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ON HERONIAN MEAN ANTI-MAGIC LABELING OF SOME GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a finite, simple, connected and undirected graph with p vertices and q edges. Let $\Phi : V(G) \to \{1, 2, 3, \cdots, q+1\}$ and the induced edge labeling $\Phi^* : E(G) \to \{1, 2, 3, \cdots, q+1\}$ is defined by $\Phi^*(e = uv) = \left\lceil \frac{\Phi(u)+\sqrt{\Phi(u)\Phi(v)}+\Phi(v)}{3} \right\rceil$ or $\left\lfloor \frac{\Phi(u)+\sqrt{\Phi(u)\Phi(v)}+\Phi(v)}{3} \right\rfloor$ for $e \in E(G)$. Then Φ is said to be a Heronian mean labeling if induced edge labels $\Phi^*(e)$ are distinct. An anti-magic labeling is a bijection from the set of edges to the set of integers $\{1, 2, 3, \cdots, q\}$ such that the weights are pairwise distinct, where the weight at one vertex is the sum of all labels of the edges incident to such vertex. A Heronian mean labeling Φ is said to be Heronian mean anti-magic if $w(v_i) \neq w(v_j)$ for all distinct vertices $v_i, v_j \in V(G)$, where $w(v) = \sum\limits_{u \in N(v)} \Phi^*(uv)$. A graph is called Heronian mean anti-magic graph if it admits a Heronian mean anti-magic labeling. In this paper, we investigate the behaviour of this labeling for graphs which contains a clique of order at least 4, $P_n \circ 2K_1$, $kC_n, n \geq 4$, $CL_n, n \geq 3$, $T \cup T^{'}$ where T and $T^{'}$ be any two trees of order at least 3. We also prove that $K_{2,n}$ is Heronian mean anti-magic for $n \leq 9$ and is not for $n \geq 10$.

Keywords: Mean labeling, Anti-magic labeling, Mean anti-magic labeling, Heronian mean anti-magic labeling.

AMS Subject Classification: 05C78.

1. Introduction

In Graph Theory, graph labeling is an important branch which can be used to solve many real life problems. It is the assignment of labels, traditionally represented by integers, to the vertices or edges of a graph. Formally, for a given graph $G = (V(G), E(G))$, the vertex labeling is a function of $V(G)$ over the set of labels. Graphs with such functions defined are called vertex-labeled graphs. Similarly, edge labeling is a function of $E(G)$ on the set of labels. In this case, the graph is called an edge-labeled graph. The term

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'labeled graph' generally refers to a vertex-labeled graph in which all labels are different. In many applications, edges or vertices are labeled meaningfully in the relevant domain. For example, edges can be assigned weights that represent the 'cost'of travelling between incident vertices. In the definition above, the graph is understood to be a finite undirected simple graph. However, the labeling concept is applicable to all extensions and generalizations of graphs. For example, in automaton theory and formal language theory, it is useful to consider labeled multigraphs. That is, a pair of vertices may be connected by some number of labeled edges. For graph theoretic terminology, we refer Harary [4] and for labeling concepts we refer the famous survey article written by Gallian[1].

Most graph labelings trace their origins to labelings presented by Alexander Rosa in his 1967 paper. He identified three types of labelings which he called α, β and ρ. The $\beta-$labelings were later renamed as 'graceful' by Solomon Golomb, and the name has been popular since. S. Somasundaram and R. Ponraj [9] originated the theory of mean labeling of graphs in 2004. In the last 60 years, over 200 types of graph labeling have been studied and almost 2500 articles have been published [2, 3, 6, 7, 8].

The concept of Heronian mean was introduced by S.S. Sandhya [10] and she has investigated the results for some graphs. In 1990, Hartsfield and Ringel [5] introduced the concept called anti-magic labeling and anti-magic graphs. An anti-magic labeling is a bijection from the set of edges to the set of integers $\{1, 2, \ldots, q\}$ such that the weights are pairwise distinct, where the weight at one vertex is the sum of all the labels of the edges incident to such vertex. A graph is called anti-magic if it admits anti-magic labeling. Hartsfield and Ringel showed that the paths, cycles, complete graphs $K_n, (n \geq 3)$ are anti-magic. They conjectured that all connected graphs besides K_2 are anti-magic which still remains unsettled.

Motivated by the above works, we introduced the concept of Heronian mean anti-magic labeling[11] and studied the existence of Heronian mean anti-magic graphs. In this paper, we continue to examine the existence of the labeling for some standard graphs and certain special classes of graphs with their generalizations. Not all graphs are Heronian mean anti-magic and we have proved the existence of Non Heronian mean anti-magic labeling of some graphs using the inequality method.

2. Preliminaries

Throughout this paper, we mean G to be a simple, finite and undirected graph.

