TWMS J. App. and Eng. Math. V.15, N.8, 2025, pp. 2140-255

A STUDY ON BI-PARAMETRIC POTENTIALS: INVERSION FORMULAS UTILIZING WAVELET-LIKE TRANSFORMATIONS IN WEIGHTED LEBESGUE SPACES

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Abstract. We introduce a new family of wavelet-like transforms based on bi-parametric semigroups associated with the Laplace-Bessel differential operator. Using these transforms, we obtain new inversion formulas for bi-parametric potentials in the framework of weighted Lebesgue spaces.

Keywords: Wavelet transforms, Abel-Poisson semigroup, Gauss-Weierstrass semigroup, inversion formulas, Bessel potentials, Modified Bessel potentials

AMS Subject Classification: 26A33,42C40,44A35

1. Introduction

The classical Bessel potentials, an important integral operator in Fourier harmonic analysis associated with the Laplace differential operator, are defined in terms of the Fourier transform by

$$(\mathcal{J}^{\alpha}\varphi)^{\wedge}(x) = (1+|x|^2)^{-\alpha/2}(\varphi)^{\wedge}(x), \quad (x \in \mathbb{R}^n,\ 0 < \alpha < \infty).$$

These potentials are interpreted as negative fractional powers of "the strictly positive" differential operator $(I-\Delta)$, (Δ is the Laplacian and I is the identity operator) that is,

$$\mathcal{J}^{\alpha}\varphi = (I-\Delta)^{-\alpha/2}\varphi.$$

Moreover Bessel potentials have the following convulution-type integral representation:

$$(\mathcal{J}^{\alpha}\varphi)(x) = \int\limits_{\mathbb{R}^n} g_{\alpha}(y)\,\varphi(x-y)\,dy, \quad \varphi \in L_p(\mathbb{R}^n), \quad (1 \le p < \infty)$$

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§ Manuscript received: June 29, 2024; accepted: November 05, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.8;

where the kernel $g_\alpha(y) = \frac{2^{(2-n-\alpha)/2}}{\pi^{n/2}\Gamma(\alpha/2)} |y|^{(\alpha-n)/2} K_{(\alpha-n)/2}(|y|)$ and $K_\nu(z)$ is known as McDonald function ([9]). An interesting modification of the classical Bessel potentials appears in Fourier-Bessel harmonic analysis, which is associated with the Bessel or Laplace-Bessel differential operators. The study of different versions of these differential operators in Fourier-Bessel harmonic analysis began with Delsarte and was further developed by researchers such as Levitan, Kipriyanov, Lyakhov, Trimeche, Gadjiev, Aliev, Guliev, Hasanov, Bayrakci, Sezer, Yıldız, Kahraman, and others ([4, 5, 6, 11, 12, 13, 16, 19, 21]).

One of the important problems concerning the Bessel potentials (in Fourier or Fourier-Bessel harmonic analysis) is obtaining an explicit inversion formula. The hypersingular integral technique, a very powerful tool for the inversion of potentials, was introduced and studied by Stein, Lizorkin, Wheeden, Samko, Rubin, Aliev,([2, 14, 18, 20, 22]), and references therein. An alternative approach to this problem has been introduced and developed by Rubin. One should also mention the papers by Aliev and Rubin [3].

In this paper, a family of the bi-parametric potentials $\mathcal{B}^\alpha_{\nu,\beta}$, $(0 < \alpha, \beta < \infty)$ that generalize the Bessel and the modified Bessel potentials associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{k=1}^{N} \left(\frac{\partial^2}{\partial x_k^2} + \frac{2\nu_k + 1}{x_k} \frac{\partial}{\partial x_k} \right) + \sum_{k=N+1}^{n} \frac{\partial^2}{\partial x_k^2}, \ (\nu_k > -1/2; \ k = 1, \cdots, N) \tag{1}$$

are introduced. These potentials are defined in terms of the Fourier-Bessel transform

$$F_\nu(\mathcal{B}^\alpha_{\nu,\beta}\varphi)(x) = (1 + |x|^\beta)^{-\alpha/\beta} F_\nu(\varphi)(x), \ (0 < \alpha, \beta < \infty)$$

and may be interpreted as negative fractional powers of order $(-\alpha/\beta)$ of the fractional differential operator $I + (-\Delta_B)^{\beta/2}$; that is, formally

$$\mathcal{B}^\alpha_{\nu,\beta}\varphi = \left(I + (-\Delta_B)^{\beta/2} \right)^{-\alpha/\beta} \varphi.$$

