b & a+(n-2)b & -b & \cdots & -b \\ -b & -b & a+(n-2)b & \cdots & -b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b & -b & -b & \cdots & a+(n-2)b \end{bmatrix}_{n\times n}$$

2. Main Results

Theorem 2.1. *Let* $K^{(k)}_{p,q}$ *be the one-point union of* k*-copies of* $K_{p,q}$. *Then* $\mathcal{E}(K^{(k)}_{p,q}) = 2\sqrt{q}[\sqrt{k+p-1} + (k-1)\sqrt{p-1}]$.

Proof. For one-point union of 2-copies of $K_{p,q}$, $G = H = K_{p,q}$
$G^* = H^* = K_{p-1,q}$ (by removing the root v_{i1}, $i = 1, 2$)

$\phi(G) = \phi(H) = (x^2 - pq)x^{p+q-2}$
$\phi(G^*) = \phi(H^*) = (x^2 - (p-1)q)x^{p+q-3}$

By lemma 1.1., $\phi(G \cdot H) = \phi(G)\phi(H^*) + \phi(G^*)\phi(H) - x\phi(G^*)\phi(H^*)$.

$\phi(G \cdot H) = 2(x^2 - pq)(x^2 - (p-1)q)x^{2p+2q-5} - x(x^2 - (p-1)q)^2x^{2p+2q-6}$
$\phi(G \cdot H) = x^{2p+2q-5}(x^2 - (p-1)q)[2(x^2 - pq) - (x^2 - (p-1)q)]$

$\phi(A(K^{(2)}_{p,q}) : x) = \phi(G \cdot H) = x^{2p+2q-5}(x^2 - (p-1)q)(x^2 - (p+1)q)$.

For one-point union of 3-copies of $K_{p,q}$, $G = K^{(2)}_{p,q}$, $H = K_{p,q}$.
$G^* = K_{p-1,q} \cup K_{p-1,q}$, $H^* = K_{p-1,q}$. (by removing the root)

$\phi(G) = x^{2p+2q-5}(x^2 - (p-1)q)(x^2 - (p+1)q), \phi(H) = (x^2 - pq)x^{p+q-2}$
$\phi(G^*) = (x^2 - pq)^2x^{2(p+q-2)}, \phi(H^*) = (x^2 - (p-1)q)x^{p+q-3}$

By lemma 1.1., $\phi(A(K^{(3)}_{p,q}) : x) = \phi(G \cdot H) = x^{3p+3q-8}(x^2 - (p-1)q)^2(x^2 - (p+2)q)$.

Let $\phi(K^{(k)}_{p,q}) = x^{kp+kq-3k+1}(x^2 - (p-1)q)^{k-1}(x^2 - (p+k-1)q)$ be true, then for $K^{(k+1)}_{p,q}$, $G = K^{(k)}_{p,q}$, $H = K_{p,q}$.
$G^* = \bigcup_{i=1}^{k} K_{p-1,q}$, $H^* = K_{p-1,q}$. (by removing the root)

$\phi(G) = x^{kp+kq-3k+1}(x^2-(p-1)q)^{k-1}(x^2-(p+k-1)q), \phi(H) = (x^2-pq)x^{p+q-2}$
$\phi(G^*) = (x^2-pq)^k x^{k(p+q-2)}, \phi(H^*) = (x^2-(p-1)q)x^{p+q-3}$

by lemma 1.1., $\phi(A(K_{p,q}^{(k+1)} : x) = \phi(G \cdot H) = x^{3kp+3kq-3k-2}(x^2-(p-1)q)^k(x^2-pq+x^2-(p+k-1)q-x^2+(p-1)q)$

$$\phi(A(K_{p,q}^{(k+1)}) : x) = x^{3kp+3kq-3k-2}(x^2-(p-1)q)^k(x^2-(p+k)q)$$

$$\phi(A(K_{p,q}^{(k)}) : x) = x^{k(p+q)-3k+1}(x^2-q(p-1))^{k-1}(x^2-q(k+p-1))$$

By the mathematical inducation, it is true for all $k \in \mathbb{N}$.
$spec(K_{p,q}^{(k)}) =$
$$\begin{pmatrix} 0 & \sqrt{q(p-1)} & -\sqrt{q(p-1)} & \sqrt{q(k+p-1)} & -\sqrt{q(k+p-1)} \\ k(p+q)-3k+1 & k-1 & k-1 & 1 & 1 \end{pmatrix}$$
$\mathcal{E}(K_{p,q}^{(k)}) = 2(k-1)\sqrt{q(p-1)} + 2(\sqrt{q(k+p-1)}) = 2\sqrt{q}[k-1(\sqrt{p-1}) + \sqrt{k+p-1}]$. □

Example 2.1. *The energy of one-point union of* 3*-copies of* $K_{2,3}$ *is* $8\sqrt{3}$.

