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INVERSE PROBLEM FOR A TWO-DIMENSIONAL WAVE EQUATION WITH A FRACTIONAL RIEMANN-LIOUVILLE TIME DERIVATIVE

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Abstract. In this paper, we consider direct and inverse problems for a two-dimensional fractional wave equation with the Riemann-Liouville time fractional derivative. The direct problem is the initial-boundary problem for this equation with nonlocal boundary conditions. In inverse problem it is required to find time variable coefficient at the lower term of equation. Using the method of separation of variables, a classical solution of direct problem was found in the form of a bi orthogonal series in terms of eigenfunctions and associated functions. A nonlocal integral condition is used as the overdetermination condition with respect to the direct problem solution. Using the Fourier method, direct problem is reduced to equivalent integral equations. Then, using the estimates for Mittag-Leffler function and the generalized singular Gronwall inequality, we obtain an a priori estimate of the solution through an unknown coefficient, which we will need to study the inverse problem. The inverse problem is reduced to a Volterra integral equation of the second kind. Based on the unique solvability of this equation in the class of continuous functions, theorems on the unique solvability of direct and inverse problems are proven. Stability estimate is also obtained.

Keywords: wave equation, Riemann-Liouville fractional integral, inverse problem, spectral method, stability, Banach fixed point theorem.

AMS Subject Classification: 34A08, 34K10, 34K37, 34M50, 35R11.

1. Introduction

In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Major directions include anomalous diffusion, vibration and control, continuous time random walk, fractional Brownian motion, power law, fractional derivative and fractals, computational fractional derivative equations, nonlocal phenomena, biomedical engineering, fractional transforms, singularities analysis and

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integral represen-tations for fractional differential systems, special functions related to fractional calculus, heat conduction, acoustic dissipation, geophysics, relaxation, creep, viscoelasticity, rheology, fluid dynamics, and groundwater problems [1, 2, 3, 4, 5].

The identification of the right hand side and the order of time fractional derivative equation in applied fractional modeling plays an important role. In the papers [6, 7, 8], inverse problems for determining these unknowns in a subdiffusion equation with an arbitrary second order elliptic differential operator are considered. It is proved that the additional information about the solution at a fixed time instant at a monitoring location, as the observation data, identified uniquely the order of the fractional derivative.

Inverse problems for classical integro-differential equations of wave propa-gation have been widely studied.

Nonlinear inverse coefficient problems with various types of overdetermination conditions are often found in the literature (eg, [9, 10, 11, 12, 13, 14, 15, 16] and references therein). In the works [17, 18, 19, 20, 21, 22], inverse problems of determining unknown coefficients in the Cauchy problem for a fractional wave-diffusion equation were studied. Local existence and global uniqueness are proved, and conditional stability estimates are obtained.

In this paper, we study the local existence and global uniqueness of the inverse problem of determining the non-stationary coefficient in a two-dimensional fractional-time wave equation with initial, non local boundary and integral redefinition conditions.

In the domain $\Omega := \mathrm{D} \times (0,T]$, $\mathrm{D} := \{(x,y) : 0 < x, y < 1,\}$ consider the time-fractional wave equation

$$\left(D^{\alpha}_{0+t}u\right)(x,y,t) - \Delta u + q(t)u(x,y,t) = f(x,y,t), \quad (x,y,t) \in \Omega, \tag{1}$$

with initial and boundary conditions

$$I^{(2-\alpha)}_{0+t}u(x,y,t)\big|_{t=0} = \varphi_1(x,y),$$

$$\frac{\partial}{\partial t}\left(I^{(2-\alpha)}_{0+t}u\right)(x,y,t)\big|_{t=0} = \varphi_2(x,y),\ (x,y) \in [0,1]\times[0,1], \tag{2}$$

$$u(0,y,t) = u(1,y,t), u_x(1,y,t) = 0,\ (y,t) \in [0,1]\times[0,T], \tag{3}$$

$$u(x,0,t) = u(x,1,t) = 0,\ (x,t) \in [0,1]\times[0,T]. \tag{4}$$

Here the Riemann-Liouville fractional differential operator $D^{\alpha}_{0+,t}$ of the order $1 < \alpha < 2$ is defined as follows [[1], pp. 69-72,[23], pp. 62-65]

$$D^{\alpha}_{0+,t}u(\cdot,\cdot,t) = \frac{\partial^2}{\partial t^2}\left(I^{(2-\alpha)}_{0+,t}u\right)(\cdot,\cdot,t),$$

$$I^{\gamma}_{0+,t}u(x,y,t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{u(x,y,\tau)}{(t-\tau)^{1-\gamma}}d\tau,\ \gamma \in (0,1),$$

is the Riemann-Liouville fractional integral of the function $u(x,y,t)$ with respect to t [1, pp. 69-72], $\Gamma(\cdot)$ is the Euler's Gamma function, Δ is the Laplace operator.

Everywhere in this paper, functions $f(x,y,t), \varphi_1(x,y), \varphi_2(x,y)$ are known functions.

We pose the inverse problem as follows: find the function $q(t) \in C[0,T]$ in (1), if the solution of the initial-boundary problem (1)-(4) satisfies condition:

$$\int_0^1\int_0^1 w(x,y)u(x,y,t)dxdy = h(t), 0 \le t \le T. \tag{5}$$

where $w(x,y), h(t)$ are known functions.

Assume that throughout this article, given functions $\varphi_1, \varphi_2, f, w$ and h satisfy the following conditions:

A1) $\{\varphi_1, \varphi_2\} \in C^3([0,1]\times[0,1])$, $\left\{\varphi_1^{(4)}, \varphi_2^{(4)}\right\} \in L_2([0,1]\times[0,1])$; $\varphi(0,y)=\varphi(1,y)=0$, $\varphi_x(0,y)=\varphi_x(1,y)=0$, $\varphi_{xx}(0,y)=\varphi_{xx}(1,y)=0$, $\varphi(x,0)=\varphi(x,1)=0$, $\varphi_y(x,0)=\varphi_y(x,1)=0$, $\varphi_{yy}(x,0)=\varphi_{yy}(x,1)=0$.

A2) $f(x,y,\cdot)\in C[0,T], t\in[0,T]$, $f(\cdot,\cdot,t)\in C^3([0,1]\times[0,1])$, $f^{(4)}(\cdot,\cdot,t)\in L_2([0,1]\times[0,1])$ $f(0,y,t)=f(1,y,t)=0$, $f_x(0,y,t)=f_x(1,y,t)=0$, $f_{xx}(0,y,t)=f_{xx}(1,y,t)=0$, $f(x,0,t)=f(x,1,t)=0$, $f_y(x,0,t)=f_y(x,1,t)=0$, $f_{yy}(x,0,t)=f_{yy}(x,1,t)=0$.

A3) $w(x,y)\in C^2([0,1]\times[0,1]); w(0,y)=0, w_x(0,y)=w_x(1,y)=0,\ w_{xx}(0,y)=w_{xx}(1,y)=0,$ and $w(x,1)=w(x,0)=0$.

A4) $\left(D^{\alpha}_{0+,t}h\right)(t)\in C[0,T], |h(t)|\geq h_0>0, h_0$ is a given number:

$$\int_0^1\int_0^1 w(x,y)\varphi_1(x,y)dxdy=\left(I^{(2-\alpha)}_{0+,t}h\right)(t)_{t=0+},$$

$$\int_0^1\int_0^1 w(x,y)\varphi_2(x,y)dxdy=\frac{\partial}{\partial t}\left(I^{(2-\alpha)}_{0+,t}h\right)(t)_{t=0+}.$$

In section 2, we will provide some necessary preliminary informations.

