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FUZZIFIED BISECTION METHOD TO FIND THE ROOT OF AN ALGEBRAIC EQUATION USING $n-$POLYGONAL FUZZY NUMBERS

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Abstract. In this paper, we introduce a fuzzified bisection method to find the root of an algebraic equation using n-polygonal fuzzy numbers. We present a new approach for finding the root of an algebraic equation with the n-polygonal fuzzy number. To solve the given algebraic equation, we consider a fuzzy interval, and the method iteratively reduces the interval containing the fuzzy root by evaluating the value of a function at each midpoint. We continue this process until the desired level of approximation is achieved. The fuzzified bisection method has broad applications in engineering, economics, and decision-making. The results demonstrate that the method provides a flexible and effective way to find root.

Keywords: Triangular fuzzy number, Trapezoidal fuzzy number, n-Polygonal fuzzy number (n-PFN), Bisection method, Fuzzy membership function, Arithmetic operations on n-PFN, α - cut.

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1. Introduction

In the real world, due to uncertainty, complexity may arise in the form of ambiguity. To handle such uncertainty and ambiguity, fuzzy logic and fuzzy numbers have been used as effective tools. Fuzzy logic was introduced by Prof. Lotfi A. Zadeh. in conjunction with the proposal of fuzzy set theory. Fuzzy logic is applied by many researchers to various fields. A value that lies in the expected range is defined by the quantitative limits and described in non-specific categories. With the help of fuzzy theory, many researchers have investigated lots of methods of numerical analysis.

Hazarika and Bora introduced a fuzzification of bisection method using triangular fuzzy numbers [3] in 2015. Kalyani Vable et al. introduced a fuzzified bisection method to

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find the root of an algebraic equation using TrFNs [5]. An approach for computing the product of various fuzzy numbers using α cut was proposed by Hassanzadeh et al. [7] in 2012. Mahmoud Alrefaei et al. solved the fully fuzzy transportation problem with n-PFN [6] in 2021. Gang Sun et al. developed a convergence and gradient algorithm for a class of neural networks based on the polygonal fuzzy number representation [4] in 2022.

The well-known bisection method is used for finding the root of an algebraic equation. It bisects the crisp interval, and the solution is iteratively reached by narrowing down the values. However, when dealing with fuzzy intervals, the traditional bisection method becomes inadequate.

The fuzzy bisection method using n-polygonal fuzzy numbers extends the applicability of the bisection method to situations where input data is fuzzy. The fuzzy bisection method is used to fuzzify the crisp intervals and the arithmetic operations involved in the bisection process. Instead of crisp intervals, fuzzy intervals are defined using n-polygonal fuzzy numbers, and the fuzzy outputs generated during the iterative process are defuzzified to obtain crisp solutions.

In this paper, we use n-PFNs to fuzzify the bisection method to obtain the root of an algebraic equation. This paper is organized as Section I is for the introduction. Section II and Section III are devoted to some preliminary concepts. Sections IV, V, and VI include methodology, convergence analysis, and stopping conditions, respectively. Sections VII and VIII are for some numerical examples and algorithms, respectively. The conclusion is in the ninth section.

Novelty: There are several methods to find the root of an algebraic equation, such as the false position method, secant method, Newton-Raphson method, classical bisection method, etc. The proposed method uses a fuzzy interval with n-polygonal fuzzy numbers, and it takes fewer iterations to obtain the desired result than the classical bisection method. The proposed technique is therefore more efficient than the classical bisection method.

2. Preliminaries

Here, we will state some basic definitions and results related to trapezoidal fuzzy numbers (TrFNs) and n-polygonal fuzzy numbers (n-PFNs) in fuzzy theory.

Definition 1. [1] *Let X be a universal set. Then, the fuzzy subset $\tilde{P}$ of X is defined by its membership function $\mu_{\tilde{P}}$: $X \to [0,1]$ which assign to each element $x \in X$ a real number $\mu_{\tilde{P}}(x)$ in the interval [0,1], where the function value of $\mu_{\tilde{P}}(x)$ represents the grade of membership of x in $\tilde{P}$. A fuzzy set $\tilde{P}$ is written as*

$$\tilde{P} = \{(x,\, \mu_{\tilde{P}}(x)),\, x \in X, \quad \mu_{\tilde{P}}(x) \in [0,1]\}.$$

Definition 2. [1] *A fuzzy set $\tilde{P}$, defined on the universal set of real number R, is said to be a fuzzy number if its membership function has the following characteristics:*

(i) $\tilde{P}$ *is convex i.e.,*
$\mu_{\tilde{P}}\ (\beta x_1 + (1-\beta)x_2) \geq min(\mu_{\tilde{P}}(x_1),\, \mu_{\tilde{P}}(x_2)),\ \forall x_1, x_2 \in R,\ \forall \beta \in [0,1].$

(ii) $\tilde{P}$ *is normal, i.e.,* $\exists\ x_0 \in R$ *such that* $\mu_{\tilde{P}}(x_0) = 1$.