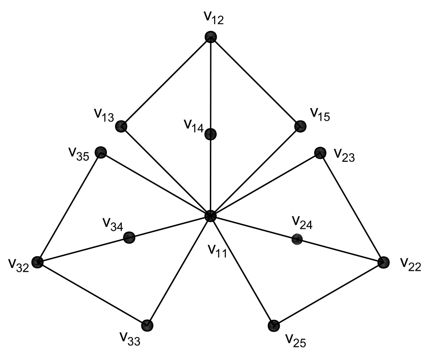

Figure 1: One-point union of 3-copies of $K_{2,3}$.

$$A(K_{2,3}^{(3)}) = \begin{array}{c} \\ v_{11} \\ v_{12} \\ v_{22} \\ v_{32} \\ v_{13} \\ v_{14} \\ v_{15} \\ v_{23} \\ v_{24} \\ v_{25} \\ v_{33} \\ v_{34} \\ v_{35} \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} v_{11} & v_{12} & v_{22} & v_{32} & v_{13} & v_{14} & v_{15} & v_{23} & v_{24} & v_{25} & v_{33} & v_{34} & v_{35} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$spec(K_{2,3}^{(3)}) = \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} & 2\sqrt{3} & -2\sqrt{3} \\ 7 & 2 & 2 & 1 & 1 \end{pmatrix}$$

$$\mathcal{E}(K_{2,3}^{(3)}) = 8\sqrt{3}$$

Theorem 2.2. *Let $K_{p,q}^{(k)}$ be the one-point union of k-copies of $K_{p,q}$. Then $ETI(K_{p,q}^{(k)}) = 2kq(\gamma(p-1)+\alpha)$, where $\alpha = [F(p,kq)]^2$ and $\gamma = [F(p,q)]^2$.*

Proof. The general extended adjacency matrix of $K_{p,q}^{(k)}$ is
$ATI(K_{p,q}^{(k)}) = \begin{pmatrix} O_{k(p-1)+1} & B \\ B^T & O_{kq} \end{pmatrix}$,
where $B_{(k(p-1)+1)\times kq}=$

$$\begin{pmatrix}
F(p,kq) & \cdots & F(p,kq) & F(p,kq) & \cdots & F(p,kq) & \cdots & F(p,kq) & \cdots & F(p,kq) \\
F(p,q) & \cdots & F(p,q) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
F(p,q) & \cdots & F(p,q) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & F(p,q) & \cdots & F(p,q) & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & F(p,q) & \cdots & F(p,q) & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & F(p,q) & \cdots & F(p,q) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & F(p,q) & \cdots & F(p,q)
\end{pmatrix}$$

The characteristic polynomial of the above matrix is given from proposition 1.1.,

$$\begin{aligned}
\phi(ATI(K_{p,q}^{(k)}) : x) = |xI_n - ATI(K_{p,q}^{(k)})| &= \begin{vmatrix} xI_{k(p-1)+1} & -B \\ -B^T & xI_{kq} \end{vmatrix} \\
&= |xI_{kq}||xI_{k(p-1)+1} - (-B)(xI_{kq})^{-1}(-B^T)| \\
&= |xI_{kq}||xI_{k(p-1)+1} - B(xI_{kq})^{-1}B^T|
\end{aligned}$$

