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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY RABOTNOV FUNCTION

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Abstract. The study of the geometric properties of analytic functions and their numerous applications in a variety of mathematical fields, including fractional calculus, probability distributions, and special functions, has drawn significant and impressive attention to Geometric Function Theory (GFT), one of the most prominent branches of complex analysis, in recent years. The focus of this article is the introduction of a new subclass of analytic functions involving Rabotnov function and obtained coefficient inequalities, convex linear combination, radii properties, Integral means inequality and neighborhood result for this class.

Keywords: analytic, convex, starlike, coefficient estimates, neighborhood.

AMS Subject Classification: 30C45; 30C50.

1. Introduction

Let A specify the category of analytical functions u represent on the unit disc $\mathbb{U} = \{z : |z| < 1\}$ with normalization $u(0) = 0$ and $u'(0) = 1$, such a function has the extension of the Taylor series on the origin in the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Indicated by S, the subclass of A be composed of functions that are univalent in $\mathbb{U}$.

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Then a $u(z)$ function of A is known as starlike and convex of order ϑ if it delights the pursing

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \vartheta,\ (z \in \mathbb{U}), \tag{2}$$

and

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \vartheta,\ (z \in \mathbb{U}), \tag{3}$$

for specific $\vartheta(0 \leq \vartheta < 1)$ respectively and we express by $S^*(\vartheta)$ and $K(\vartheta)$ the subclass of A be expressed by aforesaid functions respectively. Also, indicate by T the subclass of A made up of functions of this form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ (a_n \geq 0,\ z \in \mathbb{U}) \tag{4}$$

and let $T^*(\vartheta) = T \cap S^*(\vartheta), C(\vartheta) = T \cap K(\vartheta)$. There are interesting properties in the $T^*(\vartheta)$ and $C(\vartheta)$ classes and were thoroughly studied by Silverman [24] and others.
In [13], Goodman defined the class of uniformly convex functions, defined by UCV as follows:

Definition 1.1. *A function $u \in A$ is said to be uniformly convex in $\mathbb{U}$ if $u \in K$ and have the property that for every circular arc ρ contained in $\mathbb{U}$, with centre ζ, also in $\mathbb{U}$, the arc $u(\rho)$ is convex.*

Due to the analytic criterion for $u \in UCV$, given by Ronning [22].
A function $u \in A$ is uniformly convex in $\mathbb{U}$ if and only if

$$\mathfrak{R}\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \left|\frac{zu''(z)}{u'(z)}\right|,\ z \in \mathbb{U}. \tag{5}$$

The class of $k-$ uniformly convex functions was introduced by Kanas and Wisniowska [14], as a generalization of uniform convexity. The class of $k-$ uniformly convex functions are denoted by $k-UCV$. In [22], Ronning defined the classs of parabolic starlike functions by the following way:

$$S_p := \left\{F \in S^* \ \mid\ F(z) = zu'(z),\ u \in UCV\right\}.$$

Definition 1.2. *The class S_p of parabolic starlike functions consists of functions $u \in A$ satisfying*

$$\mathfrak{R}\left\{\frac{zu'(z)}{u(z)}\right\} > k\ \left|\frac{zu'(z)}{u(z)}\right|,\ z \in \mathbb{U}.$$

The class of $k-$ parabolic starlike functions denoted by $k-S_p$ are related to the class $k-UCV$ by well known Alexander equavalence.
For $-1 < \rho \leq 1$ and $k \geq 0$ a function $u \in A$ is said to be in the class of $k-$ parabolic starlike functions of order ρ, denoted by $k-S_p(\rho)$ if

$$\mathfrak{R}\left\{\frac{zu'(z)}{u(z)}-\rho\right\} > k\left|\frac{zu'(z)}{u(z)}-1\right|,\ z \in \mathbb{U}.$$

For the same conditions for the parameters ρ and k, the function $u \in A$ is said to be in the class of $k-$ uniformly convex functions of order ρ if

$$\mathfrak{R}\left\{1+\frac{zu''(z)}{u'(z)}-\rho\right\} > k\left|\frac{zu''(z)}{u'(z)}\right|,\ z \in \mathbb{U}.$$

We denote by $k-S_p(\rho)$ the class of $k-$parabolic starlike functions of order ρ and $k-UCV(\rho)$ the class of $k-$ uniformly convex functions of order ρ.