(iii) $\mu_{\tilde{P}}$ *is piecewise continuous.*

Definition 3. [8] *A fuzzy number $\tilde{P} = (p_1, p_2, p_3)$ is a triangular fuzzy number (TFN) in the general form if its membership function is as follows :*

$$\mu_{\tilde{P}}(x) = \begin{cases} \frac{x-p_1}{p_2-p_1}, & p_1 \le x \le p_2 \\ \frac{p_3-x}{p_3-p_2}, & p_2 \le x \le p_3 \\ 0, & otherwise \end{cases}$$

Definition 4. [1] *A fuzzy number $\tilde{P} = (p_1, p_2, p_3, p_4)$ is a TrFN in the general form if its membership function is as follows :*

$$\mu_{\tilde{P}}(x) = \begin{cases} 0, & x \le p_1 \\ \frac{x-p_1}{p_2-p_1}, & p_1 \le x \le p_2 \\ 1, & p_2 \le x \le p_3 \\ \frac{p_4-x}{p_4-p_3}, & p_3 \le x \le p_4 \\ 0, & x > p_4 \end{cases}$$

Definition 5. [6] *An n-polygonal fuzzy number (n-PFN), $\tilde{P}$ is fuzzy number with a membership function of the type :*

$$\mu_{\tilde{P}}(x) = \begin{cases} \frac{1}{n}[\frac{x-p_i}{p_{i+1}-p_i}] + \frac{i}{n}, & p_i \le x \le p_{i+1}, i = 0, 1, ..., n-1 \\ 1, & p_n \le x \le q_0 \\ \frac{-1}{n}[\frac{x-q_i}{q_{i+1}-q_i}] + \frac{n-i}{n}, & q_i \le x \le q_{i+1}, i = 0, 1, ..., n-1 \\ 0, & otherwise \end{cases}$$

which can be represented by its knots: $(p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n)$.

3. Arithmetic operations on $n-$ Polygonal fuzzy numbers.

[1], [4] Let $\tilde{P} = (p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n)$ and $\tilde{R} = (r_0, r_1, r_2, \cdots, r_n, s_0, s_1, s_2, \cdots, s_n)$ be any two n - PFNs then the corresponding operations are defines as follows:

(1) **Addition :**

$$\tilde{P} + \tilde{R} = (p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n) + (r_0, r_1, r_2, \cdots, r_n, s_0, s_1, s_2, \cdots, s_n)$$
$$= (p_0 + r_0, p_1 + r_1, p_2 + r_2, \cdots, p_n + r_n, q_0 + s_0, q_1 + s_1, q_2 + s_2, \cdots, q_n + s_n)$$

(2) **Subtraction :**

$$\tilde{P} - \tilde{R} = (p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n) - (r_0, r_1, r_2, \cdots, r_n, s_0, s_1, s_2, \cdots, s_n)$$
$$= (p_0 - s_n, p_1 - s_{n-1}, \cdots, p_n - s_0, q_0 - r_n, q_1 - r_{n-1}, \cdots, q_n - r_0)$$

(3) **Multiplication :**

$$\tilde{P}.\tilde{R} = (p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n).(r_0, r_1, r_2, \cdots, r_n, s_0, s_1, s_2, \cdots, s_n)$$
$$= (c_0, c_1, c_2, \cdots, c_n, d_0, d_1, d_2, \cdots, d_n)$$

where

$$c_i = min(p_i.r_i, p_i.s_{n-i}, q_{n-i}.r_i, q_{n-i}.s_{n-i})$$
$$d_{n-i} = max(p_i.r_i, p_i.s_{n-i}, q_{n-i}.r_i, q_{n-i}.s_{n-i})$$

(4) **Scalar Multiplication :**

$$k.\tilde{P} = \begin{cases} (k.p_0, k.p_1, \cdots, k.p_n, k.q_0, \cdots, k.q_{n-1}, k.q_n), k \geq 0 \\ (k.q_n, k.q_{n-1}, \cdots, k.q_0, k.p_n, \cdots, k.p_1, k.p_0), k < 0 \end{cases}$$

(5) **Sign of n - PFN :**

The sign of n - PFN $\tilde{P} = (p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n)$ can be classified as follows:

(a) $\tilde{P}$ is positive iff $\forall p_i, q_i \geq 0$
(b) $\tilde{P}$ is negative iff $\forall p_i, q_i \leq 0$
(c) $\tilde{P}$ is zero iff $\forall p_i, q_i = 0$
(d) $\tilde{P}$ is near zero iff $p_0 \leq 0 \leq q_n$

(6) α **- cut of n - PFN :**

Let $\tilde{P} = (p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n)$ *be* $n - PFN$.

$$[\tilde{P}]^{\alpha} = [(p_{i+1} - p_i)(n\alpha - i) + p_i, q_i - (n(\alpha - 1) + i)(q_{i+1} - q_i)]$$

is a α - cut set of n - PFN.