Definition 2.1. *A set $C \subseteq V(G)$ is said to be a clique of a graph G if the subgraph induced by C is complete.*

Definition 2.2. *The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.*

Definition 2.3. *Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2 is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1, p_1 copies of G_2 and joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 where $1 \leq i \leq p_1$.*

Definition 2.4. *A kC_n graph is obtained by considering a path P_{k+1} on the vertices $u_1, u_2, \ldots, u_{k+1}$ where each edge u_iu_{i+1} is replaced by a cycle of length $l > 3$. The vertex set is given by $V(G) = \{u_i | 1 \leq i \leq k+1\} \cup \{u_{ij} | 1 \leq i \leq k, 1 \leq j \leq n-2\}$ and the edge set is given by $E(G) = \{u_iu_{ij} | 1 \leq i \leq k, 1 \leq j \leq 2\} \cup \{u_{ij}u_{i(j+2)} | 1 \leq i \leq k, 1 \leq j \leq n-4\} \cup \{u_{i+1}u_{i(n-3)}, u_{i+1}u_{i(n-2)} | 1 \leq i \leq k\}$.*

Definition 2.5. *The Circular ladder graph CL_n is graph obtained by the cartesian product of $C_n \square K_2$, where K_2 is the complete graph on two vertices and C_n is the cycle graph on n vertices.*

Definition 2.6. *An injective function $\Phi : V(G) \to \{1, 2, 3, \ldots, q+1\}$ is said to be a Heronian mean labeling if the induced edge labeling $\Phi^* : E(G) \to \{1, 2, 3, \ldots, q+1\}$ defined by $\Phi^*(e = uv) = \left\lceil \frac{\Phi(u)+\sqrt{\Phi(u)\Phi(v)}+\Phi(v)}{3} \right\rceil$ or $\left\lfloor \frac{\Phi(u)+\sqrt{\Phi(u)\Phi(v)}+\Phi(v)}{3} \right\rfloor$ gives distinct labels for distinct edges. A Heronian mean labeling Φ is said to be Heronian mean anti-magic if $w(v_i) \neq w(v_j)$ for all distinct vertices $v_i, v_j \in V(G)$, where $w(v) = \sum_{u \in N(v)} \Phi^*(uv)$. A graph is called Heronian mean anti-magic graph if it admits Heronian mean anti-magic labeling.*

Let us see two examples which are not Heronian mean anti-magic. The first example do not satisfy the anti-magic condition and the second example do not satisfy the Heronian mean condition. Thus both the graphs are not Heronian mean anti-magic.

Example 2.1. *Consider the graph G given in Figure 1. The vertex set $V(G) = \{u_1, u_2\}$ and the edge set $E(G) = \{u_1u_2\}$. Here $|V(G)| = 2$ and $|E(G)| = 1$. Define a function $\Phi : V(G) \to \{1, 2\}$. Without loss of generality we can assume that Φ is defined by $\Phi(u_1) = 1$ and $\Phi(u_2) = 2$. Then the induced edge labels are taken as, $\Phi^*(u_1u_2) = \left\lceil \frac{1+\sqrt{2}+2}{3} \right\rceil$ or $\left\lfloor \frac{1+\sqrt{2}+2}{3} \right\rfloor$. Thus the edge $\Phi^*(u_1u_2)$ will receive either 1 or 2. Since the degree of both the vertices are 1, the weights of u_1 and u_2 are equal (i.e.) $w(u_1) = w(u_2)$ and so they cannot be distinct. Hence the graph P_2 is not Heronian mean anti-magic.*

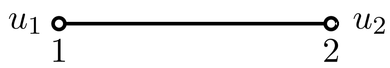

FIGURE 1. P_2

Example 2.2. *Consider the complete bipartite graph $K_{1,6}$ as shown in Figure 2. Let the vertex set be $V(K_{1,6}) = \{v, v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set be $E(K_{1,6}) = \{vv_1, vv_2, vv_3, vv_4, vv_5, vv_6\}$. Here $|V(K_{1,6})| = 7$ and $|E(K_{1,6})| = 6$. Let the apex vertex be v. Define a function $\Phi : V(K_{1,6}) \to \{1, 2, 3, 4, 5, 6, 7\}$. Since $p = q+1$, 1 must be a label for some vertex in $K_{1,6}$. If $\Phi(v) = 1$, then the remaining 6 vertices will receive labels $2, 3, 4, 5, 6, 7$ in some order. Without loss of generality, let $\Phi(v_1) = 2, \Phi(v_2) = 3, \Phi(v_3) = 4, \Phi(v_4) = 5, \Phi(v_5) = 6, \Phi(v_6) = 7$. Then the induced Heronian mean edge label for vv_1 will be 1 or 2. Similarly, the edge label for vv_2 will be either 1 or 2, the edge label for vv_3 will be either 2 or 3 and the edge label for vv_4 will be either 2 or 3. Then there exist two vertices v_i and v_j such that $\Phi^*(vv_i) = \Phi^*(vv_j)$ and so they cannot be distinct.*

If $\Phi(v) = a$, $2 \leq a \leq 5$, by similar arguments there exist vertices v_i and v_j such that $\Phi^*(vv_i) = \Phi^*(vv_j)$ and hence the edges cannot be distinct.

If $\Phi(v) > 5$, by calculating Heronian mean edge labels, one can see that no edge shall receive 1 or 2 as their edge labels. Since we can never label 6 distinct edges with 5 values $\{3, 4, \ldots, 7\}$, the graph does not satisfy the Heronian mean condition.