In [14], the authors generalized the class of $k-$ parabolic starlike, respectively $k-$ uniformlly convex functions, of order ρ, for $0 \leq \rho < 1$.

For $0 \leq \mu < 1, 0 \leq \rho < 1$ and $k \geq 0$ the function $u \in A$ belongs to the class $k - S_p(\mu, \rho)$ if

$$\Re\left\{\frac{zu'(z)}{(1-\mu)u(z)+\mu zu'(z)} - \rho\right\} > k\left|\frac{zu'(z)}{(1-\mu)u(z)+\mu zu'(z)} - 1\right|, \ z \in \mathbb{U}. \tag{6}$$

For the same conditions to the parameters μ, ρ and k, the function $u \in A$ belongs to the class $k - UCV(\mu, \rho)$ if

$$\Re\left\{\frac{u'(z)+zu''(z)}{u'(z)+\mu zu''(z)} - \rho\right\} > k\left|\frac{u'(z)+zu''(z)}{u'(z)+\mu zu''(z)} - 1\right|, \ z \in \mathbb{U}. \tag{7}$$

It is easily seen that, $k - S_p(0, \rho) = k - S_p(\rho), k - S_p(0, 0) = k - S_p, k - UCV(0, \rho) = k - UCV(\rho)$ and $k - UCV(0, 0) = k - UCV$, where $0 \leq \rho < 1$.

The Rabotnov function is used to model the time-dependent deformation of materials subjected to constant stress, such as in creep testing or in the analysis of long-term structural behavior under load. It provides a mathematical framework for understanding how materials deform over time, incorporating both instantaneous and time-dependent deformation mechanisms.

The Rabotnov function finds applications primarily in the fields of materials science, mechanical engineering, and structural analysis, where understanding the time-dependent behavior of materials under load is crucial. Rabotnov function plays a key role in understanding and predicting the time-dependent behavior of materials under load, with applications ranging from materials characterization and testing to structural analysis and design in various engineering disciplines. The Rabotnov function holds significant importance in several areas of materials science, mechanical engineering, and structural analysis due to its ability to model the time-dependent deformation of materials under load.

It serves as a fundamental tool for engineers and scientists involved in materials research, mechanical design, and structural analysis, enabling them to address the challenges associated with time-dependent deformation and ensure the reliability and performance of engineered systems and components.

The aims of the Rabotnov function revolve around providing a quantitative framework for understanding and predicting the time-dependent deformation of materials, with implications for material selection, design optimization, and structural analysis in engineering applications.The scope of the Rabotnov function encompasses a wide range of applications and disciplines, playing a crucial role in understanding, predicting, and mitigating the effects of time-dependent deformation in materials and structures across various engineering fields.

In 1948, Yu. N. Rabotnov [20], who worked in solid mechanics included plasticity, creep theory, hereditary mechanics, failure machanics, nonelastic stability, composites and shell theory, introduced a special function applied in viscoelasticity. This function, known today as the Rabotnov fractional exponential function or briefly Rabotnov function, is defined as follows:

$$\Phi_{\wp,\varsigma}(z) = z^{\wp}\sum_{n=0}^{\infty}\frac{(\varsigma)^n z^{n(1+\wp)}}{\Gamma\left((n+1)(1+\wp)\right)}, \quad (\wp, \varsigma, z \in \mathbb{C}). \tag{8}$$

The convergence of this series at any values of the argument is evident. Noting that for $\wp = 0$ it reduces to the standard exponential exp (βz). The Rabotnov function is a particular case of the familier Mittag-Lefflor [19] widely used in the solution of fractional order or fractional differential equations. The relation between the Rabotnov and Mittag-Lefflor functions [19] can be written as follows:

$$\Phi_{\wp,\varsigma}(z) = z^{\wp} E_{1+\wp,1+\wp}(\varsigma z^{1+\wp}),$$

where E is Mittag-Lefflor and $\wp, \varsigma, z \in \mathbb{C}$. Several sufficient conditions are established for Mittag-Leffler functions to exhibit specific geometric properties, including univalence, starlikeness, convexity, and close-to-convexity, as well as for integral operators within the open unit disk. Additionally, subordination results, partial sums, and various other properties of the generalized Mittag-Leffler function are discussed (see [1, 2, 4, 5, 7, 11]) . It is clear that the Rabotnov function $\Phi_{\wp,\varsigma}(z)$ does not belong to A. Thus, it is natural to consider the normalization of the Rabotnov function for $\wp \geq 0$ and $\varsigma > 0$ defined by

$$\begin{aligned} \mathbb{R}_{\wp,\varsigma}(z) &= z^{1/1+\wp}\Gamma(1+\wp)\Phi_{\wp,\varsigma}(z^{1/1+\wp}) \\ &= z + \sum_{n=2}^{\infty} \frac{\varsigma^{n-1}\Gamma(1+\wp)}{\Gamma\left((1+\wp)n\right)} z^n, \quad z \in \mathbb{U}. \end{aligned}$$

Note that some special cases of $\mathbb{R}_{\wp,\varsigma}(z)$ are:

$$\begin{aligned} \mathbb{R}_{0,\frac{1}{3}}(z) &= ze^{-z/3}, \\ \mathbb{R}_{1,\frac{1}{2}}(z) &= \sqrt{2z}\ sinh\sqrt{\frac{z}{2}}, \\ \mathbb{R}_{1,-\frac{1}{4}}(z) &= 2\sqrt{z}\ sin\frac{\sqrt{z}}{2}, \\ \mathbb{R}_{1,1}(z) &= \sqrt{z}\ sinh\sqrt{z}, \\ \mathbb{R}_{1,2}(z) &= \frac{\sqrt{2z}\ sinh\sqrt{2z}}{2}. \end{aligned}$$

Geometric properties including starlikeness, convexity, close-to-convexity for the normalized Rabotnov function $\mathbb{R}_{\wp,\varsigma}(z)$ were investigated by Eker and Ece in [8] and also see [15, 16, 21] .

$$\begin{aligned} \Delta_{\wp}^{\varsigma}u(z) = \mathbb{R}_{\wp,\varsigma}(z) * u(z) &= z + \sum_{n=2}^{\infty} \frac{\varsigma^{n-1}\Gamma(1+\wp)}{\Gamma\left((1+\wp)n\right)} a_n z^n, \quad z \in \mathbb{U} \\ &= z + \sum_{n=2}^{\infty} \Theta(n,\wp,\varsigma) a_n z^n \end{aligned} \tag{9}$$

where $\Theta(n,\wp,\varsigma) = \frac{\varsigma^{n-1}\Gamma(1+\wp)}{\Gamma((1+\wp)n)}$ and $\wp \geq 0$, $\varsigma > 0$.

Now we define the following new subclass motivated by Amourah et al. [3], Eker et al. [8, 9], Frasin [10], Thirupathi Reddy and Venkateswarlu [27].

Definition 1.3. *The function $u(z)$ of the form* (1) *is in the class $\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$, if it satisfies the inequality*

$$\Re\left\{\frac{z(\Delta_{\wp}^{\varsigma}u(z))'}{(1-\mu)z+\mu\Delta_{\wp}^{\varsigma}u(z)} - \gamma\right\} > k\left|\frac{z(\Delta_{\wp}^{\varsigma}u(z))'}{(1-\mu)z+\mu\Delta_{\wp}^{\varsigma}u(z)} - 1\right|$$

for $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$and $k \geq 0$.

Further we define $T\Omega_{\wp}^{\varsigma,k}(\mu,\gamma,k) = \Omega_{\wp}^{\varsigma}(\mu,\gamma,k) \cap T$.

The aim of present paper is to study the coefficient bounds, radii of close-to-convex and starlikeness convex linear combinations and integral means inequalities of the $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$.

2. Coefficient bounds

Theorem 2.1. *Let the function $u(z)$ of the form (1) be in $\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$. Then*

$$\sum_{n=2}^{\infty}[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)|a_n| \leq 1-\gamma \tag{10}$$

where $0\leq\mu\leq 1$, $0\leq\gamma\leq 1, k\geq 0$ and $\Theta(n,\wp,\varsigma)$ is given by (9).