2. Preliminaries

In this section, we present some useful definitions and results of fractional calculus.

The two parameter function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha,\beta,z\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\alpha)-$denotes the real part of the complex number α. The Mittag-Leffler function has been studied by many authors who have proposed and studied various generalizations and applications.

Proposition 2.1. Let $0<\alpha<2$ and $\beta\in\mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2<\kappa<\min\{\pi,\pi\alpha\}$. Then there exists a constant $C=C(\alpha,\beta,\kappa)>0$ such that

$$|E_{\alpha,\beta}(z)|\leq\frac{C}{1+|z|},\quad \kappa\leq|\arg(z)|\leq\pi.$$

For the proof, we refer to [[4].pp. 40-45], for example. We consider the weighted spaces of continuous functions [[1].pp. 4-5, 162-163].

$$C_\gamma[a,b]:=\{g:(a,b]\to R:(t-a)^\gamma g(t)\in C[a,b],\ 0\leq\gamma<1,\},$$

$$C^{\alpha}_{\gamma}(\Omega)=\left\{g(t):\ D^{\alpha}_{0+,t}g(t)\in C_\gamma(0,T];\ 1<\alpha\leq 2,\ \right\},$$

$$C^{2,\alpha}_{\gamma}(\Omega)=\Big\{u(x,y,t):\ u(\cdot,\cdot,t)\in C^2(0,1);\ t\in[0,T]$$

and

$$D^{\alpha}_{0+,t}u(x,y,\cdot)\in C_\gamma(0,T];\ x,y\in[0,1],\ 1<\alpha\leq 2,\ \Big\},$$

$$C^0_\gamma[a,b]=C_\gamma[a,b],$$

with the norms

$$\|f\|_{C_\gamma}=\|(t-a)^\gamma f(t)\|_C,\quad \|f\|_{C^n_\gamma}=\sum_{k=0}^{n-1}\|f^{(k)}\|_C+\|f^{(n)}\|_{C_\gamma}.$$

Lemma 2.1. *[[24],pp.188]. Suppose $b \geq 0$, $\alpha > 0$ and $a(t)$ nonnegative function locally integrable on $0 < t \leq T$ (some $T \leq +\infty$) and suppose $u(t)$ is nonnegative and locally integrable on $0 < t \leq T$ with*

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} u(s) ds$$

then

$$u(t) \leq a(t) + b\Gamma(\alpha) \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(b\Gamma(\alpha)(t-s)^{\alpha} \right) a(s) ds.$$

Lemma 2.2. *[[24],pp.189]. Suppose $b \geq 0$, $\alpha > 0$, $\gamma > 0$, $\alpha + \gamma > 1$ and $a(t)$ nonnegative function locally integrable on $0 < t \leq T$ and suppose $t^{\gamma-1} u(t)$ is nonnegative and locally integrable on $0 < t \leq T$ with*

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} u(s) ds,$$

then

$$u(t) \leq a(t) E_{\alpha,\gamma} \left((b\Gamma(\alpha))^{\frac{1}{\alpha+\gamma-1}} t \right),$$

where

$$E_{\alpha,\gamma}(t) = \sum_{m=0}^{\infty} c_m t^{m(\alpha+\gamma-1)}, \ c_0 = 1, \ \frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)}$$

for $m \geq 0$. As $t \to +\infty$, $E_{\alpha,\gamma}(t) = O\left(t^{\frac{1}{2}\frac{\alpha+\gamma-1}{\alpha} - \gamma} \exp\left(\frac{\alpha+\gamma-1}{\alpha} t^{\frac{\alpha+\gamma-1}{\alpha}} \right) \right)$.

3. Investigation of the Spectral Problem

Consider the following equation

$$\left(D_{0+,t}^{\alpha} u \right)(x,y,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \ (x,y,t) \in \Omega. \tag{6}$$

We will look for particular solutions to the problem (6), (3) - (4) in the form

$$u(x,y,t) = Z(x,y) v(t). \tag{7}$$

Substituting this expression into equation (6) and boundary conditions (3), (4) and separating the variables, we obtain the problem for finding the eigenfunctions

$$\partial^2 Z / \partial x^2 + \partial^2 Z / \partial y^2 + \mu Z = 0, \ 0 < x, y < 1, \tag{8}$$

$$Z(0,y) = Z(1,y), \frac{\partial Z(1,y)}{\partial x} = 0, \tag{9}$$

$$Z(x,0) = Z(x,1) = 0, \ 0 \leq x, y \leq 1, \tag{10}$$

here μ is the split parameter. Note that problem (8)-(9) is not self-adjoint in the sense of the scalar product $(\psi, \xi) = \int_0^1 \int_0^1 \psi(x,y) \xi(x,y) dx dy$. It is easy to verify that the problem associated with it will be

$$\partial^2 W / \partial x^2 + \partial^2 W / \partial y^2 + \mu W = 0, 0 < x, y < 1, \tag{11}$$

$$W(0,y) = 0, \ \frac{\partial W(0,y)}{\partial x} = \frac{\partial W(1,y)}{\partial x}, \tag{12}$$

$$W(x,0) = W(x,1) = 0, \ 0 \leq x, y \leq 1. \tag{13}$$

Let's solve problem (8)-(10). To do this, we represent its solution in the form

$$Z(x,y)=X(x)Y(y). \tag{14}$$

Substituting this expression into equation (8) and boundary conditions (9),(10), we arrive at the problems

$$X''(x)+\lambda X(x)=0,\quad 0<x<1,\quad X(0)=X(1),\quad X'(1)=0, \tag{15}$$

$$Y''(y)+\gamma Y(y)=0,\quad 0<y<1,\quad Y(0)=Y(1)=0, \tag{16}$$

where $\gamma=\mu-\lambda$.

The solution to problem (16), as is known, has the form $\gamma_k=(\pi k)^2, Y_k(y)=\sqrt{2}\sin(\sqrt{\gamma_k}y)$ $k=1,2,\ldots$. Here and below, the constants for eigenfunctions and associated functions are chosen from the normalization conditions.

The eigenvalues and the corresponding eigenfunctions of problem (15) were found in [25];

$$X_0=2,\quad X_m(x)=4\cos(2\pi mx),\quad \lambda_m=(2\pi m)^2,\quad m=1,2,\ldots.$$

Therefore, according to representation (7), the eigenvalues and eigenfunctions of problem (8)-(9) have the form $\mu_{mk}=\lambda_m+\gamma_k=(2\pi m)^2+(\pi k)^2, Z_{mk}(x,y)=X_m(x)Y_k(y)$, $m=0,1,2,\ldots$, $k=1,2,\ldots$. Note that the collection of eigenfunctions $Z_{mk}(x,y)$ is incomplete in the space $L_2([0,1]\times[0,1])$, therefore, by analogy with [26], we complete the set of eigenfunctions with the associated functions $\tilde{Z}_{mk}(x,y), m=0,1,2,\ldots,k=1,2,\ldots$, which are the solution to the problem

$$\partial^2\tilde{Z}_{mk}/\partial x^2+\partial^2\tilde{Z}_{mk}/\partial y^2+\mu_{mk}\tilde{Z}_{mk}=p_m Z_{mk}, 0<x,y<1,$$

$$\tilde{Z}_{mk}(0,y)=\tilde{Z}_{mk}(1,y),\quad \partial\tilde{Z}_{mk}(1,y)/\partial x=0,\quad 0\le y\le 1,$$

$$\tilde{Z}_{mk}(x,0)=\tilde{Z}_{mk}(x,1)=0,\quad 0\le x\le 1, \tag{17}$$

where $p_m\neq 0$ is some constant. For $m=0$, $k=1,2\ldots$, this problem has no solution. Setting $p_m=-2\sqrt{\lambda_m}$, for m, $k=1,2,\ldots$, we get

$$\tilde{Z}_{mk}=4\sqrt{2}(1-x)\sin(2\pi mx)\sin(\pi ky).$$

Let us redesignate the system of eigenfunctions and associated functions of problem (8)-(10) as follows:

$$Z_{0k}(x,y)=X_0(x)Y_k(y),\quad Z_{2m-1k}(x,y)=X_{2m-1}(x)Y_k(y),\quad Z_{2mk}(x,y)=\tilde{Z}_{mk}. \tag{18}$$