Definition 6. ***Defuzzification :*** [8] *The fundamental reason is the indirect comparison of fuzzy numbers. The comparison over fuzzy sets has no universal consensus over several literatures. The fuzzy numbers have to be mapped initially to real values that can be compared for computing the magnitude values. The real value assignment over a fuzzy set is called the defuzzification process. The computation may take several forms; however, the standard form is the usage of the centroid rule. The computation of the defuzzification process requires integrating the membership function over the fuzzy sets. This improves the effect over direct computation of the centroid rule with center of gravity that describes the fuzzy quantity. The defuzzification over trapezoid or triangular fuzzy numbers using median is evaluated at the risk rate.*

Proposition 1. *For the trapezoid fuzzy number* $\tilde{P} = (p_0, p_1, q_0, q_1)$*, by the bisection of area , the median* $M_{\tilde{P}} = \frac{(p_0+p_1+q_0+q_1)}{4}$ *, only if ,* $p_1 \leq M_{\tilde{P}} \leq q_0$

Remark: *For the triangular fuzzy number* $\tilde{P} = (p_0, p_1, p_1, q_0)$*, we have median of*

$$\tilde{P} \;\; is \; M_{\tilde{P}} = \frac{(p_0 + 2p_1 + q_0)}{4}$$

4. Methodology

[3], [5] Let us consider a continuous function $G(x) = 0$ which changes sign over an interval $(\tilde{P}, \tilde{R})$, where $\tilde{P}$ and $\tilde{R}$ are n - PFNs. i.e. $\tilde{P} = (p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n)$ and $\tilde{R} = (r_0, r_1, r_2, \cdots, r_n, s_0, s_1, s_2, \cdots, s_n)$. Then the roots of $G(x) = 0$ lying between $\tilde{P}$ and $\tilde{R}$. The fuzzy membership function (f.m.f) of $\tilde{P}$ and $\tilde{R}$ are as follows respectively,

$$\mu_{\tilde{P}}(x) = \begin{cases} \frac{1}{n}[\frac{x-p_i}{p_{i+1}-p_i}] + \frac{i}{n}, & p_i \le x \le p_{i+1}, i = 0, 1, ..., n-1 \\ 1, & p_n \le x \le q_0 \\ \frac{-1}{n}[\frac{x-q_i}{q_{i+1}-q_i}] + \frac{n-i}{n}, & q_i \le x \le q_{i+1}, i = 0, 1, ..., n-1 \\ 0, & otherwise \end{cases}$$

$$\mu_{\tilde{R}}(x) = \begin{cases} \frac{1}{n}[\frac{x-r_i}{r_{i+1}-r_i}] + \frac{i}{n}, & r_i \le x \le r_{i+1}, i = 0, 1, ..., n-1 \\ 1, & r_n \le x \le s_0 \\ \frac{-1}{n}[\frac{x-s_i}{s_{i+1}-s_i}] + \frac{n-i}{n}, & s_i \le x \le s_{i+1}, i = 0, 1, ..., n-1 \\ 0, & otherwise \end{cases}$$

with respect to α - cut as

$$[\tilde{P}]^{\alpha} = [(p_{i+1} - p_i)(n\alpha - i) + p_i, q_i - (n(\alpha - 1) + i)(q_{i+1} - q_i)]$$

$$[\tilde{R}]^{\alpha} = [(r_{i+1} - r_i)(n\alpha - i) + r_i, s_i - (n(\alpha - 1) + i)(s_{i+1} - s_i)]$$

As a first approximation, the root $G(x) = 0$ is

$$\tilde{x_0} = \frac{\tilde{P} + \tilde{R}}{2} = \frac{(p_0, p_1, p_2, \cdots, p_n, q_0, q_1, q_2, \cdots, q_n) + (r_0, r_1, r_2, \cdots, r_n, s_0, s_1, s_2, \cdots, s_n)}{2}$$

Let us consider, $\tilde{x_0} = (x_{00}, x_{01}, \cdots, x_{0n}, y_{00}, y_{01}, \cdots, y_{0n})$