Proof. It suffices to show that

$$k\left|\frac{z(\Delta_{\wp}^{\varsigma}u(z))'}{(1-\mu)z+\mu\Delta_{\wp}^{\varsigma}u(z)}-1\right|-\Re\left\{\frac{z(\Delta_{\wp}^{\varsigma}u(z))'}{(1-\mu)z+\mu\Delta_{\wp}^{\varsigma}u(z)}-1\right\}$$
$$\leq 1-\gamma.$$

We have

$$k\left|\frac{z(\Delta_{\wp}^{\varsigma}u(z))'}{(1-\mu)z+\mu\Delta_{\wp}^{\varsigma}u(z)}-1\right|-\Re\left\{\frac{z(\Delta_{\wp}^{\varsigma}u(z))'}{(1-\mu)z+\mu\Delta_{\wp}^{\varsigma}u(z)}-1\right\}$$
$$\leq(1+k)\left|\frac{z(\Delta_{\wp}^{\varsigma}u(z))'}{(1-\mu)z+\mu\Delta_{\wp}^{\varsigma}u(z)}-1\right|$$
$$\leq\frac{(1+k)\sum\limits_{n=2}^{\infty}(n-\mu)\Theta(n,\wp,\varsigma)|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}\mu\Theta(n,\wp,\varsigma)|a_n||z|^{n-1}}$$
$$\leq\frac{(1+k)\sum\limits_{n=2}^{\infty}(n-\mu)\Theta(n,\wp,\varsigma)|a_n|}{1-\sum\limits_{n=2}^{\infty}\mu\Theta(n,\wp,\varsigma)|a_n|}.$$

The last expression is bounded above by $(1-\gamma)$, if

$$\sum_{n=2}^{\infty}[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)|a_n| \leq 1-\gamma$$

and the proof is complete. □

Theorem 2.2. *Let $0\leq\mu\leq 1$, $0\leq\gamma\leq 1$ and $k\geq 0$. Then a function u of the form (4) to be in the class $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)|a_n| \leq 1-\gamma \tag{11}$$

where $\Theta(n,\wp,\varsigma)$ is given by (9) .

Proof. In view of Theorem 2.1, we need only to prove the necssity. If $u\in T\Omega_{\wp}^{\varsigma}(\mu,\gamma.k)$ and z is real, then

$$\Re\left\{\frac{1-\sum\limits_{n=2}^{\infty}n\Theta(n,\wp,\varsigma)a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}\mu\Theta(n,\wp,\varsigma)a_nz^{n-1}}-\gamma\right\}>k\left|\frac{\sum\limits_{n=2}^{\infty}(n-\mu)\Theta(n,\wp,\varsigma)a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}\mu\Theta(n,\wp,\varsigma)a_nz^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+k)-\mu(\gamma+1)]\Theta(n,\wp,\varsigma)|a_n| \leq 1-\gamma.$$

□

Corollary 2.1. *If $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$, then*

$$|a_n| \leq \frac{1-\gamma}{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)} \tag{12}$$

where $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$, $k \geq 0$ and $\Theta(n,\wp,\varsigma)$ is given by (9). Equality holds for the function

$$u(z) = z - \frac{1-\gamma}{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}z^n. \tag{13}$$

Theorem 2.3. *Let $u_1(z) = z$ and*

$$u_n(z) = z - \frac{1-\gamma}{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}z^n, \; n \geq 2. \tag{14}$$

Then $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$ if and only if it can be expressed in the form

$$u(z) = \sum_{n=1}^{\infty} w_n u_n(z), \; w_n \geq 0, \; \sum_{n=1}^{\infty} w_n = 1. \tag{15}$$

Proof. Suppose $u(z)$ can be written as in (15). Then

$$u(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}z^n.$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{(1-\gamma)[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}$$
$$= \sum_{n=2}^{\infty} w_n = 1 - w_1 \leq 1.$$

Thus $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$.
Conversely, let $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$. Then by using (12), we get

$$w_n = \frac{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{(1-\gamma)}a_n, \; n \geq 2$$

and $w_1 = 1 - \sum\limits_{n=2}^{\infty} w_n$. Then we have $u(z) = \sum\limits_{n=1}^{\infty} w_n u_n(z)$ and hence this completes the proof of theorem. □