Thus, there is no Heronian mean labeling for $K_{1,6}$. Hence $K_{1,6}$ is not a Heronian mean anti-magic graph.

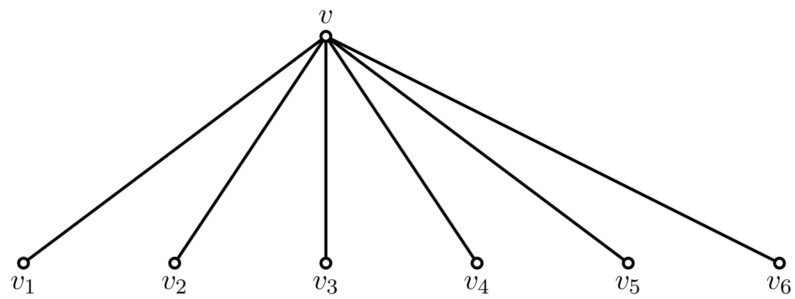

Figure 2. $K_{1,6}$

3. Main Results

In this section we investigate the behaviour of certain classes of graphs like Clique, $P_n \circ 2K_1$, kC_n, CL_n, $T \cup T'$ and $K_{2,n}$.

The following theorem shows that a graph which contains a clique of order at least 4 can never be labeled with the consecutive vertex labels.

Theorem 3.1. *Let G be any graph with q edges and C be a clique of order at least 4. If $\Phi : V(G) \to \{1, 2, \ldots, q+1\}$ is a Heronian mean anti-magic function, then any 4 vertices in C can never be labeled with the consecutive vertex labels.*

Proof. Let $C = \{u_1, u_2, \ldots, u_k\}$ be the clique of order $k \geq 4$. Let $\Phi : V(G) \to \{1, 2, \ldots, q+1\}$ be a Heronian mean anti-magic function.
Suppose not, then there exist 4 vertices $u_1, u_2, u_3, u_4 \in C$ such that $\Phi(u_1) = i, \Phi(u_2) = i+1, \Phi(u_3) = i+2, \Phi(u_4) = i+3$.
Then the induced edge labels are,
$\Phi^*(u_1u_2) = (i, i+1)$ or $[i, i+1]$
$\Phi^*(u_1u_3) = (i, i+2)$ or $[i, i+2]$
$\Phi^*(u_1u_4) = (i, i+3)$ or $[i, i+3]$
$\Phi^*(u_2u_3) = (i+1, i+2)$ or $[i+1, i+2]$ which is not distinct.
This is a contradiction to Heronian mean anti-magic function. Therefore, any 4 vertices in C cannot be labeled consecutively. □

The following theorem proves the existence of Heronian mean anti-magic labeling for the corona product of path graph with two copies of K_1.

Theorem 3.2. *For every positive integer n, the graph $P_n \odot 2K_1$ is a Heronian mean anti-magic graph.*

Proof. Let $G \cong P_n \odot 2K_1$ for $n \geq 1$. Let $V(G) = \{u_i, x_i, v_i | 1 \leq i \leq n\}$ be the vertex set where $u_i (1 \leq i \leq n)$ are the vertices of P_n and $v_i, x_i (1 \leq i \leq n)$ are the pendent vertices. The edge set is $E(G) = \{u_ix_i, u_iv_i | 1 \leq i \leq n\} \cup \{u_iu_{i+1} | 1 \leq i \leq n-1\}$.
Here $|V(G)| = 3n$ and $|E(G)| = 3n - 1$.
Let us define a vertex function $\Phi : V(G) \to \{1, 2, \ldots, (3n-1)+1\}$ by,
$\Phi(u_i) = 3i - 1,\ 1 \leq i \leq n$
$\Phi(v_i) = 3i - 2,\ 1 \leq i \leq n$
$\Phi(x_i) = 3i,\ 1 \leq i \leq n.$
If the induced edge labels are taken as,
$\Phi^*(u_iu_{i+1}) = 3i,\ 1 \leq i \leq n-1$
$\Phi^*(u_iv_i) = 3i - 2,\ 1 \leq i \leq n$
$\Phi^*(u_ix_i) = 3i - 1,\ 1 \leq i \leq n-1$

$$\Phi^*(u_n x_n) = \begin{cases} 3n-1 & \text{for } n \not\equiv 0(mod\ 4) \\ 3n & \text{for } n \equiv 0(mod\ 4) \end{cases}$$

then the obtained edge labels are all distinct.
We observe that,

$w(u_i) = 12i - 6,\ 1 \le i \le n-1$

$$w(u_n) = \begin{cases} 9n-6 & \text{for } n \not\equiv 0(mod\ 4) \\ 9n-5 & \text{for } n \equiv 0(mod\ 4) \end{cases}$$