The f.m.f of x_0 is

$$\mu_{\tilde{x_0}}(x) = \begin{cases} \frac{1}{n}[\frac{x-x_{0i}}{x_{0(i+1)}-x_{0i}}] + \frac{i}{n}, & x_{0i} \le x \le x_{0(i+1)}, i = 0, 1, ..., n-1 \\ 1, & x_{0n} \le x \le y_{00} \\ \frac{-1}{n}[\frac{x-y_{0i}}{y_{0(i+1)}-y_{0i}}] + \frac{n-i}{n}, & y_{0i} \le x \le y_{0(i+1)}, i = 0, 1, ..., n-1 \\ 0, & otherwise \end{cases}$$

with respect to α - cut as

$$[\tilde{x_0}]^{\alpha} = [(x_{0(i+1)} - x_{0i})(n\alpha - i) + x_{0i}, y_{0i} - (n(\alpha - 1) + i)(y_{0(i+1)} - y_{0i})]$$

Suppose $G(\tilde{P})$ and $G(\tilde{x_0})$ are of opposite signs then the root lies between $\tilde{P}$ and $\tilde{x_0}$ while if $G(\tilde{x_0})$ and $G(\tilde{R})$ are of opposite signs then the root lies between $\tilde{x_0}$ and $\tilde{R}$. Suppose $G(\tilde{x_0}) < 0$ then the root lies between $\tilde{x_0}$ and $\tilde{R}$.
Then the second approximation is,

$$\tilde{x_1} = \frac{\tilde{x_0} + \tilde{R}}{2} = \frac{(x_{00}, x_{01}, \cdots, x_{0n}, y_{00}, y_{01}, \cdots, y_{0n}) + (r_0, r_1, r_2, \cdots, r_n, s_0, s_1, s_2, \cdots, s_n)}{2}$$

Let us consider, $\tilde{x_1} = (x_{10}, x_{11}, \cdots, x_{1n}, y_{10}, y_{11}, \cdots, y_{1n})$
The f.m.f of x_1 is

$$\mu_{\tilde{x_1}}(x) = \begin{cases} \frac{1}{n}[\frac{x-x_{1i}}{x_{1(i+1)}-x_{1i}}] + \frac{i}{n}, & x_{1i} \le x \le x_{1(i+1)}, i = 0, 1, ..., n-1 \\ 1, & x_{1n} \le x \le y_{10} \\ \frac{-1}{n}[\frac{x-y_{1i}}{y_{1(i+1)}-y_{1i}}] + \frac{n-i}{n}, & y_{1i} \le x \le y_{1(i+1)}, i = 0, 1, ..., n-1 \\ 0, & otherwise \end{cases}$$

with respect to α - cut as

$$[\tilde{x_1}]^\alpha = [(x_{1(i+1)} - x_{1i})(n\alpha - i) + x_{1i}, y_{1i} - (n(\alpha - 1) + i)(y_{1(i+1)} - y_{1i})]$$

and so on. We repeat above steps until the value of function at $\tilde{x}$ is either near zero or equals to zero, where $\tilde{x} \in (\tilde{P}, \tilde{Q})$.

5. Convergence Analysis

Theorem 1. [2] *Let f be a continuous on the closed interval [a,b] and suppose that $f(a).f(b) < 0$. The bisection method generates a sequence of approximation $\{p_n\}$ which converges to a root $p \in (a, b)$ with the property*

$$|p_n - p| \le \frac{b-a}{2^n}$$

Proof: [2] Since the quantity $b - a$ is constant and $2^{-n} \to 0$ as $n \to \infty$, establishing the error bound will be sufficient to prove convergence of the bisection method sequence. By construction of the bisection algorithm, for each $n, p \in (a_n, b_n)$ and p_n is taken as the midpoint of (a_n, b_n). This implies that p_n can differ from p by no more than half the length of (a_n, b_n); that is,

$$|p_n - p| \le \tfrac{1}{2}(b_n - a_n).$$

However, again by construction,

$$b_n - a_n = \tfrac{1}{2}(b_{n-1} - a_{n-1}) = \tfrac{1}{4}(b_{n-2} - a_{n-2}) = \cdots = \tfrac{1}{2^{n-1}}(b_1 - a_1).$$

Recalling that $b_1 = b$ and $a_1 = a$ and combining the last two equations produces the desired error bound

$$|p_n - p| \le \tfrac{(b-a)}{2^n}.$$

6. Stopping Condition

[2] Let ϵ be a specified convergence tolerance for any root finding technique, there are three primary measures of convergence with which to construct the stopping condition. These are

(1) **The absolute error in the location of the root.**
Terminate the iteration when $|p_n - p| < \epsilon$.
(2) **The relative error in the location of root.**
Terminate the iteration when $|p_n - p| < \epsilon |p_n|$.
(3) **The test for a root.**
Terminate the iteration when $|f(p_n)| < \epsilon$.

There is no general rule of thumb for selecting one stopping condition over another, and it is worth noting that none of these conditions works well in all cases. This result was stated by the author Brian Bradie in 2007. From the proof of the bisection method convergence theorem, we know that,

$$|p_n - p| \leq \frac{(b-a)}{2^n}.$$

We can therefore terminate the bisection method when $\frac{(b_n-a_n)}{2} < \epsilon$.