The eigenfunctions and associated functions of the adjoint problem (8) have the form

$$W_{0k}(x,y)=xY_k(y),\quad W_{2mk}(x,y)=\sin(2\pi mx)Y_k(y),$$

$$W_{2m-1k}(x,y)=x\cos(2\pi mx)Y_k(y),\quad m,\,k=1,2,\ldots. \tag{19}$$

Note that the sequences of functions (18), (19) form a biorthonormal relationship: $(Z_{mk},W_{lp})=1$ for $m=l$, $k=p$, otherwise $(Z_{mk},W_{lp})=0$.

4. Estimate for solving the direct problem.

According to (7), particular solutions to problem (8)-(10) have the form

$$u(x,y,t)=\sum_{k=1}^{\infty}Z_{0k}(x,y)v_{0k}(t)+$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left(Z_{2m-1k}(x,y)v_{2m-1k}(t)+Z_{2mk}(x,y)v_{2mk}(t)\right). \tag{20}$$

The coefficients $v_{0k}(t),\ v_{2m-1k}(t),\ v_{2m-1k}(t)$ are to be found by making use of the orthogonality of the eigenfunctions. Namely, we multiply (1) by the eigenfunctions of (19) and integrate over $(0,1)$. Recall that the scalar product in $L_2([0,1]\times[0,1])$ is defined by

$$(f,g)=\int_0^1\int_0^1 f(x,y)g(x,y)dxdy.$$

Let us note the expansion coefficients of $f(x,y,t)$ and $\varphi(x,y)$ in the eigenfunctions of (19) for respectively by

$$\begin{cases}(f(x,y,t),W_{0k}(x,y))=f_{0k}(t),\\(f(x,y,t),W_{2m-1k}(x,y))=f_{2m-1k}(t),\\(f(x,y,t),W_{2mk}(x,y))=f_{2mk}(t),\end{cases}$$

$$\begin{cases}(\varphi_i(x,y),W_{0k}(x,y))=\varphi_{0k,i},\\(\varphi_i(x,y),W_{2m-1k}(x,y))=\varphi_{2m-1k,i}, \qquad i=1,2.\\(\varphi_i(x,y),W_{2mk}(x,y))=\varphi_{2mk,i},\end{cases} \tag{21}$$

We obtain in view of (1) and with $(u(x,y,t),W_{0k}(x,y))=v_{0k}(t)$, and first component of (21), we may write

$$\begin{cases}\left(D^{\alpha}_{0+,t}v_{0k}\right)(t)+\gamma_k^2v_{0k}+q(t)v_{0k}(t)=f_{0k}(t),\\ I^{(2-\alpha)}_{0+,t}v_{0k}(t)\Big|_{t=0}=\varphi_{0k,1},\ \frac{d}{dt}\left(I^{(2-\alpha)}_{0+,t}v_{0k}\right)(t)\Big|_{t=0}=\varphi_{0k,2}.\end{cases} \tag{22}$$

Also, the linear fractional differential equations satisfied by $(u(x,y,t),W_{2mk}(x,y))=v_{2km}(t)$, $m,k\geq 1$; are

$$\begin{cases}\left(D^{\alpha}_{0+,t}v_{2mk}\right)(t)+\mu_{mk}^2v_{2mk}(t)+q(t)v_{2mk}(t)=f_{2mk}(t),\\ I^{(2-\alpha)}_{0+,t}v_{2mk}(t)\Big|_{t=0}=\varphi_{2mk,1},\ \frac{d}{dt}\left(I^{(2-\alpha)}_{0+,t}v_{2mk}\right)(t)\Big|_{t=0}=\varphi_{2mk,2}.\end{cases} \tag{23}$$

For $v_{2m-1k}(t)=(u(x,y,t),W_{2m-1k}(x,y))\,;m,k\geq 1$, in view of (1) we have

$$\begin{cases}\left(D^{\alpha}_{0+,t}v_{2m-1k}\right)(t)+\mu_{mk}^2v_{2m-1k}+2\lambda_m u_{2mk}+q(t)v_{2m-1k}(t)=f_{2m-1k}(t),\\ I^{(2-\alpha)}_{0+,t}v_{2m-1k}(t)\Big|_{t=0}=\varphi_{2m-1k,1},\ \frac{d}{dt}\left(I^{(2-\alpha)}_{0+,t}v_{2m-1k}\right)(t)\Big|_{t=0}=\varphi_{2m-1k,2}.\end{cases} \tag{24}$$

We solve problems (22)-(24).

Based [[27], pp. 61-114], we have that the initial problem (22) is equivalent in the space $C^{\alpha}_{\gamma}[0,T]$ to the Volterra integral equation of the second kind

$$v_{0k}(t)=t^{\alpha-2}E_{\alpha,\alpha-1}\left(-\gamma_k^2t^{\alpha}\right)\varphi_{0k,1}+t^{\alpha-1}E_{\alpha,\alpha}\left(-\gamma_k^2t^{\alpha}\right)\varphi_{0k,2}+$$

$$+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\gamma_k^2(t-\tau)^{\alpha}\right)f_{0k}(\tau)d\tau-$$

$$-\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\gamma_k^2(t-\tau)^{\alpha}\right)q(\tau)v_{0k}(\tau)d\tau. \tag{25}$$

We prove the following assertions for $v_{0k}(t)$:

Lemma 4.1. *We have the estimates*

$$t^{\gamma}\left|v_{0k}\right| \leq \left(t^{\gamma+\alpha-2} M_1\left|\varphi_{0k,1}\right|+t^{\gamma+\alpha-1} M_2\left|\varphi_{0k,2}\right|+\right.$$
$$\left.+\frac{\left\|f_{0k}\right\|_{C_\gamma[0,T]} t^\alpha B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}} t\right);$$

$$t^{\gamma}\left|\left(D_{0+,t}^{\alpha} v_{0k}\right)(t)\right| \leq\left\|f_{0k}\right\|_{C_\gamma[0,T]}+\left(\gamma_k^2+\|q\|_{C[0,T]}\right)\left(t^{\gamma+\alpha-2} M_1\left|\varphi_{0k,1}\right|+\right.$$
$$\left.+t^{\gamma+\alpha-1} M_2\left|\varphi_{0k,2}\right|+\frac{\left\|f_{0k}\right\|_{C_\gamma[0,T]} t^\alpha B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}} t\right),$$

where $1>\gamma>2-\alpha$.

Proof. The solution of (22) is bounded in $C_{\gamma}^{\alpha,\beta}[0,T]$ in view of A1), A2).