Theorem 2.4. *The class $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$ is a convex set.*

Proof. Let the function

$$u_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \; a_{n,j} \geq 0, \; j = 1, 2 \tag{16}$$

be in the class $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi u_1(z) + (1-\xi)u_2(z), \; 0 \leq \xi < 1,$$

in the class $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$. Since

$$h(z) = z - \sum_{n=2}^{\infty}[\xi a_{n,1} + (1-\xi)a_{n,2}]z^n.$$

An easy computation with the aid of Theorem 2.2 gives

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+k)-\mu(\gamma+k)]\xi\Theta(n,\wp,\varsigma)a_{n,1} + \sum_{n=2}^{\infty}[n(1+k)-\mu(\gamma+k)](1-\xi)\Theta(n,\wp,\varsigma)a_{n,2}\\
&\le \xi(1-\gamma) + (1-\xi)(1-\gamma)\\
&\le (1-\gamma)
\end{aligned}$$

which implies that $h \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$.
Hence $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$ is convex.

□

3. Radii of Close-to-Convexity, Starlikeness and Convexity

In this section, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$.

Theorem 3.1. *Let the function $u(z)$ defined by (4) belong to the class $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$. Then $u(z)$ is close-to-convex of order $\delta (0 \le \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{n\ge 2}\left[\frac{(1-\delta)\sum\limits_{n=2}^{\infty}[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{n(1-\gamma)}\right]^{1/n-1}, n \ge 2. \quad (17)$$

The result is sharp with the external function $u(z)$ is given by (14).

Proof. Given $u \in T$ and u is close-to-convex of order δ, we have

$$|u'(z) - 1| < 1 - \delta. \quad (18)$$

For the left hand side of (18), we have

$$|u'(z) - 1| \le \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1 - \delta$

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta}a_n|z|^{n-1} \le 1.$$

Using the fact, that $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{1-\gamma}a_n \le 1.$$

We can see that (18) is true, if

$$\frac{n}{1-\delta}|z|^{n-1} \le \frac{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{1-\gamma}$$

or, equivalently

$$|z| \leq \left\{ \frac{(1-\delta)[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{n(1-\gamma)} \right\}^{1/n-1}$$

which completes the proof. □

Theorem 3.2. *Let the function $u(z)$ defined by* (4) *belong to the class $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$. Then $u(z)$ is starlike of order $\delta (0 \leq \delta < 1)$ in the disc $|z| < r_2$, where*

$$r_2 = \inf_{n\geq 2} \left[\frac{(1-\delta)\sum\limits_{n=2}^{\infty}[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{(n-\delta)(1-\gamma)} \right]^{1/n-1}. \tag{19}$$

The result is sharp with external function $u(z)$ is given by (14).

Proof. Given $u \in T$ and u is starlike of order δ, we have

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| < 1-\delta. \tag{20}$$

For the left hand side of (20), we have

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| \leq \sum_{n=2}^{\infty} \frac{(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} a_n|z|^{n-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n|z|^{n-1} < 1.$$

Using the fact that $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,k)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{1-\gamma} a_n \leq 1.$$

We can say (20) is true, if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta}|z|^{n-1} \leq \frac{[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{1-\gamma}$$

or equivalently

$$|z|^{n-1} \leq \frac{(1-\delta)[n(1+k)-\mu(\gamma+k)]\Theta(n,\wp,\varsigma)}{(n-\delta)(1-\gamma)}$$

which yields the starlikeness of the family. □

4. Integral Means Inequalities

In [24], Silverman found that the function $u_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjunctured [25] and setteled in [26], that

$$\int_0^{2\pi} |u(re^{i\varphi})|^{\tau} d\varphi \leq \int_0^{2\pi} |u_2(re^{i\varphi})^n|^{\tau} d\varphi$$

for all $u \in T$, $\tau > 0$ and $0 < r < 1$. In [26], he also proved his conjuncture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now, we prove Silverman's conjecture for the class of functions $T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k)$.
We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [18].