7. Numerical Examples

Let us consider an algebraic equation $G(x) = x^3 + x^2 - 3x - 3$

First we solve this example for $n = 1$-PFN.

Choose a fuzzy interval $(a, b) = (\tilde{P}, \tilde{R})$ where $f(\tilde{P})$ and $f(\tilde{R})$ have opposite signs.

Let $\tilde{P} = (p_0, p_1, q_0, q_1) = (1.6025, 1.6035, 1.6045, 1.6055)$ and

$\tilde{R} = (r_0, r_1, s_0, s_1) = (2.1025, 2.1035, 2.1045, 2.1055)$

The function $G(x)$ changes sign over an interval $(\tilde{P}, \tilde{R})$. Here $G(\tilde{P}) < 0$ and $G(\tilde{R}) > 0$.

The root of $G(x) = 0$ lying between $\tilde{P}$ and $\tilde{R}$. The fuzzy membership function (f.m.f) of $\tilde{P}$ and $\tilde{R}$ are as follows respectively,

$$\mu_{\tilde{P}}(x) = \begin{cases} [\frac{x-p_0}{p_1-p_0}], & p_0 \leq x \leq p_1, \\ 1, & p_1 \leq x \leq q_0, \\ [\frac{q_1-x}{q_1-q_0}], & q_0 \leq x \leq q_1, \\ 0, & otherwise \end{cases} = \begin{cases} [\frac{x-1.6025}{1.6035-1.6025}], & 1.6025 \leq x \leq 1.6035, \\ 1, & 1.6035 \leq x \leq 1.6045, \\ [\frac{1.6055-x}{1.6055-1.6045}], & 1.6045 \leq x \leq 1.6055, \\ 0, & otherwise \end{cases}$$

$$\mu_{\tilde{R}}(x) = \begin{cases} \left[\frac{x-2.1025}{2.1035-2.1025}\right], & 2.1025 \leq x \leq 2.1035, \\ 1, & 2.1035 \leq x \leq 2.1045, \\ \left[\frac{2.1055-x}{2.1055-2.1045}\right], & 2.1045 \leq x \leq 2.1055, \\ 0, & otherwise \end{cases}$$

with respect to α - cut

$$[\tilde{P}]^\alpha = [(1.6035 - 1.6025)\alpha + 1.6025, 1.6055 - (1.6055 - 1.6045)\alpha]$$

$$[\tilde{R}]^\alpha = [(2.1035 - 2.1025)\alpha + 2.1025, 2.1055 - (2.1055 - 2.1045)\alpha]$$

Then the first approximation is

$$\tilde{x_0} = \frac{\tilde{P} + \tilde{R}}{2} = (1.8525, 1.8535, 1.8545, 1.8555)$$

Now f.m.f of $\tilde{x_0}$ is

$$\mu_{\tilde{x_0}}(x) = \begin{cases} \left[\frac{x-1.8525}{1.8535-1.8525}\right], & 1.8525 \leq x \leq 1.8535, \\ 1, & 1.8535 \leq x \leq 1.8545, \\ \left[\frac{1.8555-x}{1.8555-1.8545}\right], & 1.8545 \leq x \leq 1.8555, \\ 0, & otherwise \end{cases}$$

with respect to α - cut

$$[\tilde{x_0}]^\alpha = [(1.8535 - 1.8525)\alpha + 1.8525, 1.8555 - (1.8555 - 1.8545)\alpha]$$

$G(\tilde{x_0}) = G(1.8525, 1.8535, 1.8545, 1.8555) > 0$, so root lies between $\tilde{P}$ and $\tilde{x_0}$. Then the second approximation is,

$$\tilde{x_1} = \frac{\tilde{P} + \tilde{x_0}}{2} = (1.7275, 1.7285, 1.7295, 1.7305)$$

Now f.m.f of $\tilde{x_1}$ is

$$\mu_{\tilde{x_1}}(x) = \begin{cases} \left[\frac{x-1.7275}{1.7285-1.7275}\right], & 1.7275 \le x \le 1.7285, \\ 1, & 1.7285 \le x \le 1.7295, \\ \left[\frac{1.7305-x}{1.7305-1.7295}\right], & 1.7295 \le x \le 1.7305, \\ 0, & otherwise \end{cases}$$

with respect to α - cut

$$[\tilde{x_1}]^\alpha = [(1.7285 - 1.7275)\alpha + 1.7275, 1.7305 - (1.73055 - 1.7295)\alpha]$$

$G(\tilde{x_1}) = G(1.7275, 1.7285, 1.7295, 1.7305) < 0$, so root lies between $\tilde{x_1}$ and $\tilde{x_0}$. Then the third approximation is,

$$\tilde{x_2} = \frac{\tilde{x_1} + \tilde{x_0}}{2} = (1.79, 1.791, 1.792, 1.793)$$