Multiplying the last equation (22) by t^{γ}, we get

$$t^{\gamma}\left|v_{0k}\right| \leq t^{\gamma+\alpha-2} M_1\left|\varphi_{0k,1}\right|+t^{\gamma+\alpha-1} M_2\left|\varphi_{0k,2}\right|+$$
$$+\frac{\left\|f_{0k}\right\|_{C_\gamma[0,T]} t^\alpha B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}+\frac{|q|_{C[0,T]} t^\gamma}{\Gamma(\alpha)} \int_0^t(t-\tau)^{\alpha-1}\left|v_{0k}(\tau)\right| d\tau, \tag{26}$$

where $B(\alpha,1-\gamma)$ is Euler's beta function. Next, according to Lemma 2.2, we get

$$t^{\gamma}\left|v_{0k}\right| \leq \left(t^{\gamma+\alpha-2} M_1\left|\varphi_{0k,1}\right|+t^{\gamma+\alpha-1} M_2\left|\varphi_{0k,2}\right|+\frac{\left\|f_{0k}\right\|_{C_\gamma[0,T]} t^\alpha B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) \times$$
$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}} t\right)=: \Psi_{0k}(t), t \in[0,T]. \tag{27}$$

We get the second part of the lemma 4.1, from the equation in problem (22) and the first estimate of lemma 4.1:

$$t^{\gamma}\left|\left(D_{0+,t}^{\alpha} v_{0k}\right)(t)\right| \leq\left\|f_{0k}\right\|_{C_\gamma[0,T]}+\left(\gamma_k^2+\|q\|_{C[0,T]}\right)\left(t^{\gamma+\alpha-2} M_1\left|\varphi_{0k,1}\right|+\right.$$
$$\left.+t^{\gamma+\alpha-1} M_2\left|\varphi_{0k,2}\right|+\frac{\left\|f_{0k}\right\|_{C_\gamma[0,T]} t^\alpha B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}} t\right).$$

From the last two inequalities we immediately obtain the estimates of lemma 4.1 for any $t\in[0,T]$. Lemma 4.1 proven. □

In view of [[27], pp. 61-114], we have that the initial problems (23), (24) are equivalent in the space $C_{\gamma}^{\alpha}[0,T]$ to the Volterra integral equations of the second kind

$$v_{2mk}(t)=t^{\alpha-2} E_{\alpha,\alpha-1}\left(-\mu_{mk}^2 t^\alpha\right) \varphi_{2mk,1}+t^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{mk}^2 t^\alpha\right) \varphi_{2mk,2}+$$
$$+\int_0^t(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{mk}^2(t-\tau)^\alpha\right) f_{2mk}(\tau) d\tau-$$
$$-\int_0^t(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{mk}^2(t-\tau)^\alpha\right) q(\tau) v_{2mk}(\tau) d\tau, \tag{28}$$

$$v_{2m-1k}(t)=t^{\alpha-2} E_{\alpha,\alpha-1}\left(-\mu_{mk}^2 t^\alpha\right) \varphi_{2m-1k,1}+t^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{mk}^2 t^\alpha\right) \varphi_{2m-1k,2}+$$
$$+\int_0^t(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{mk}^2(t-\tau)^\alpha\right) f_{2m-1k}(\tau) d\tau-$$
$$-2\lambda_m \int_0^t(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{mk}^2(t-\tau)^\alpha\right) v_{2mk}(\tau) d\tau-$$

$$-\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{mk}^2 (t-\tau)^{\alpha}\right) q(\tau) v_{2m-1k}(\tau) d\tau. \tag{29}$$

Estimating the integral equation (25), analogous, we get estimates for equations (28), (29).

$$t^{\gamma}\left|v_{2mk}\right| \leq \left(t^{\gamma+\alpha-2} M_1 \left|\varphi_{2mk,1}\right| + t^{\gamma+\alpha-1} M_2 \left|\varphi_{2mk,2}\right| + \right.$$
$$\left. + \frac{\left\|f_{2mk}\right\|_{C_{\gamma}[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right) =: \Psi_{2m}(t), t \in [0,T],$$
$$t^{\gamma}\left|v_{2m-1k}\right| \leq \left(t^{\gamma+\alpha-2} M_1 \left|\varphi_{2m-1k,1}\right| + t^{\gamma+\alpha-1} M_2 \left|\varphi_{2m-1k,2}\right| + \right.$$
$$\left. +2\lambda_m \frac{\Psi_{2mk}(t) t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} + \frac{\left\|f_{2m-1k}\right\|_{C_{\gamma}[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) \times$$
$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right) =: \Psi_{2m-1k}(t), t \in [0,T], \tag{30}$$

and

$$t^{\gamma}\left|\left(D_{0+,t}^{\alpha} v_{2mk}\right)(t)\right| \leq \left\|f_{2mk}\right\|_{C_{\gamma}[0,T]} + \left(\mu_m^2 + \|q\|_{C[0,T]}\right)\left(t^{\gamma+\alpha-2)} M_1 \left|\varphi_{2mk,1}\right| + \right.$$
$$\left. + t^{\gamma+\alpha-1)} M_2 \left|\varphi_{2mk,2}\right| + \frac{\left\|f_{2mk}\right\|_{C_{\gamma}[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) \times$$
$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right) =: \bar{\Psi}_{2mk}(t), t \in [0,T],$$
$$t^{\gamma}\left|\left(D_{0+,t}^{\alpha} v_{2m-1k}\right)(t)\right| \leq \left\|f_{2m-1k}\right\|_{C_{\gamma}[0,T]} + 2\lambda_m \left|u_{2mk}\right|_{C_{\gamma}[0,T]} +$$
$$+ \left(\mu_m^2 + \|q\|_{C[0,T]}\right)\left(t^{\gamma+\alpha-2} M_1 \left|\varphi_{2m-1k,1}\right| + t^{\gamma+\alpha-1} M_2 \left|\varphi_{2m-1k,2}\right| + \right.$$
$$\left. + \frac{\left\|f_{2m-1k}\right\|_{C_{\gamma}[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)} + 2\lambda_m \frac{\bar{\Psi}_{2mk}(t) t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)}\right) \times$$
$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right), t \in [0,T], \tag{31}$$

where $1 > \gamma > 2 - \alpha$.

From the last two inequalities we obtain the estimates of lemma 4.1 for any $t \in [0,T]$. Formally, from (20) by term-by-term differentiation we compose the series

$$\left(D_{0+,t}^{\alpha} u\right)(x,y,t) = 2\sqrt{2} \sum_{k=1}^{\infty}\left(D_{0+,t}^{\alpha} v_{0k}\right)(t) \sin\left(\gamma_k y\right) +$$
$$+4\sqrt{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty}\left(D_{0+,t}^{\alpha} v_{2m-1k}\right)(t) \cos\left(\lambda_m x\right) \sin\left(\gamma_k y\right) +$$
$$+4\sqrt{2}(1-x) \sum_{k=1}^{\infty} \sum_{m=1}^{\infty}\left(D_{0+,t}^{\alpha} v_{2mk}\right)(t) \sin\left(\lambda_m x\right) \sin\left(\gamma_k y\right), \tag{32}$$

$$u_{xx}(x,y,t) = -4\sqrt{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \lambda_m^2 v_{2m-1k}(t) \cos\left(\lambda_m x\right) \sin\left(\gamma_k y\right) -$$
$$-8\sqrt{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \lambda_m v_{2mk}(t) \cos\left(\lambda_m x\right) \sin\left(\gamma_k y\right) -$$

$$-4\sqrt{2}(1-x)\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\lambda_m^2 v_{2mk}(t)\sin(\lambda_m x)\sin(\gamma_k y), \quad (33)$$

$$u_{yy}(x,y,t) = -2\sqrt{2}\sum_{k=1}^{\infty}\gamma_k^2 v_{0k}(t)\sin(\gamma_k y) -$$

$$-4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\gamma_k^2 v_{2m-1k}(t)\cos(\lambda_m x)\sin(\gamma_k y) -$$

$$-4\sqrt{2}(1-x)\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\gamma_k^2 v_{2mk}(t)\sin(\lambda_m x)\sin(\gamma_k y). \quad (34)$$