Two functions u and v, which are analytic in $\mathbb{U}$, the function u is said to be subordinate to v in $\mathbb{U}$, if there exists a function w analytic in $\mathbb{U}$ with $w(0) = 0, |w(z)| < 1, (z \in \mathbb{U})$ such that $u(z) = v(w(z)), (z \in \mathbb{U})$. We denote this subordination by $u(z) \prec v(z), (\prec$ denote subordination).

Lemma 4.1. *If the function u and v are analytic in $\mathbb{U}$ with $u(z) \prec v(z)$, then for $\tau > 0$ and $z = re^{i\varphi}$, $0 < r < 1$*

$$\int_0^{2\pi} |v(re^{i\varphi})|^\tau d\varphi \leq \int_0^{2\pi} |u(re^{i\varphi})|^\tau d\varphi.$$

Now, we discuss the integral means inequalities for functions u in $T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k)$.

Theorem 4.1. *$u \in T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k)$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$ and $u_2(z)$ be defined by*

$$u_2(z) = z - \frac{1-\gamma}{\phi(2)} z^2(\phi(2)), \tag{21}$$

where $\phi(n) = [n(1 + k) - \mu(\gamma + 1)]\Theta(n, \wp, \varsigma)$.

Proof. For $u(z) = z - \sum\limits_{n=2}^{\infty} a_n z^n$, (21) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\tau d\varphi \leq \int_0^{2\pi} \left|1 - \frac{1-\gamma}{\phi(2)} z\right|^\tau d\varphi.$$

By Lemma 4.1, it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\phi(2)} z.$$

Assuming

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\gamma}{\phi(2)} w(z)$$

and using (11), we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\phi(n)}{1-\gamma} a_n z^{n-1}\right| \leq |z| \sum_{n=2}^{\infty} \frac{\phi(n)}{1-\gamma} a_n \leq |z| < 1,$$

where

$$\phi(n) = [n(1 + k) - \mu(\gamma + 1)]\Theta(n, \wp, \varsigma).$$

This completes the proof. □

5. Neighborhood Property

Following the earlier investigations by Darwish et al. [6], Goodman [12], Kazimogulu [17] and Ruscheweyh [23], and others. We define the (n, δ)- neighborhood of a function $u(z) \in T$ by

$$N_\delta(u) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n\,|a_n - b_n| \leq \delta \right\}. \tag{22}$$

In particular, if $e(z) = z$, we have

$$N_\delta(e) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n\,|b_n| \leq \delta \right\}. \tag{23}$$

Now we determine the neighborhood for each of the class $T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k)$ which we define as follows.
A function $u \in T$ is said to be in the class $T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k, \xi)$ if there exists a function $g \in T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k)$ such that

$$\left| \frac{u(z)}{g(z)} - 1 \right| \leq 1 - \xi, \quad (z \in \mathbb{U},\ 0 \leq \xi < 1). \tag{24}$$

Theorem 5.1. *If $g \in T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k)$ and*

$$\xi = 1 - \frac{\delta[2(1+k) - \mu(\gamma+1)]\Theta(2, \wp, \varsigma)}{2[(2(1+k) - \mu(\gamma+k)) - (1-\gamma)]} \tag{25}$$

then $N_\delta(g) \subset T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k, \xi)$.

Proof. Suppose $u(z) \in N_\delta(g)$. We find from (2.1) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta$$

which implies that

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2}, \quad (n \in \mathbb{N}). \tag{26}$$

Next, since $g(z) \in T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k)$, we have

$$\sum_{n=2}^{\infty} |b_n| \leq \frac{1-\gamma}{[2(1+k) - \mu(\gamma+k)]\Theta(2, \wp, \varsigma)} \tag{27}$$

so that

$$\begin{aligned} \left| \frac{u(z)}{g(z)} - 1 \right| &< \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} |b_n|} \\ &\leq \frac{\delta}{2} \left[\frac{[2(1+k) - \mu(\gamma+k)]\Theta(2, \wp, \varsigma)}{[2(1+k) - \mu(\gamma+k)]\Theta(2, \wp, \varsigma) - (1-\gamma)} \right] \\ &\leq 1 - \xi, \end{aligned}$$

provided that ξ is given by (25) . Thus, $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu, \gamma, k, \xi)$ for ξ given by (25). This completes the proof of the Theorem. □

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