$w(v_i) = 3i - 2,\ 1 \le i \le n$

$w(x_i) = 3i - 1,\ 1 \le i \le n.$

Now we show that $w(u) \ne w(v)$ for $u, v \in V(G)$. It is clear that for $1 \le i \le n$, $w(v_i) = 3i-2$ and $w(x_i) = 3i-1$ are distinct. Also for $1 \le i \le n-1$, $w(u_i) = 12i-6$ is a multiple of 6 and $w(v_i) = 3i-2$ is not a multiple of 6. Therefore $w(u_i) \ne w(v_i)$. It is also clear that when $n \not\equiv 0(mod\ 4)$, $w(v_i) \ne w(u_n)$ for $1 \le i \le n$ because $w(v_i) = 3i-2 \ne w(u_n) = 9n-6$. When $n \equiv 0(mod\ 4)$, if $w(v_i) = w(u_n)$, then it implies that $3i - 2 = 9n - 5$ which gives $i = 3n - 1$ which is not possible because $i \le n$. Hence $w(v_i) \ne w(u_n)$. We know that, $w(u_i)$ is multiple of 6 but $w(x_i)$ is not. Hence $w(x_i) \ne w(u_i)$. Also $w(u_n) \ne w(u_i)$ for $1 \le i \le n-1$. If $w(x_i) = w(u_n)$ for $1 \le i \le n$, then $3i-1 = 9n-5$ which gives $3i = 9n-4$ which is impossible because the right hand side is multiple of 3 but the left hand side is not. Similarly $3i - 1 \ne 9n - 6$.
Hence in all the cases we see that $w(u) \ne w(v)$ for $u, v \in V(G)$. Thus $P_n \odot 2K_1$ is a Heronian mean anti-magic graph. □

As described in the proof of Theorem 3.2, a Heronian mean anti-magic labeling for $P_7 \odot 2K_1$ is shown in Figure 3.

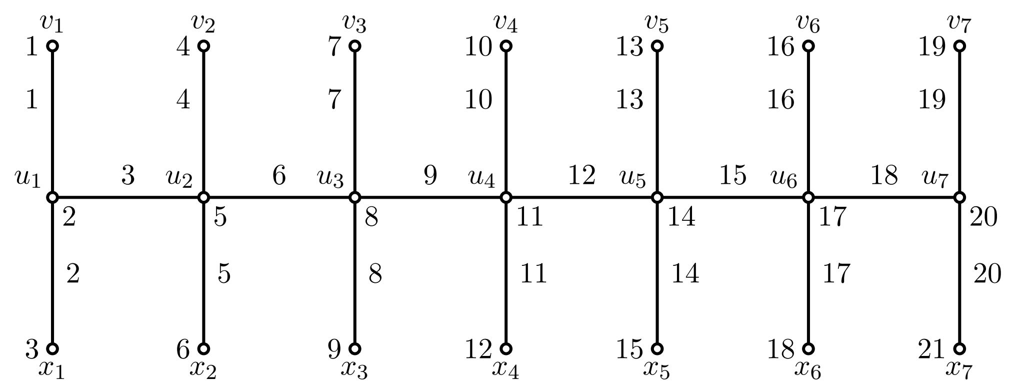

FIGURE 3. $P_7 \odot 2K_1$

In the next theorem, we prove the Heronian mean anti-magic labeling for k copies of the cycle C_n.

Theorem 3.3. *For every positive integer k and $n \ge 4$, the graph kC_n is a Heronian mean anti-magic graph.*

Proof. Let $G \cong kC_n$ be a graph where k is the number of cycles attached and n is the length of the cycle. Let $V(G) = \{u_i | 1 \le i \le k+1\} \cup \{u_{ij} | 1 \le i \le k, 1 \le j \le n-2\}$ be the vertex set and the edge set be $E(G) = \{u_i u_{ij} | 1 \le i \le k, 1 \le j \le 2\} \cup \{u_{ij} u_{i(j+2)} | 1 \le i \le k, 1 \le j \le n-4\} \cup \{u_{i+1} u_{i(n-3)}, u_{i+1} u_{i(n-2)} | 1 \le i \le k\}$. Here $|V(G)| = k(n-1)+1$ and $|E(G)| = kn$.

Let us define a vertex function $\Phi : V(G) \to \{1, 2, \ldots, kn+1\}$ by,
$\Phi(u_1) = 1$
$\Phi(u_i) = ni - n,\ 2 \leq i \leq k+1$
$\Phi(u_{1j}) = j+1,\ 1 \leq j \leq n-2$
$\Phi(u_{ij}) = ni - n + j + 1,\ 2 \leq i \leq k, 1 \leq j \leq n-2$
If the induced edge labels are taken as,

$$\Phi^*(u_i u_{ij}) = \begin{cases} j & \text{for } i = 1, 1 \leq j \leq 2 \\ ni - n + j & \text{for } 2 \leq i \leq k, 1 \leq j \leq 2 \end{cases}$$

$$\Phi^*(u_{ij} u_{i(j+2)}) = \begin{cases} j+2 & \text{for } i = 1, 1 \leq j \leq n-4 \\ ni - n + j + 2 & \text{for } 2 \leq i \leq k, 1 \leq j \leq n-4 \end{cases}$$