Let us prove the uniform convergence of series (20), (32)-(34) in the domain $\bar{\Omega}$. This series for any $(x,y,t)\in\bar{\Omega}$ is majorized as follows

$$2\sum_{k=1}^{\infty}\Psi_{0k}(T) + 4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Psi_{2m-1k}(T) + 4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Psi_{2mk}(T),$$

$$2\sqrt{2}\sum_{k=1}^{\infty}\Big(\|f_{0k}\|_{C_\gamma[0,T]} + \left(\gamma_k^2 + \|q\|_{C[0,T]}\right)\Psi_{0k}(T)\Big) +$$

$$+4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Big(\|f_{2m-1k}\|_{C_\gamma[0,T]} + 2\lambda_m\Psi_{2mk}(T) + \left(\mu_{mk}^2 + \|q\|_{C[0,T]}\right)\Psi_{2m-1k}(T)\Big) +$$

$$+4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Big(\|f_{2mk}\|_{C_\gamma[0,T]} + \left(\mu_{mk}^2 + \|q\|_{C[0,T]}\right)\Psi_{2mk}(T)\Big),$$

$$4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\lambda_m^2\Psi_{2m-1k}(T) + 4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left(\lambda_m^2 + 2\lambda_m\right)\Psi_{2mk}(T),$$

$$2\sqrt{2}\sum_{k=1}^{\infty}\gamma_k^2\Psi_{0k}(T) + 4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\gamma_k^2\Psi_{2m-1k}(T) + 4\sqrt{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\gamma_k^2\Psi_{2mk}(T),$$

where $\bar{\Omega} := \{(x,t) : 0 \le x, y \le 1, 0 \le t \le T\}$.

Lemma 4.2. *If conditions A1), A2) are satisfied, then the equalities*

$$\varphi_{0k,i} = \frac{\varphi_{0k,i}^{(0,4)}}{\gamma_k^4}, \varphi_{2m-1k,i} = \frac{\varphi_{2m-1k,i}^{(2,2)}}{\gamma_k^2\left(\lambda_m^2 + 2\lambda_m\right)}, \varphi_{2mk,i} = \frac{\varphi_{2mk,i}^{(2,2)}}{\lambda_m^2\gamma_k^2}, i = 1, 2,$$

$$f_{0k}(t) = \frac{f_{0k}^{(0,4)}(t)}{\gamma_k^4}, f_{2m-1k}(t) = \frac{f_{2m-1k}^{(2,2)}(t)}{\gamma_k^2\left(\lambda_m^2 + 2\lambda_m\right)}, f_{2mk}(t) = \frac{f_{2mk}^{(2,2)}(t)}{\lambda_m^2\gamma_k^2}, \quad (35)$$

here $\varphi_{0k,i}^{(0,4)}$, $\varphi_{2m-1k,i}^{(2,2)}$, $\varphi_{2mk,i}^{(2,2)}$, $f_{0k}^{(0,4)}$, $f_{2m-1k}^{(2,2)}$, $f_{2mk}^{(2,2)}-$ *expansion coefficients of the functions* $\varphi(x,y), f(x,y,t)$ *in a Fourier series with the following estimates:*

$$\sum_{k\ge1}\left|\varphi_{0k,i}^{(0,4)}\right| \le \iint_D \left(\varphi_{yyyy,i}(x,y)\right)^4 dxdy, 0 \le x, y \le 1,$$

$$\sum_{m,k\ge1}\left|\varphi_{2m-1k,i}^{(2,2)}\right|^4 \le \iint_D \left(\varphi_{xxyy,i}(x,y)\right)^4 dxdy, 0 \le x, y \le 1,$$

$$\sum_{m,k\ge1}\left|\varphi_{2mk,i}^{(2,2)}\right|^4 \le \iint_D \left(\varphi_{xxyy,i}(x,y)\right)^4 dxdy, 0 \le x, y \le 1,$$

$$\sum_{k\geq 1}\left|f_{0k}^{(0,4)}\right|^4 \leq \iint_D (f_{yyyy}(x,y,t))^4\,dxdy, 0\leq t\leq T,$$

$$\sum_{m,k\geq 1}\left|f_{2m-1k}^{(2,2)}\right|^4 \leq \iint_D (f_{xxyy}(x,y,t))^4\,dxdy, 0\leq t\leq T,$$

$$\sum_{m,k\geq 1}\left|f_{2mk}^{(2,2)}\right|^4 \leq \iint_D (f_{xxyy}(x,y,t))^4\,dxdy, 0\leq t\leq T. \tag{36}$$

If the functions $\varphi(x,y), f(x,y,t)$ satisfy the conditions of lemma 4.2, then, due to representations (35) and (36), series (20), (32)-(34) converge uniformly in the rectangle $\bar{\Omega}$, so the function $u(x,y,t)$ satisfies relations (1)-(4).

Using the above results, we obtain the following assertion.

Theorem 4.1. *Let $q(t)\in C[0,T]$, A1), A2) hold, then there exists a unique solution to the direct problem (1)–(4) $u(x,y,t)\in C^{2,\alpha}_{xy,t}(\bar{\Omega})$.*

5. Continuous Dependence on the Data

We use the topological product of Banach spaces $K = C^\alpha_\gamma[0,T]\times C^\alpha_\gamma[0,T]\times C^\alpha_\gamma[0,T]$ with its norm to prove the existence and uniqueness of a solution in this form $(v_{0k}(t), v_{2mk}(t), v_{2m-1k}(t))\in K$. Define an operator $\mathcal{A}$ on K formula

$$\mathcal{A}\left(v_{0k}(t), v_{2mk}(t), v_{2m-1k}(t)\right) = \left(P_{0k}v_{0k}(t), P_{2mk}v_{2mk}(t), P_{2m-1k}v_{2m-1k}(t)\right)$$

where the operators $P_{0k}, P_{2mk}, P_{2m-1k}$ are defined on $C^\alpha_\gamma[0,T]$ by the right-hand side of (25),(28) and (29), respectively. In view of (27), (30) , (31) $\mathcal{A}: K\to K$.