Now f.m.f of $\tilde{x_2}$ is

$$\mu_{\tilde{x_2}}(x) = \begin{cases} \left[\frac{x-1.79}{1.791-1.79}\right], & 1.79 \le x \le 1.791, \\ 1, & 1.791 \le x \le 1.792, \\ \left[\frac{1.793-x}{1.793-1.792}\right], & 1.792 \le x \le 1.793, \\ 0, & otherwise \end{cases}$$

with respect to α - cut

$$[\tilde{x_2}]^\alpha = [(1.791 - 1.79)\alpha + 1.79, 1.793 - (1.793 - 1.792)\alpha]$$

$G(\tilde{x_2}) = G(1.79, 1.791, 1.792, 1.793) > 0$, so root lies between $\tilde{x_1}$ and $\tilde{x_2}$. Then the fourth approximation is,

$$\tilde{x_3} = \frac{\tilde{x_1} + \tilde{x_2}}{2} = (1.75875, 1.75975, 1.76075, 1.76175)$$

Now f.m.f of $\tilde{x_3}$ is

$$\mu_{\tilde{x_3}}(x) = \begin{cases} \left[\frac{x-1.75875}{1.75975-1.75875}\right], & 1.75875 \le x \le 1.75975, \\ 1, & 1.75975 \le x \le 1.76075, \\ \left[\frac{1.76175-x}{1.76175-1.76075}\right], & 1.76075 \le x \le 1.76175, \\ 0, & otherwise \end{cases}$$

with respect to α - cut

$$[\tilde{x_3}]^\alpha = [(1.75975 - 1.75875)\alpha + 1.75875, 1.76175 - (1.76175 - 1.76075)\alpha]$$

$G(\tilde{x_3}) = G(1.75875, 1.75975, 1.76075, 1.76175) > 0$, so root lies between $\tilde{x_1}$ and $\tilde{x_3}$. Then the fifth approximation is,

$$\tilde{x_4} = \frac{\tilde{x_1} + \tilde{x_3}}{2} = (1.743125, 1.744125, 1.745125, 1.746125)$$

Now f.m.f of $\tilde{x_4}$ is

$$\mu_{\tilde{x_4}}(x) = \begin{cases} \left[\frac{x-1.743125}{1.744125-1.743125}\right], & 1.743125 \leq x \leq 1.744125, \\ 1, & 1.744125 \leq x \leq 1.745125, \\ \left[\frac{1.746125-x}{1.746125-1.745125}\right], & 1.745125 \leq x \leq 1.746125, \\ 0, & otherwise \end{cases}$$

with respect to α - cut

$$[\tilde{x_4}]^{\alpha} = [(1.744125 - 1.743125)\alpha + 1.743125, 1.746125 - (1.746125 - 1.745125)\alpha]$$

$G(\tilde{x_4}) = G(1.743125, 1.744125, 1.745125, 1.746125) > 0$, so root lies between $\tilde{x_1}$ and $\tilde{x_4}$. Then the sixth approximation is,

$$\tilde{x_5} = \frac{\tilde{x_1} + \tilde{x_4}}{2} = (1.7353125, 1.7363125, 1.7373125, 1.7383125)$$

Now f.m.f of $\tilde{x_5}$ is

$$\mu_{\tilde{x_5}}(x) = \begin{cases} \left[\frac{x-1.7353125}{1.7363125-1.7353125}\right], & 1.7353125 \leq x \leq 1.7363125, \\ 1, & 1.7363125 \leq x \leq 1.7373125, \\ \left[\frac{1.7383125-x}{1.7383125-1.7373125}\right], & 1.7373125 \leq x \leq 1.7383125, \\ 0, & otherwise \end{cases}$$

with respect to α - cut

$$[\tilde{x_5}]^{\alpha} = [(1.7363125 - 1.7353125)\alpha + 1.7353125, 1.7383125 - (1.7383125 - 1.7373125)\alpha]$$

$G(\tilde{x_5}) = G(1.7353125, 1.7363125, 1.7373125, 1.7383125) > 0$, so root lies between $\tilde{x_1}$ and $\tilde{x_5}$. Then the seventh approximation is,

$$\tilde{x_6} = \frac{\tilde{x_1} + \tilde{x_5}}{2} = (1.73140625, 1.73240625, 1.73340625, 1.73440625)$$

Now f.m.f of $\tilde{x_6}$ is

$$\mu_{\tilde{x_6}}(x) = \begin{cases} \left[\frac{x-1.73140625}{1.73240625-1.73140625}\right], & 1.73140625 \leq x \leq 1.73240625, \\ 1, & 1.73240625 \leq x \leq 1.73340625, \\ \left[\frac{1.73440625-x}{1.73440625-1.73340625}\right], & 1.73340625 \leq x \leq 1.73440625, \\ 0, & otherwise \end{cases}$$

with respect to α - cut

$$[\tilde{x_6}]^{\alpha} = [(1.73240625-1.73140625)\alpha+1.73140625, 1.73440625-(1.73440625-1.73340625)\alpha]$$

$G(\tilde{x_6}) = G(1.73140625, 1.73240625, 1.73340625, 1.73440625)$ *is near* 0, so root of $G(x) = 0$ *is* $(1.73140625, 1.73240625, 1.73340625, 1.73440625)$. Now by Defuzzification method, the defuzzified value is 1.73290625.