$\Phi^*(u_{i+1} u_{i(n-3)}) = ni - 1,\ 1 \leq i \leq k$
$\Phi^*(u_{i+1} u_{i(n-2)}) = ni,\ 1 \leq i \leq k$
then the obtained edge labels are all distinct.
We observe that,
$w(u_1) = 3$
$w(u_i) = 4ni - 4n + 2, 2 \leq i \leq k$
$w(u_{m+1}) = 2kn - 1$

$$w(u_{ij}) = \begin{cases} 2i + 2j & \text{for } i = 1, 1 \leq j \leq n-2 \\ 2ni - 2n + 2j + 2 & \text{for } 1 \leq i \leq k, 1 \leq j \leq n-2 \end{cases}$$

Therefore, for every vertex v in $V(G)$, $w(v)$ are distinct. Hence, the graph kC_n is Heronian mean anti-magic. □

As described in the proof of Theorem 3.3, a Heronian mean anti-magic labeling for $4C_6$ is shown in Figure 4.

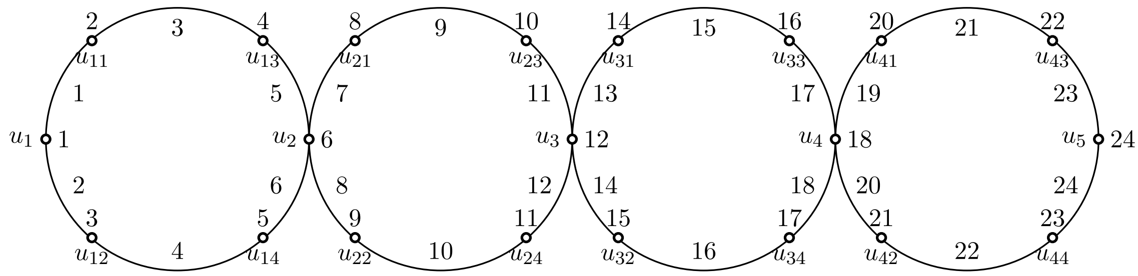

FIGURE 4. $4C_6$

Theorem 3.4. *For every positive integer $n \geq 3$, the Circular ladder graph CL_n is a Heronian mean anti-magic graph.*

Proof. Let $G \cong CL_n$ be a graph. Let $V(G) = \{u_i, v_i | 1 \leq i \leq n\}$ be the vertex set and the edge set be $E(G) = \{u_1u_2, v_1v_2, u_{n-1}u_n, v_{n-1}v_n\} \cup \{u_iv_i | 1 \leq i \leq n\} \cup \{u_iu_{i+2}, v_iv_{i+2} | 1 \leq i \leq n-2\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n$.
Let us define a vertex function by $\Phi : V(G) \to \{1, 2, \ldots, (3n)+1\}$ by,
$\Phi(u_1) = 1$
$\Phi(u_i) = 3i - 1,\ 2 \leq i \leq n$
$\Phi(v_i) = 3i,\ 1 \leq i \leq n$
If the induced edge labels are taken as,

$\Phi^*(u_1u_2) = 2$, $\Phi^*(u_1u_3) = 3$
$\Phi^*(u_iu_{i+2}) = 3i + 1,\ 2 \leq i \leq n - 2$
$\Phi^*(u_{n-1}u_n) = 3n - 2$
$\Phi^*(v_1v_2) = 4$
$\Phi^*(v_iv_{i+2}) = 3i + 2,\ 1 \leq i \leq n - 2$
$\Phi^*(v_{n-1}v_n) = 3n - 1$
$\Phi^*(u_1v_1) = 1$
$\Phi^*(u_iv_i) = 3i,\ 2 \leq i \leq n$

then the obtained edge labels are all distinct.
We observe that,
When $n = 3$,
$w(u_i) = 9i - 3,\ 1 \leq i \leq n - 1$
$w(u_n) = 19$
$w(v_i) = 8i + 2,\ 1 \leq i \leq n - 1$
$w(v_n) = 22$

When $n = 4$,
$w(u_1) = 6$
$w(u_i) = 7i + 1,\ 2 \leq i \leq n$
$w(v_1) = 10$, $w(v_n) = 31$
$w(v_i) = 7i + 4,\ 2 \leq i \leq n - 1$

When $n \geq 5$,
$w(u_1) = 6$, $w(u_2) = 15$, $w(u_3) = 22$
$w(u_i) = 9i - 4,\ 4 \leq i \leq n - 1$
$w(u_n) = 9n - 7$
$w(v_1) = 10$, $w(v_2) = 18$
$w(v_i) = 9i - 2,\ 3 \leq i \leq n - 1$
$w(v_n) = 9n - 5$

Therefore, for every vertex v in $V(G)$, $w(v)$ are distinct. Hence, the graph $CL_n, n \geq 3$ is Heronian mean anti-magic. □

As described in the proof of Theorem 3.4, a Heronian mean anti-magic labeling for CL_7 and is shown in Figure 5.

The following theorem shows that union of trees cannot be Heronian mean anti-magic.