Prove that $\mathcal{A}$ is a constraint on K. So for each

$$\left((v_{0k}(t), v_{2m-1k}(t), v_{2mk}(t))\,;\,(\tilde{v}_{0k}(t), \tilde{v}_{2m-1k}(t), \tilde{v}_{2mk}(t))\right)\in K$$

we have

$$\left\|\mathcal{A}\left(v_{0k}, v_{2mk}, v_{2m-1k}\right) - \mathcal{A}\left(\tilde{v}_{0k}, \tilde{v}_{2mk}, \tilde{v}_{2m-1k}\right)\right\|_K \leq$$

$$\leq \max\left\{\,\|P_{0k}v_{0k} - P_{0k}\tilde{v}_{0k}\|_{C^\alpha_\gamma[0,T]}\,,\right.$$

$$\left.\|P_{2mk}v_{2mk} - P_{2mk}\tilde{v}_{2mk}\|_{C^\alpha_\gamma[0,T]}\,, \|P_{2m-1k}v_{2m-1k} - P_{2m-1k}\tilde{v}_{2m-1k}\|_{C^\alpha_\gamma[0,T]}\right\}.$$

First, we easily get

$$\|P_{0k}v_{0k}(t) - P_{0k}\tilde{v}_{0k}(t)\|_{C^\alpha_\gamma} \leq \frac{\|q\|_{C[0,T]}t^\gamma}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|v_{0k}(\tau) - \tilde{v}_{0k}(\tau)\right|d\tau \leq$$

$$\leq \frac{\|q\|_{C[0,T]}T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha)}\,\|v_{0k} - \tilde{v}_{0k}\|_{C^\alpha_\gamma[0,T]}\,.$$

For P_{2mk} we have for every $t\in[0,T]$

$$\|P_{2mk}v_{2mk} - P_{2mk}\tilde{v}_{2mk}\|_{C^\alpha_\gamma[0,T]} \leq \frac{\|q\|_{C[0,T]}T^\gamma B(\alpha, 1-\gamma)}{\Gamma(\alpha)}\,\|v_{2mk} - \tilde{v}_{2mk}\|_{C^\alpha_\gamma[0,T]}$$

where $m,k\geq 1$.

Similarly, for each $t\in[0,T]$

$$\|P_{2m-1k}v_{2m-1k}(t) - P_{2m-1k}\tilde{v}_{2m-1k}(t)\|_{C^\alpha_\gamma[0,T]} \leq \frac{2\lambda_m t^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)}\times$$

$$\times \left\| v_{2mk} - \tilde{v}_{2mk} \right\|_{C_\gamma[0,T]} + \frac{\|q\|_{C[0,T]} t^\gamma}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left| v_{2m-1k}(\tau) - \tilde{v}_{2m-1k}(\tau) \right| d\tau \leq$$

$$\leq \frac{2\lambda_m T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \left\| v_{2mk} - \tilde{v}_{2mk} \right\|_{C_\gamma^\alpha[0,T]} +$$

$$+ \frac{\|q\|_{C[0,T]} T^\gamma B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \left\| v_{2m-1k} - \tilde{v}_{2m-1k} \right\|_{C_\gamma^\alpha[0,T]}$$

which gives for $m, k \geq 1$.

As a result

$$\left\| \mathcal{A}\left(v_{0k}, v_{2mk}, v_{2m-1k}\right) - \mathcal{A}\left(\tilde{v}_{0k}, \tilde{v}_{2mk}, \tilde{v}_{2m-1k}\right) \right\|_K \leq$$

$$\leq \max \left\{ \frac{\|q\|_{C[0,T]} T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \left\| v_{0k} - \tilde{v}_{0k} \right\|_{C_\gamma^\alpha[0,T]}, \right.$$

$$\frac{\|q\|_{C[0,T]} T^\gamma B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \left\| v_{2mk} - \tilde{v}_{2mk} \right\|_{C_\gamma^\alpha[0,T]},$$

$$\frac{2\lambda_m T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \left\| v_{2mk} - \tilde{v}_{2mk} \right\|_{C_\gamma^\alpha[0,T]} +$$

$$\left. + \frac{\|q\|_{C[0,T]} T^\gamma B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \left\| v_{2m-1k} - \tilde{v}_{2m-1k} \right\|_{C_\gamma^\alpha[0,T]} \right\} \leq$$

$$\leq \max \left\{ \frac{\|q\|_{C[0,T]} T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha)} + \frac{2\lambda_m T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \right\} \times$$

$$\times \left\| \left(v_{0k}, v_{2mk}, v_{2m-1k}\right) - \left(\tilde{v}_{0k}, \tilde{v}_{2mk}, \tilde{v}_{2m-1k}\right) \right\|_{C_\gamma^\alpha[0,T]}$$

If

$$\max \left\{ \frac{\|q\|_{C[0,T]} T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha)} + \frac{2\lambda_m T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \right\} < 1$$

then, $\mathcal{A}$ is a contraction on K and has a unique fixed point which is the coefficients $v_{0k}(t)$, $v_{2mk}(t)$, $v_{2m-1k}(t)$ of the solution (20). Then, there exists a unique solution of (1)-(4) for arbitrary $q(t)$ in $C[0,T]$.

6. Study of the inverse problem (1)-(5).

Multiply (1) by $w(x,y)$ and integrate in x, y from 0 to 1. As a result we have

$$\int_0^1 \int_0^1 w(x,y) \left\{ \left(D_{0+,t}^\alpha u\right)(x,y,t) - \Delta u + q(t) u(x,y,t) \right\} dxdy =$$

$$= \int_0^1 \int_0^1 w(x,y) f(x,y,t) dxdy. \tag{37}$$

The future after simple transformations, taking into account A3), we get

$$\left(D_{0+,t}^\alpha h\right)(t) + q(t) h(t) - \int_0^1 \int_0^1 \Delta w u(x,y,t) dxdy = \int_0^1 \int_0^1 w(x,y) f(x,y,t) dxdy,$$

which gives

$$q(t) = \frac{1}{h(t)} \left(\int_0^1 \int_0^1 w(x,y) f(x,y,t) dxdy - \left(D_{0+,t}^\alpha h\right)(t) \right) +$$

$$+ \frac{1}{h(t)} \int_0^1 \int_0^1 \Delta w \left(\sum_{k=1}^\infty Z_{0k}(x,y) v_{0k}(t) + \sum_{k=1}^\infty \sum_{m=1}^\infty Z_{2m-1k}(x,y) v_{2m-1k}(t) + \right.$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} Z_{2mk}(x,y)v_{2mk}(t)\Bigg) dxdy.$$

The functions $u_{mk}(t)$ depend on $q(t)$, i.e., $u_{mk}(t;q)$. After a simple transformation, we obtain the following integral equation for determining $q(t)$:

$$q(t) = q_0(t) + \frac{1}{h(t)}\left(\sum_{k=1}^{\infty} w_{0k}v_{0k}(t;q) + \right.$$

$$\left. + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} w_{2m-1k}v_{2m-1k}(t;q) + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} w_{2mk}v_{2mk}(t;q)\right), \tag{38}$$

where

$$q_0(t) = \frac{1}{h(t)}\left(\int_0^1\int_0^1 w(x,y)f(x,y,t)dxdy - \left(D_{0+,t}^{\alpha}h\right)(t)\right),$$

$$w_{0k} = \int_0^1\int_0^1 \Delta w Z_{0k}(x,y)dxdy,\ w_{2m-1k} = \int_0^1\int_0^1 \Delta w Z_{2m-1k}(x,y)dxdy,$$

$$w_{2mk} = \int_0^1\int_0^1 \Delta w Z_{2mk}(x,y)dxdy,$$

$v_{0k}, v_{2m-1k}, v_{2mk}$ are determined by the right-hand sides of (25), (28), (29), respectively. Let us introduce the operator F, which defines it by the right side of (38):

$$F[q](t) = q_0(t) + \frac{1}{h(t)}\left(\sum_{k=1}^{\infty} w_{0k}v_{0k}(t;q) + \right.$$

$$\left. + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} w_{2m-1k}v_{2m-1k}(t;q) + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} w_{2mk}v_{2mk}(t;q)\right). \tag{39}$$

Then equation (39) can be written in a more convenient form as

$$F[q](t) = q(t). \tag{40}$$

Let

$$q_{00} := \max_{t\in[0;T]} |q_0(t)| = \left\| \frac{1}{h(t)}\left(\int_0^1\int_0^1 wf(t)dxdy - \left(D_{0+,t}^{\alpha}h\right)(t)\right)\right\|_{C[0,T]}.$$