$xI_{k(p-1)} - B(xI_{kq})^{-1}B^T =$

$$\frac{1}{x}\left(\begin{array}{c|cccccccccc}
x^2-kq\alpha & -q\beta & \cdots & -q\beta & -q\beta & \cdots & -q\beta & \cdots & -q\beta & \cdots & -q\beta \\
\hline
-q\beta & x^2-q\gamma & \cdots & -q\gamma & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-q\beta & -q\gamma & \cdots & x^2-q\gamma & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
-q\beta & 0 & \cdots & 0 & x^2-q\gamma & \cdots & -q\gamma & \cdots & 0 & \cdots & 0 \\
\vdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots & \ddots & \cdots & \ddots & \cdots \\
-q\beta & 0 & \cdots & 0 & -q\gamma & \cdots & x^2-q\gamma & \cdots & 0 & \cdots & 0 \\
\vdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots & \ddots & \cdots & \ddots & \cdots \\
-q\beta & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & x^2-q\gamma & \cdots & -q\gamma \\
\vdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots & \ddots & \cdots & \ddots & \cdots \\
-q\beta & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & -q\gamma & \cdots & x^2-q\gamma
\end{array}\right),$$

$= \begin{pmatrix} S & T \\ T^T & V \end{pmatrix}$,
where $\alpha = [F(p,kq)]^2, \beta = F(p,kq)F(p,q)$ and $\gamma = [F(p,q)]^2$
and $V = P \bigotimes I_k, P= \begin{pmatrix} x^2-q\gamma & \cdots & -q\gamma \\ \vdots & \ddots & \vdots \\ -q\gamma & \cdots & x^2-q\gamma \end{pmatrix}_{(p-1)\times(p-1)}$

$$|xI_{k(p-1)} - B(xI_{kq})^{-1}B^T| = det(V) \cdot det(S - TV^{-1}T^T)$$
$$det(V) = (x^2 - (p-1)q\gamma)^k (x^2)^{k(p-2)}$$

From proposition 1.2. and 1.3.,

$$P^{-1} = \frac{1}{(x^2-(p-1)q\gamma)(x^2)} \begin{pmatrix} (x^2-(p-2)q\gamma) & q\gamma & \cdots & q\gamma \\ \vdots & \vdots & \ddots & \vdots \\ q\gamma & q\gamma & \cdots & (x^2-(p-2)q\gamma) \end{pmatrix}_{(p-1)\times(p-1)}$$

$$TV^{-1}T^T = \frac{k(p-1)q^2\beta^2}{(x^2-(p-1)q\gamma)}$$
$$S - TV^{-1}T^T = [\frac{(x^2-kq\alpha)(x^2-(p-1)q\gamma) - k(p-1)q^2\beta^2}{(x^2-(p-1)q\gamma)}]$$

$$\phi(ATI(K_{p,q}^{(k)}) : x) = x^{kq+2k(p-2)} \frac{1}{x^{k(p-1)+1}} (x^2-(p-1)q\gamma)^{k-1}(x^2)(x^2-((p-1)q\gamma+kq\alpha))$$
$$\phi(ATI(K_{p,q}^{(k)}) : x) = x^{k(p+q)-3k+1}[(x^2-(p-1)q\gamma)^{k-1}(x^2-((p-1)q\gamma+kq\alpha))]$$

$$spec(K_{p,q}^{(k)}) = \begin{pmatrix} 0 & \pm\sqrt{(p-1)q\gamma} & \pm((p-1)q\gamma+kq\alpha) \\ k(p+q)-3k+1 & k-1 & 1 \end{pmatrix}$$

$ETI(K_{p,q}^{(k)}) = 2kq(\gamma(p-1)+\alpha)$,
where $\alpha = [F(p,kq)]^2$ and $\gamma = [F(p,q)]^2$. □

Corollary 2.1. *The Randić energy of one-point union of k-copies of the complete bipartite graph $K_{p,q}$ obtained by identifying k vertices of degree q is $2 + 2(k-1)\sqrt{\frac{p-1}{p}}$.*

Proof. For the Randić matrix of $(K_{p,q}^{(k)})$,

$\alpha = [F(p,kq)]^2 = \frac{1}{kpq}, \beta = F(p,kq)F(p,q) = \frac{1}{pq\sqrt{k}}$ and $\gamma = [F(p,q)]^2 = \frac{1}{pq}$

$$\phi(R(K_{p,q}^{(k)}) : x) = x^{k(p+q)-3k+1}[((x^2-\tfrac{p-1}{p})^{k-1})(x^2-1)]$$
$$spec(K_{p,q}^{(k)}) = \begin{pmatrix} 0 & -\sqrt{\frac{p-1}{p}} & \sqrt{\frac{p-1}{p}} & 1 & -1 \\ k(p+q)-3k+1 & k-1 & k-1 & 1 & 1 \end{pmatrix}$$
$$RE(K_{p,q}^{(k)}) = 2 + 2(k-1)\sqrt{\frac{p-1}{p}}$$ □