Theorem 3.5. *If T and $T^{'}$ be any two trees of order at least 3, then $T \cup T^{'}$ is not a Heronian mean anti-magic graph.*

Proof. Let T be a tree with $n_1 \geq 3$ vertices and $T^{'}$ be another tree with $n_2 \geq 3$ vertices. Then $|E(T)| = n_1 - 1$ and $|E(T^{'})| = n_2 - 1$. Now $|V(T \cup T^{'})| = n_1 + n_2$ and $|E(T \cup T^{'})| = n_1 + n_2 - 2$. Thus no function, $\Phi : V(T \cup T^{'}) \to \{1, 2, \ldots, (n_1 + n_2 - 2) + 1\}$ can be injective, since $|V(T \cup T^{'})| = n_1 + n_2 > |\{1, 2, \ldots, (n_1 + n_2 - 2) + 1\}|$. □

We need the following lemma to prove the non existence of Heronian mean anti-magic labeling for $n \geq 10$.

Lemma 3.1. *The following inequality holds.*

(i) *For $n \geq 10, (n - 5)^2 > 2n + 1$.*
(ii) *For $n \geq 14, (n - 6)^2 > 4n + 2$.*
(iii) *For $n \geq 18, (n - 7)^2 > 6n + 3$.*
(iv) *For $n \geq 17, (n - 5)^2 > 8n + 4$.*

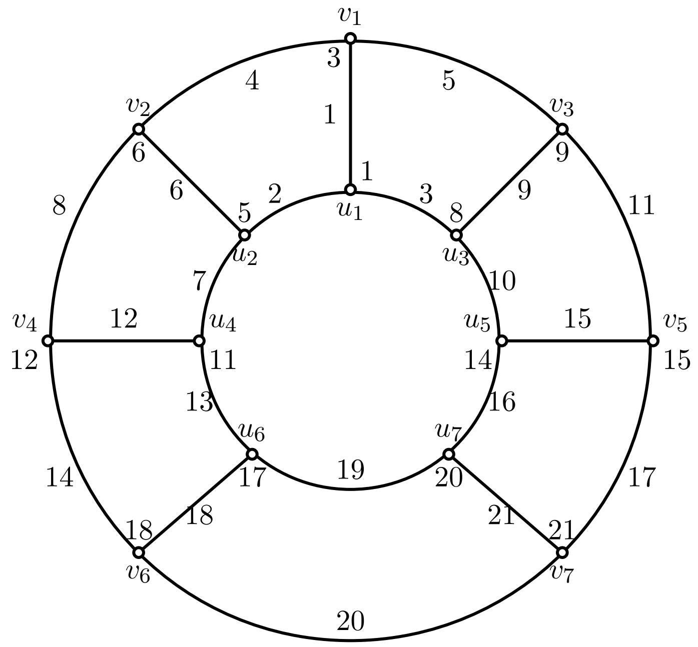

FIGURE 5. CL_7

(v) *For* $n \geq 21, (n-6)^2 > 10n+5$.

Proof. We prove all the inequalities using mathematical induction.
1. When $n = 10$, $(n-5)^2 = 25 > 2(10)+1$
Hence (1) holds for $n = 10$.
Assume that the result is true for $n = m > 10$ so that $(m-5)^2 > 2m+1$
To Prove : $(m+1-5)^2 > 2(m+1)+1$
$((m-5)+1)^2 = (m-5)^2 + 2(m-5) + 1$
$> 2m+1+2m-10+1$
$= 4m-8$.
Now, to prove : $4m-8 > 2m+3$.
i.e., to prove, $2m > 11$
i.e., to prove, $m > \frac{11}{2}$, which is true.
Hence the result. Similarly, we can prove the remaining inequalities by using mathematical induction. □

The following theorem proves that the complete bipartite graph $K_{2,n}$ is Heronian mean anti-magic for $n \leq 9$ and is not for $n \geq 10$.

Theorem 3.6. (i) *For every positive integer* $n \leq 9$*,* $K_{2,n}$ *is a Heronian mean anti-magic graph.*
(ii) *For every positive integer* $n \geq 10$*,* $K_{2,n}$ *is not a Heronian mean anti-magic graph.*

Proof. (i) Let $G \cong K_{2,n}, n \leq 9$. Let $V(G) = \{u, v, u_i | 1 \leq i \leq n\}$ be the vertex set and the edge set be $E(G) = \{uu_i, vu_i | 1 \leq i \leq n\}$. Here $|V(G)| = n+2$ and $|E(G)| = 2n$.
Define a vertex function $\Phi : V(G) \to \{1, 2, \ldots, 2n+1\}$ as follows:
For $n = 1$, $\Phi(u) = 1$, $\Phi(v) = 2$, $\Phi(u_1) = 3$
For $2 \leq n \leq 5$,

$\Phi(u) = 1$, $\Phi(v) = 2n$, $\Phi(u_n) = 2n+1$.
$\Phi(u_i) = 2i$, $1 \leq i \leq n-1$.