We fix a number $\rho > 0$ and consider the ball

$$\mathcal{B}(q_0,\rho) := \{q(t) \in C[0,T]; \|q - q_0\| \le \rho\}.$$

Theorem 6.1. *Let A1)-A4) be satisfied. Then there exists a number $T^* \in (0,T)$ such that there exists a unique solution $q(t) \in C[0,T^*]$ of the inverse problem (1)-(5).*

Proof. First we prove that for sufficiently small $T > 0$ the operator F maps the ball $\mathcal{B}(q_0,\rho)$ and implies that $F[q](t) \in \mathcal{B}(q_0,\rho)$. Indeed, for any continuous function $q(t)$, the function $F[q](t)$ calculated by formula (40) will be continuous. At the same time, estimating the norm of differences, we find that

$$\|F[q](t) - q_0(t)\| \le \frac{w_0}{h_0}\sum_{k=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1|\varphi_{0k,1}| + T^{\gamma+\alpha-1}M_2|\varphi_{0k,2}| +$$

$$+\frac{\|f_{0k}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$

$$+\frac{w_0}{h_0}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2m-1k,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2m-1k,2}\right|+$$
$$+\frac{\|f_{2m-1k}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}+$$
$$+2\lambda_m\frac{\Psi_{2mk}(T)T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$
$$+\frac{w_0}{h_0}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2mk,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2mk,2}\right|+$$
$$+\frac{\|f_{2mk}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

Here we have used the estimate for $v_{0k}, v_{2m-1k}, v_{2mk}$ given in (25), (28), (29). In view of the above lemmas, the last series is a convergent series. Note that the function occurring on the right-hand side of this inequality is monotone increases with T, and the fact that the function $q(t)$ belongs to the ball $\mathcal{B}(q_0,\rho)$ implies the inequality $|q\| \leq \|q_0\|+\rho$. Therefore, we only strengthen the inequality if we replace $\|q\|$ in this inequality with the expression $\|q\|+\rho$. Performing these replacements, we obtain the estimate

$$\|F[q](t)-q_0(t)\| \leq \frac{w_0}{h_0}\sum_{k=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{0k,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{0k,2}\right|+$$
$$+\frac{\|f_{0k}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|+\rho\right)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$
$$+\frac{w_0}{h_0}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2m-1k,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2m-1k,2}\right|+$$
$$+\frac{\|f_{2m-1k}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}+$$
$$+2\lambda_m\frac{\Psi_{2mk}(T)T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|+\rho\right)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$
$$+\frac{w_0}{h_0}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2mk,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2mk,2}\right|+$$
$$+\frac{\|f_{2mk}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|+\rho\right)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)=:s_1(T).$$

Let T_1 be the positive root of the equation. Therefore, if we denote by T_1 the positive root of the equation (for T), then $\|F[q](t)-q_0(t)\| \leq \rho$ for $T \leq T_1$; those $F[q](t) \in \mathcal{B}(q_0,\rho)$. □

Now we take any functions $q(t), \tilde{q}(t) \in \mathcal{B}(q_0,\rho)$ and estimate the distance between their images $F[q](t)$ and $F[\tilde{q}](t)$ in the space $C[0,T]$. The function $v_{mk}(t)=\tilde{v}_{mk}(t)$ corresponding to $\tilde{q}(t)$ satisfies integral equations (25), (28) and (29) for $\varphi_{mk,i}=\tilde{\varphi}_{mk,i}$, and $f_{mk}=\tilde{f}_{mk}$.

Compiling the difference $F[q](t)-F[\tilde{q}](t)$ using equations (22)-(24) and then estimating its norm, we obtain

$$\|F[q](t)-F[\tilde{q}](t)\| \leq \frac{w_0\|q\|_{C[0,T]}T^{\gamma}}{h_0\Gamma(\alpha)}\Bigg[\sum_{k=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{0k,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{0k,2}\right|+$$

$$+\frac{\|f_{0k}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2m-1k,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2m-1k,2}\right|+$$

$$+\frac{\|f_{2m-1k}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}+$$

$$+2\lambda_m\frac{\Psi_{2mk}(T)T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2mk,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2mk,2}\right|+$$

$$+\frac{\|f_{2mk}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg]\|q-\widetilde{q}\|_{C[0,T]}. \quad (41)$$

The functions $q(t)$ and $\tilde{q}(t)$ belong to the ball $\mathcal{B}(q_{0,}\rho)$, so for each of these functions, we have the inequality $\|q\|\leq\|q_0\|+\rho$. Note that the function on the right side of inequality (41) with the multiplier $\|q\|-\|\tilde{q}\|$ monotonically increases with $\|q\|$,$\|\tilde{q}\|$ and T. Therefore, replacing $\|q\|$ and $\|\tilde{q}\|$ in inequality (41) with $\|q\|+\rho$ will only strengthen the inequality. This, we have

$$\|F[q](t)-F[\tilde{q}](t)\|\leq\frac{w_0\|q\|_{C[0,T]}T^\gamma}{h_0\Gamma(\alpha)}\Bigg[\sum_{k=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{0k,1}\right|+$$

$$+T^{\gamma+\alpha-1}M_2\left|\varphi_{0k,2}\right|+\frac{\|f_{0k}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)\times$$

$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2m-1k,1}\right|+$$

$$+T^{\gamma+\alpha-1}M_2\left|\varphi_{2m-1k,2}\right|+\frac{\|f_{2m-1k}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}+$$

$$+2\lambda_m\frac{\Psi_{2mk}(T)T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2mk,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2mk,2}\right|+$$

$$+\frac{\|f_{2mk}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\|q-\widetilde{q}\|_{C[0,T]}.$$

Therefore, if T_2 is a positive root of the equation (for T)

$$\frac{w_0\|q\|_{C[0,T]}T^\gamma}{h_0\Gamma(\alpha)}\Bigg[\sum_{k=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{0k,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{0k,2}\right|+$$

$$+\frac{\|f_{0k}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|+\rho\right)T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2m-1k,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2m-1k,2}\right|+$$

$$+\frac{\|f_{2m-1k}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}+$$
$$+2\lambda_m\frac{\Psi_{2mk}(T)T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Big)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|+\rho\right)T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)+$$
$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(T^{\gamma+\alpha-2}M_1\left|\varphi_{2mk,1}\right|+T^{\gamma+\alpha-1}M_2\left|\varphi_{2mk,2}\right|+$$
$$+\frac{\|f_{2mk}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|+\rho\right)T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg]=1,$$

then for $T\in(0,T_2)$ the operator F shortens the distance between the elements $q(t),\tilde{q}(t)\in\mathcal{B}(q_0,\rho)$. Therefore, if we choose $T^*<\min(T_1,T_2)$, then the operator F is a contraction in the ball $\mathcal{B}(q_0,\rho)$. However, in accordance with the Banach theorem [[28], p. 87-97], the operator F has a unique fixed point in the ball $\mathcal{B}(q_0,\rho)$, i.e., there is a unique solution to Eq. (40). Theorem 6.1 is proven.

Let $T-$ be a positive fixed number. Consider the set D_{μ_0} of given functions $(\varphi_1,\varphi_2,h,f)$ for which all conditions from A1)-A4) are satisfied and

$$\max\left\{\|\varphi_1\|_{C^4[0,1]},\|\varphi_2\|_{C^4[0,1]},\|h\|_{C^\alpha_\gamma[0,T]},\|f\|_{C^4_\gamma(\bar{\Omega})}\right\}\le\mu_0$$

Denote by G_{v_1} the set of functions $q(t)$ that for some $T>0$ satisfy the following condition $\|q\|_{C[0,T]}\le\mu_1,\mu_1>0$.