Corollary 2.2. *The Extended energy of one-point union of k-copies of the complete bipartite graph $K_{p,q}$ obtained by identifying k vertices of degree q is* $\frac{k(k-1)\sqrt{q(p-1)}(p^2+q^2)+\sqrt{k^2q(p-1)(p^2+q^2)^2+kq(k^2q^2+p^2)^2}}{kpq}$.

Proof. The the extended matrix of $K_{p,q}^{(k)}$,

$\alpha = [F(p,kq)]^2 = \frac{(p^2+q^2)^2}{4p^2q^2}, \beta = F(p,kq)F(p,q) = \frac{(p^2+q^2)(p^2+k^2q^2)}{4kp^2q^2}$
and $\gamma = [F(p,q)]^2 = \frac{(p^2+k^2q^2)^2}{4k^2p^2q^2}$.

$$\phi(A_{ext}(K_{p,q}^{(k)}) : x) = x^{kq}(x^2-\gamma(p-1))^k(x^2)^{k(p-2)} \frac{1}{x^{k(p-1)}} \frac{x^2}{(x^2-\gamma(p-1))}[x^2-(p-1)\gamma-\alpha]$$

$$= x^{k(p+q)-3k+1}\left[\left(x^2-\frac{q(p-1)(p^2+q^2)^2}{4p^2q^2}\right)^{k-1}\left(x^2-\frac{k^2q(p-1)(p^2+q^2)^2+kq(k^2q^2+p^2)^2}{4k^2p^2q^2}\right)\right]$$

$$spec(K_{p,q}^{(k)}) = \begin{pmatrix} 0 & \pm\frac{\sqrt{q(p-1)(p^2+q^2)^2}}{2pq} & \pm\frac{\sqrt{k^2q(p-1)(p^2+q^2)^2+kq(k^2q^2+p^2)^2}}{2kpq} \\ k(p+q)-3k+1 & k-1 & 1 \end{pmatrix}$$

$$E_{ext}(K_{p,q}^{(k)}) = \frac{k(k-1)\sqrt{q(p-1)}(p^2+q^2)+\sqrt{k^2q(p-1)(p^2+q^2)^2+kq(k^2q^2+p^2)^2}}{kpq}. \qquad \square$$

Example 2.2. *The Randić energy of one-point union of* 3*-copies of* $K_{3,2}$ *is* $2+4\sqrt{\frac{2}{3}}$.

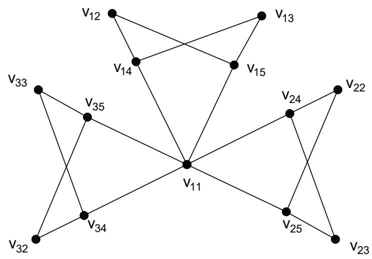

Figure 2: One-point union of 3-copies of $K_{3,2}$.

the Randić matrix of the graph is $R(K_{3,2}^{(3)}) =$

$$\begin{array}{c|ccccccccccccc}
 & v_{11} & v_{12} & v_{13} & v_{22} & v_{23} & v_{32} & v_{33} & v_{14} & v_{15} & v_{24} & v_{25} & v_{34} & v_{35} \\
\hline
v_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\
v_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 \\
v_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 \\
v_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 \\
v_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 \\
v_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
v_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
v_{14} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{15} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{24} & \frac{1}{3\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{25} & \frac{1}{3\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{34} & \frac{1}{3\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 \\
v_{35} & \frac{1}{3\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$spec(K_{3,2}^{(3)}) = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 1 & -1 \\ 7 & 2 & 2 & 1 & 1 \end{pmatrix}$$

$$RE(K_{3,2}^{(3)}) = 2+4\sqrt{\frac{2}{3}}.$$

3. MATLAB Program

The generalized adjacency matrix, its determinant and its energy can be found in the following MATLAB program:

p=input(”Enter the q degree vertices:”);
q=input(”Enter the p degree vertices:”);

```
n=input(”Enter the no. of copies of graph:”);
A = zeros(n*(p-1)+1,n*(p-1)+1);
B1=ones(1,n*q);
A1 =ones(p-1,q);
N = n;
Ar = repmat(A1, 1, N);
Ac = mat2cell(Ar, size(A1,1), repmat(size(A1,2),1,N));
B2 = blkdiag(Ac:);
B=[B1;B2];
C=B.’;
D= zeros(n*q,n*q);
K = [A,B;C,D];
det(K);
E=sum(abs(eig(K)))
```

The generalized Randić matrix, its determinant and its Randić energy can be found in the following MATLAB program:

```
p=input(”Enter the q degree vertices:”);
q=input(”Enter the p degree vertices:”);
n=input(”Enter the no. of copies of graph:”);
A = zeros(n*(p-1)+1,n*(p-1)+1);
B1= (1/sqrt(n*p*q))*ones(1,n*q);
A1 =ones(p-1,q);
N = n;
Ar = repmat(A1, 1, N);
Ac = mat2cell(Ar, size(A1,1), repmat(size(A1,2),1,N));
B2 = 1/(sqrt(p*q))*blkdiag(Ac:);
B=[B1;B2];
C=B.’;
D= zeros(n*q,n*q);
K = [A,B;C,D];
det(K);
E=sum(abs(eig(K)))
```

The generalized extended adjacency matrix, its determinant and extended energy can be found in the following MATLAB program:

```
p=input(”Enter the q degree vertices:”);
q=input(”Enter the p degree vertices:”);
n=input(”Enter the no. of copies of graph:”);
A = zeros(n*(p-1)+1,n*(p-1)+1);
B1=1/2*(((n*q)/p)+(p/(n*q)))*ones(1,n*q);
A1 =ones(p-1,q);
N = n;
Ar = repmat(A1, 1, N);
Ac = mat2cell(Ar, size(A1,1), repmat(size(A1,2),1,N));
B2 = 1/2*((p/q)+(q/p))*blkdiag(Ac:);
B=[B1;B2];
C=B.’;
D= zeros(n*q,n*q);
```

```
K = [A,B;C,D];
det(K);
E=sum(abs(eig(K)))
```

4. Conclusions

Conclusion 4.1. $\mathcal{E}(\bigcup_k K_{p,q}) > \mathcal{E}(K_{p,q}^{(k)})$

Proof. Here from lemma 1.2., $\mathcal{E}(\bigcup_k K_{p,q}) = 2k\sqrt{pq}$ and from theorem 2.1.
$\mathcal{E}(K_{p,q}^{(k)}) = 2\sqrt{q}[\sqrt{k+p-1} + (k-1)\sqrt{p-1}]$.
Let us assume $2k\sqrt{pq} < 2\sqrt{q}[\sqrt{k+p-1} + (k-1)\sqrt{p-1}]$
$k\sqrt{p} < \sqrt{k+p-1} + (k-1)\sqrt{p-1}$
$k\sqrt{p} + \sqrt{p-1} < \sqrt{k+p-1} + k\sqrt{p-1}$
but $k\sqrt{p} > k\sqrt{p-1}$ and $\sqrt{p-1} > \sqrt{k+p-1}$
Therefore, $k\sqrt{p} + \sqrt{p-1} > \sqrt{k+p-1} + k\sqrt{p-1}$
from contradiction, $2k\sqrt{pq} < 2\sqrt{q}[\sqrt{k+p-1} + (k-1)\sqrt{p-1}]$ is not possible.
$2k\sqrt{pq} > 2\sqrt{q}[\sqrt{k+p-1} + (k-1)\sqrt{p-1}]$
That is, $\mathcal{E}(\bigcup_k K_{p,q}) > \mathcal{E}(K_{p,q}^{(k)})$. □

In this paper, the energy and degree based energies of k-copies of one point union of $K_{p,q}$ graph are obtained. Their MATLAB code have been discussed. The energy of simple union of $K_{p,q}$ is higher than energy of one point union of $K_{p,q}$ with the same number of copies have been investigated.

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