For $6 \le n \le 9$,

$\Phi(u) = 1,\ \Phi(v) = 2n,\ \Phi(u_1) = 3.$

$$\Phi(u_i) = \begin{cases} 2i & \text{for } 2 \le i \le 4 \\ 2i+1 & \text{for } 5 \le i \le n \end{cases}$$

If the induced edge labels are taken as,
For $n = 1$, $\Phi^*(uu_1) = 1$, $\Phi^*(vu_1) = 2$
For $n \ge 2$,

$\Phi^*(uu_i) = i,\ 1 \le i \le n.$

$\Phi^*(vu_i) = n + i,\ 1 \le i \le n$

then the obtained edge labels are all distinct and so the labeling Φ satisfies the Heronian mean condition. Now, we have to show that $w(a) = \sum_{x \in N(a)} \Phi^*(xa)$ are distinct for all the vertices in G. We have
For $n = 1$, $w(u) = 1$, $w(v) = 2$, $w(u_1) = 3$
For $n \ge 2$,

$w(u) = \frac{n(n+1)}{2}$

$w(v) = \frac{3n^2+n}{2}$ and

$w(u_i) = n + 2i, 1 \le i \le n.$

Thus $w(a)$ is distinct for every $a \in V(G)$. So Φ is a Heronian mean anti-magic labeling. Hence, $K_{2,n}, n \le 9$ is Heronian mean anti-magic graph.

As described in the proof of Theorem 3.6, Heronian mean anti-magic labeling for $K_{2,7}$ and is shown in Figure 6.

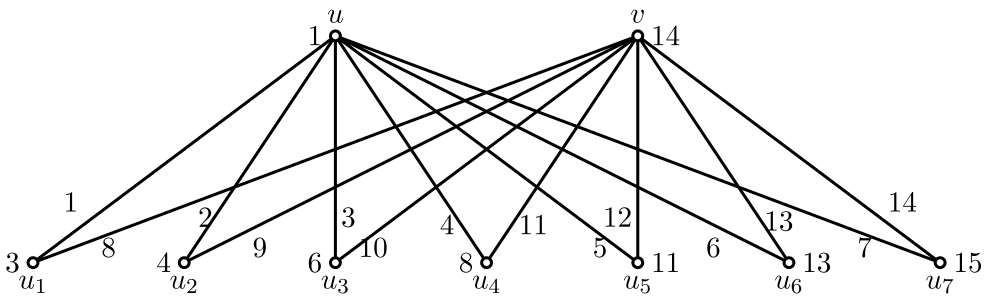

FIGURE 6. $K_{2,7}$

(ii) Let $n \ge 10$. Let $\Phi : V(K_{2,n}) \to \{1, 2, \ldots, 2n+1\}$ be any mapping such that the induced edge mapping $\Phi^* : E(K_{2,n}) \to \{1, 2, \ldots, 2n+1\}$ is given by $\Phi^*(xy) = \left\lceil \frac{\Phi(x)+\Phi(y)+\sqrt{\Phi(x)\Phi(y)}}{3} \right\rceil$ or $\Phi^*(xy) = \left\lfloor \frac{\Phi(x)+\Phi(y)+\sqrt{\Phi(x)\Phi(y)}}{3} \right\rfloor$, both Φ, Φ^* are injective and $w(x) \neq w(y)$ for any $x, y \in V(K_{2,n})$ where $w(x) = \sum_{u \in N(x)} \Phi^*(ux)$. We have to show that if such a mapping Φ exists, then the induced edge mapping coincide for some edges. Let $X = \{u, v\}$ and $Y = \{u_1, u_2, \ldots, u_n\}$ be the partition of $V(K_{2,n})$. Let e be any edge of $K_{2,n}$

Case 1. $\Phi^*(e) = 1$.

Since $\Phi^*(e) = 1$, one of the end points of e must receive the label 1.
We claim that $\Phi(u) \neq 1$ and $\Phi(v) \neq 1$.

If $\Phi(u) = 1$, then there are n edges incident with the vertex u whose label can be one of the values in the set $\{1, 2, \ldots, 2n+1\}$. The largest possible value for a vertex is $2n+1$. Even when that label is assigned for a vertex, by part (1) of Lemma 3.1 the corresponding edge label is $\frac{1+2n+1+\sqrt{2n+1}}{3} < \frac{1+2n+1+n-5}{3} < n-1$ for all $n \geq 10$. Hence it is impossible to get n distinct labels for the edges incident with u and so $\Phi(u) > 1$. Analogously, $\Phi(v) > 1$.

Therefore there exists a vertex $u_i \in Y$ such that $e = uu_i$ and $\Phi(u_i) = 1$. As $\Phi^*(e) = 1$, $\Phi(u)$ should have the label 2 or 3.

If $\Phi(u) = 2$, the largest possible value is $2n+1$. But by part (2) of Lemma 3.1 the corresponding edge label is, $\frac{2+2n+1+\sqrt{4n+2}}{3} < \frac{2+2n+1+n-6}{3} < n-1$ for all $n \geq 14$. Thus we cannot distinctly label all the n edges incident with u.

If $10 \leq n \leq 13$, there are n distinct labels for n edges incident with u. Since $\Phi(u_i) = 1$, by part (1) of Lemma 3.1 we have $\Phi^*(vu_i) < n$. But there exist a vertex u_j such that $\Phi^*(vu_i) = \Phi^*(uu_j)$. Thus the edges cannot be labeled distinctly.