Theorem 6.2. *Let $(\varphi_1,\varphi_2,h,f)\in D_{v_0}$, $\left(\tilde{\varphi}_1,\tilde{\varphi}_2,\tilde{h},\tilde{f}\right)\in D_{\mu_0}$ and $q,\tilde{q}\in G_{\mu_4}$ Then the following estimate holds for the solution of the inverse problem (1)-(4) sustainability:*

$$\|q-\tilde{q}\|_{C[0,T]}\le r\left[\|\varphi_1-\tilde{\varphi}_1\|_{C^4[0,1]}+\|\varphi_2-\tilde{\varphi}_2\|_{C^4[0,1]}+\|h-\tilde{h}\|_{C^\alpha_\gamma[0,T]}+\|f-\tilde{f}\|_{C^4_\gamma(\bar{\Omega})}\right]$$

where the constant ρ depends only on $\mu_0,\mu_1,T,\alpha,\beta$ and $\Gamma(\alpha),B(\alpha,1-\gamma)$.

Proof. To prove this theorem, using (38), we write out the equations for $\tilde{q}(t)$ and compose the difference $\hat{q}=q(t)-\tilde{q}(t)$. Then after evaluating this expression and using the estimates $v_n(t),\hat{v}_n(t)$, we obtain the following estimates

$$\|q-\tilde{q}\|_{C[0,T]}\le\max_{0\le t\le T}\Bigg|\frac{1}{h(t)}\Bigg(\int_0^1\int_0^1 w(x,y)f(x,y,t)dxdy-\left(D^\alpha_{0+,t}h\right)(t)+$$
$$+\int_0^1\int_0^1\Delta w\Bigg(\sum_{k=1}^{\infty}Z_{0k}(x,y)v_{0k}(t)+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}Z_{2m-1k}(x,y)v_{2m-1k}(t)+$$
$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}Z_{2mk}(x,y)v_{2mk}(t)\Bigg)dxdy\Bigg)-$$
$$-\frac{1}{\tilde{h}(t)}\left(\int_0^1\int_0^1 w(x,y)\tilde{f}(x,y,t)dxdy-\left(D^\alpha_{0+,t}\tilde{h}\right)(t)\right)-$$
$$-\frac{1}{\tilde{h}(t)}\int_0^1\int_0^1\Delta W\Bigg(\sum_{k=1}^{\infty}Z_{0k}(x,y)\tilde{v}_{0k}(t)+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}Z_{2m-1k}(x,y)\tilde{v}_{2m-1k}(t)+$$
$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}Z_{2mk}(x,y)\tilde{v}_{2mk}(t)\Bigg)dxdy\Bigg|\le$$

$$\leq \max_{0\leq t\leq T}\left\{\frac{w_0}{h_0^2}\left|\int_0^1\int_0^1 [h(t)(f(x,y,t)-\tilde{f}(x,y,t))-\tilde{f}(x,y,t)(h(t)-\tilde{h}(t))]dxdy+\right.\right.$$

$$\left.\left.\tilde{h}(t)\left(\left(D^{\alpha}_{0+,t}h\right)(t)-\left(D^{\alpha}_{0+,t}\tilde{h}\right)(t)\right)+\left(D^{\alpha}_{0+,t}\tilde{h}\right)(t)(h(t)-\tilde{h}(t))\right|\right\}+$$

$$+\max_{0\leq t\leq T}\left\{\frac{w_0}{h_0^2}\left|\int_0^1\int_0^1 2\sqrt{2}\left[h(t)\left(v_{0k}(t)-\tilde{v}_{0k}(t)\right)-\tilde{v}_{0k}(t)(h(t)-\tilde{h}(t))\right]dxdy+\right.\right.$$

$$+4\sqrt{2}\left[h(t)\left(v_{2m-1k}(t)-\tilde{v}_{2m-1k}(t)\right)-\tilde{v}_{2m-1k}(t)(h(t)-\tilde{h}(t))\right]+$$

$$\left.\left.+4\sqrt{2}\left[h(t)\left(v_{2mk}(t)-\tilde{v}_{2mk}(t)\right)-\tilde{v}_{2mk}(t)(h(t)-\tilde{h}(t))\right]\right|\right\}\leq$$

$$\leq r_0\Bigg(\|\varphi_1-\tilde{\varphi}_1\|_{C^4[0,1]}+\|\varphi_2-\tilde{\varphi}_2\|_{C^4[0,1]}+\|f-\tilde{f}\|_{C^4_{\gamma}(\bar{\Omega})}+\left\|\left(D^{\alpha}_{0+,t}h\right)-\left(D^{\alpha}_{0+,t}\tilde{h}\right)\right\|+$$

$$+\|h-\tilde{h}\|_{C^{\alpha}_{\gamma}[0,T]}\Bigg)+r_1\int_0^t (t-\tau)^{\alpha-1}\|q(\tau)-\tilde{q}(\tau)\|_{C[0,T]}d\tau, t\in[0,T], \tag{42}$$

where r_0, r_1 depends only on μ_0, μ_1, T, α and $\Gamma(\alpha), B(\alpha, 1-\gamma$. From (41), using lemma 2.1, we obtain the estimate

$$\|q-\tilde{q}\|_{C[0,T]}\leq r_0\Big(\|\varphi_1-\tilde{\varphi}_1\|_{C^4[0,1]}+\|\varphi_2-\tilde{\varphi}_2\|_{C^4[0,1]}+\|f-\tilde{f}\|_{C^4_{\gamma}(\bar{\Omega})}+$$

$$\|h-\tilde{h}\|_{C^{\alpha}_{\gamma}[0,T]}\Big)E_{\alpha,1}\left(r_1\Gamma(\alpha)t^{\alpha}\right), t\in[0,T]. \tag{43}$$

where $\bar{\Omega}=\bar{D}\times[0,T]$.
Estimate (41) follows from this inequality if we set $r=r_0E_{\alpha,1}\left(r_1\Gamma(\alpha)t^{\alpha}\right)$. □

From theorem 6.2 also implies the following statement about the uniqueness of the solution of the inverse problem as a whole.

Theorem 6.3. *Let the functions $\varphi_1, \varphi_2, h, f$ and $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{h}, \tilde{f}$ have the same meaning , as in theorem 6.2 and conditions A1)-A4). Moreover, if $\varphi_1=\tilde{\varphi}_1, \varphi_2=\tilde{\varphi}_2, h=\tilde{h}, f=\tilde{f}$, for $t\in[0,T]$, then $q(t)=\tilde{q}(t), t\in[0,T]$.*

7. Conclusion

In this paper, we study the solvability of a nonlinear inverse problem for a two-dimensional wave equation with a fractional Riemann-Liouville time derivative with initial non-local boundary and integral redefinition conditions. The problem is replaced by the equivalent of the integral equation in the form of Volterra.The local existence and global uniqueness of the solution to the stability of the direct problem is proved. Non-local boundary conditions, the fractional Riemann-Liouville derivative, and the control coefficient complicated our task. Existence conditions, uniqueness and continuous dependence on the data of the problem were established using the Fourier method with some biorthogonal system, the Riemann-Liouville fractional derivative associated with it, containing the initial data, and the Banach fixed point theorem for the product of Banach spaces.

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