$\therefore$ The crisp root is 1.73290625.

Now for n=2-PFN which is either a pentagonal fuzzy number or hexagonal fuzzy number we find the root of above equation .
First we consider for pentagonal fuzzy number.
Let $\tilde{P} = (p_0, p_1, p_2, q_0, q_1, q_2)$, here $p_2 = q_0$.
Consider $\tilde{P} = (1.6025, 1.6035, 1.6045, 1.6055, 1.6065)$ and
$\tilde{R} = (r_0, r_1, r_2, s_0, s_1, s_2)$, here $r_2 = s_0$ then $\tilde{R} = (2.1025, 2.1035, 2.1045, 2.1055, 2.1065)$
The function $G(x)$ changes sign over an interval $(\tilde{P}, \tilde{R})$. Here $G(\tilde{P}) < 0$ and $G(\tilde{R}) > 0$.

The root of $G(x) = 0$ lying between $\tilde{P}$ and $\tilde{R}$. The fuzzy membership function (f.m.f) of $\tilde{P}$ and $\tilde{R}$ are as follows respectively,

$$\mu_{\tilde{P}}(x) = \begin{cases} \frac{1}{2}\left[\frac{x-p_i}{p_{i+1}-p_i}\right] + \frac{i}{2}, & p_i \le x \le p_{i+1}, i = 0, 1. \\ 1, & p_2 \le x \le q_0, \\ \frac{-1}{2}\left[\frac{x-q_i}{q_{i+1}-q_i}\right] + \frac{2-i}{2}, & q_i \le x \le q_{i+1}, i = 0, 1. \\ 0, & otherwise \end{cases}$$

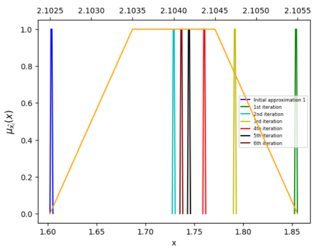
$$\mu_{\tilde{P}}(x) = \begin{cases} \frac{1}{2}\left[\frac{x-p_0}{p_1-p_0}\right], & p_0 \le x \le p_1 \\ \frac{1}{2}\left[\frac{x-p_1}{p_2-p_1}\right] + \frac{1}{2}, & p_1 \le x \le p_2 \\ 1, & p_2 \le x \le q_0 \\ \frac{-1}{2}\left[\frac{x-q_0}{q_1-q_0}\right] + 1, & q_0 \le x \le q_1 \\ \frac{-1}{2}\left[\frac{x-q_1}{q_2-q_1}\right] + \frac{1}{2}, & q_1 \le x \le q_2 \\ 0, & otherwise \end{cases}$$

by using proposed method we get the root in two iterations and the fuzzy root is (1.7275, 1.7285, 1.7295, 1.7305,1.7315). Now by the defuzzification, crisp root is 1.7295.
Now we consider hexagonal fuzzy number.
Let $\tilde{P} = (p_0, p_1, p_2, q_0, q_1, q_2), \tilde{R} = (r_0, r_1, r_2, s_0, s_1, s_2)$
Consider $\tilde{P} = (1.6025, 1.6035, 1.6045, 1.6055, 1.6065, 1.6075)$ and
$\tilde{R} = (2.1025, 2.1035, 2.1045, 2.1055, 2.1065, 2.1075)$
The function $G(x)$ changes sign over an interval $(\tilde{P}, \tilde{R})$. Here $G(\tilde{P}) < 0$ and $G(\tilde{R}) > 0$. Continue the procedure in same manner, we get fuzzy root in two iteration which is (1.7275, 1.7285, 1.7295, 1.7305,1.7315, 1.7325) and the crisp root is 1.73.