If $\Phi(u) = 3$, by part (3) of Lemma 3.1 and by a similar argument as above the edge values of all the n edges cannot be distinct if $n \geq 18$. For $10 \leq n \leq 12$, we cannot label the edges distinctly even if we have $n+1$ labels for n edges incident with u. Since $\Phi(u_i) = 1$, by part (1) of Lemma 3.1 we have $\Phi^*(vu_i) < n$. Suppose $\Phi^*(vu_i) \neq \Phi^*(uu_j)$ for any u_j. Then there exist a vertex $u_k \in Y$ such that $\Phi^*(uu_k) = 2$, otherwise $\Phi^*(vu_i) = \Phi^*(uu_j)$ for some $u_j \in Y$. Since $\Phi^*(uu_k) = 2$ and $\Phi(u) = 3$, we have $\Phi(u_k) = 2$. Now by applying part (2) of Lemma 3.1 $\Phi^*(vu_k) < n$. Therefore $\Phi^*(vu_k) = \Phi^*(uu_t)$ and so the edges cannot be distinctly labeled. For $13 \leq n \leq 17$, the maximum edge value is n. But there exist a vertex u_k such that $\Phi^*(uu_k) = \Phi^*(vu_i)$, which is not possible.

Hence we arrive at a contradiction if $\Phi^*(e) = 1$.

Case 2. $\Phi^*(e) \neq 1$.

Since $\Phi^*(e) \neq 1$, it follows that each edge must receive one of the value from the set $\{2, 3, \ldots, 2n+1\}$.

Let e, e' be two edges such that $\Phi^*(e) = 2$ and $\Phi^*(e') = 2n+1$ where $e' = uu_i$. In order to get an edge labeled $2n+1$, one of the end points must receive $2n+1$ and the other must receive $2n$. Then either $\Phi(u_i) = 2n+1$ and $\Phi(u) = 2n$ or $\Phi(u) = 2n+1$ and $\Phi(u_i) = 2n$. Without loss of generality, let $\Phi(u) = 2n+1$ and $\Phi(u_i) = 2n$. Since $\Phi^*(e) = 2$, e is never incident with the vertex u. Let $e = vu_j$ and $\Phi(v)$ can be one of $\{2, 3, 4, 5\}$ and $\Phi(u_j)$ can be one of $\{1, 2, 3, 4\}$. But by Case 1, we get a contradiction if $\Phi(v) = 2$ or 3 and $\Phi(u_i) = 2n+1$.

If $\Phi(v) = 4$, then $\Phi(u_j)$ must be either 1 or 2. By applying part (4) of Lemma 3.1, any edge incident with v have the edge value $\Phi^*(vu_i) \leq n-1$ for all $n \geq 17$.

For $n = 10, 11$, the maximum edge label is $n+2$ and the minimum edge label is 2 and so we have $n+1$ labels for n edges incident with v. Since $\Phi(u_j) = 1$ or 2, by part (1) or (2) of Lemma 3.1 we have $\Phi^*(uu_j) \leq n+1$. Since $\Phi^*(e) \neq 1$, there exist an edge e_i such that $\Phi^*(e_i) = 3$ and e_i must be incident with the vertex v, otherwise $\Phi^*(e_i) = 3$ is not possible. Let $e_i = vu_r$. Then the possible vertex labels for u_r can be one of $\{1, 2, 3\}$. Now by part (1),(2) or (3) of Lemma 3.1, $\Phi^*(uu_r) \leq n+1$. Then there exist a vertex $u_i \in Y$ such that either $\Phi^*(vu_i) = \Phi^*(uu_j)$ or $\Phi^*(vu_i) = \Phi^*(uu_r)$ and so the edges cannot be distinctly labeled. For $12 \leq n \leq 16$, the minimum edge label is 2 and maximum edge value is $n+1$ and so we have n labels for n edges incident with v. Since $\Phi(u_j) = 1$ or 2, by part (1) or (2) of Lemma 3.1 we have $\Phi^*(uu_j) \leq n+1$. Then there exist a vertex u_i such that $\Phi^*(vu_i) = \Phi^*(uu_j)$. Hence we cannot label the edges distinctly.

If $\Phi(v) = 5$, by applying part (5) of Lemma 3.1, we get a similar contradiction as in the above discussion. So, if $\Phi^*(e) \neq 1$, two edges receive the same edge label. Thus no such function exists. Hence for $n \geq 10$, the graph $K_{2,n}$ is not a Heronian mean anti-magic graph. □

4. Conclusion

In this paper, we have proved that $P_n \odot 2K_1$, kC_n, CL_n are all Heronian mean anti-magic and we have proved that the union of two trees of order at least 3 does not satisfy the Heronian mean anti-magic labeling. Added to this, using the inequality method we have also proved for $n \geq 10$, $K_{2,n}$ is not a Heronian mean anti-magic graph. In this paper, we propose the following open problem:

(1) Characterize for all positive integer m, n for which $K_{m,n}$ is not Heronian mean anti-magic labeling

5. Authors Contribution

Both the authors contributed to design and implementation of this work. B. Sivaranjani drafted the analysis of the results and the first draft of the manuscript. R. Kala directed the interpretation of the results.

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