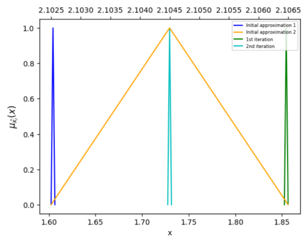
In particular, when n = 1, the 1-PFN degenerates into a trapezoid fuzzy number or triangular fuzzy number. For an n-PFN (n $\ge$ 2), if its membership function image is regarded as the superposition of n small trapezoids or triangles, then the n-PFN can be seen as a generalization of trapezoid fuzzy numbers or triangular fuzzy numbers [4]. This result was stated by the author Gang Sun in 2022.
From the above table, we observed that, for n-polygonal fuzzy numbers, the complexity increases with the number of vertices, n. More vertices generally mean a more complex

Table 1. Fuzzified bisection method using n-PFNs

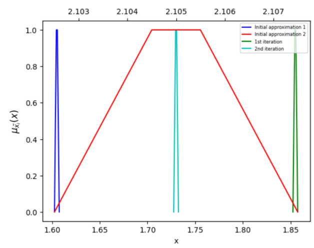
n-PFN	No. of iterations	Crisp root	Error
1-PFN (trapezoidal)	6	1.73290625	$8.55442X10^{-4}$
2-PFN (pentagonal)	2	1.7295	$2.550807X10^{-3}$
2-PFN (hexagonal)	2	1.73	$2.050807X10^{-3}$
3-PFN (heptagonal)	2	1.7305	$1.550807X10^{-3}$
3-PFN (octagonal)	2	1.7301	$1.050807X10^{-3}$
4-PFN (nonagonal)	2	1.7315	$5.508075X10^{-4}$
4-PFN (decagonal)	2	1.732	$5.080756X10^{-5}$
5-PFN (dodecagonal)	16	1.732057423	$6.61543112X10^{-6}$



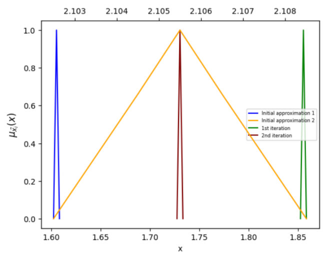
(A) 1-PFN (trapezoidal)



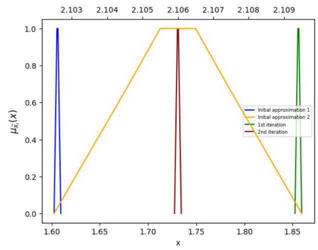
(B) 2-PFN (pentagonal).



(C) 2-PFN (hexagonal).



(D) 3-PFN (heptagonal).



(E) 5-PFN (dodecagonal).

Figure 1. Graphical representation of a solution

fuzzy number representation, probably leading to a slower convergence rate for the bisection method. In general, as n increases, it may need more iterations to achieve the accuracy. But there is no specific value of n that decides when to stop; it is determined based on the specific problem and required accuracy.

The traditional bisection method takes 18^{th} iterations, whereas the fuzzified bisection method with n-PFNs requires fewer iterations. So the proposed method is more effective.

8. Algorithm

Step 1 : Define a function, $f(x) = 0$. Choose a fuzzy interval $[a, b] = [\tilde{P}, \tilde{R}]$ where $f(\tilde{P})$ *and* $f(\tilde{R})$ have opposite signs.
Step 2 : Initialize the interval. Set $a_0 = \tilde{P}$ and $b_0 = \tilde{R}$ as boundaries of initial interval.
Step 3 : Bisect the interval. $\tilde{x_0} = \frac{\tilde{P}+\tilde{R}}{2}$.
Step 4 : Compute the function $f(\tilde{x_0})$ to determine the function value at midpoint.
Step 5 : Determine the new interval. Depending on the sign of $f(\tilde{x_o})$:

- if $f(\tilde{x_o})$ has the same sign as $f(\tilde{P})$, set $a_1 = \tilde{x_0}$ and $b_1 = \tilde{R}$.

- if $f(\tilde{x_o})$ has the same sign as $f(\tilde{R})$, set $a_1 = \tilde{P}$ and $b_1 = \tilde{x_0}$.

Step 6 : Check convergence. if $f(\tilde{x_n})$ is close to zero then stop the iteration otherwise, proceed to the next step.
Step 7 : Repeat steps 3 to 6 using the new interval $[a_1, b_1]$ until the convergence criteria are met.
Step 8 : Defuzzification: once the iteration stops that is, when we get fuzzy root of an equation then we apply defuzzification (centroid, median etc.) and we get crisp root of the function $f(x)$ with in the given tolerance.

9. Conclusion

In this paper, we discuss a fuzzified bisection method using n-polygonal fuzzy numbers. To find the root of an algebraic equation, we use the fuzzified bisection method with n-polygonal fuzzy numbers. We conclude that, for n-polygonal fuzzy numbers, the complexity increases with n. More vertices mean a more complex fuzzy number representation, which leads to a slower convergence rate for the bisection method. In general, as n increases in n-PFNs, it may need more iterations to achieve the exact solution. The roots obtained by using both methods—the fuzzified bisection with n polygonal fuzzy numbers and the classical bisection—are nearly the same. However, by comparing the roots obtained from both methods, it was shown that the fuzzified bisection method produces the root more rapidly than the classical bisection method.

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