The rest of the paper is organized as follows: Section 2 provides necessary definitions and auxiliary facts. Here, we introduce the concept of a bi-parametric semigroup and discuss its properties. Section3 defines bi-parametric potentials and wavelet-like transforms, presenting the explicit inversion formulas for these potentials.

2. Preliminaries

Let

$$\mathbb{R}^n_{N,+} = \left\{ x = (x', x'') \in \mathbb{R}^n, \ x' \in \mathbb{R}^N, \ x'' \in \mathbb{R}^{n-N}, \ x_1, x_2, \cdots, x_N > 0 \right\}$$

and define $\nu = (\nu_1, \nu_2, \cdots, \nu_N)$ such that $\nu_k > -1/2$ for $k = 1, \cdots, N$ and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_N$. For a measurable $E \subset \mathbb{R}^n_{N,+}$,

$$|E|_\nu = \int_E (x')^{2\nu+1} dx; \quad (x')^{2\nu+1} dx = x_1^{2\nu_1+1} \cdots x_N^{2\nu_N+1} dx_1 \cdots dx_n$$

is the Lebesgue measure. Let $E(x,r) = \{y \in \mathbb{R}^n_{N,+} : |x - y| < r\}$, denote the ball of radius $r > 0$ centered at $x \in \mathbb{R}^n_{N,+}$, and let $S\left(\mathbb{R}^n_{N,+}\right)$ represent the space of functions that are restrictions to $\mathbb{R}^n_{N,+}$ of the Schwartz test functions on $\mathbb{R}^n$ which are even in the variables $x_1, \cdots, x_N$.

The weighted Lebesgue space of $L_{p,\nu}$, $(1 \leq p < \infty)$ Lebesgue measurable functions is defined by

$$L_{p,\nu} \equiv L_{p,\nu}\left(\mathbb{R}^n_{N,+}\right) = \left\{ f : \|f\|_{p,\nu} = \left(\int\limits_{\mathbb{R}^n_{N,+}} |f(x)|^p \left(x'\right)^{2\nu+1} dx \right)^{\frac{1}{p}} < \infty \right\}$$

where $(x')^{2\nu+1} dx = x_1^{2\nu_1+1} \cdots x_N^{2\nu_N+1} dx_1 \cdots dx_n$.

Let T^y denote the generalized translation operator associated with the Laplace-Bessel differential operator Δ_B, which acts according to the law

$$T^y f(x) = \prod_{k=1}^{N} \frac{\Gamma(\nu_k+1)}{\sqrt{\pi}\Gamma\left(\nu_k + \frac{1}{2}\right)} \int\limits_0^{\pi} \cdots \int\limits_0^{\pi} f\left((x', y')_\theta, x'' - y'' \right) d_\nu(\theta)$$

where $(x', y')_\theta = \left((x_1, y_1)_{\theta_1}, \cdots, (x_N, y_N)_{\theta_N} \right)$, $(x_k, y_k)_{\theta_k} = \left(x_k^2 - 2x_k y_k \cos\theta_k + y_k^2 \right)^{1/2}$, $k = 1, ..., N$, and

$$x'' - y'' = (x_{k+1} - y_{k+1}, ..., x_n - y_n), \qquad d_\nu(\theta) = \prod_{i=1}^{k} \sin^{2\nu_i+1}\theta_i \; d\theta_i.$$

It is known (see e.g. [15]) that

$$\begin{cases} \|T^y f\|_{p,\nu} \leq \|f\|_{p,\nu}, \quad (\ 1 \leq p \leq \infty,\ y \in \mathbb{R}^n_{N,+}) \\ \\ \|T^y f - f\|_{p,\nu} \to 0,\ |y| \longrightarrow 0, \quad (\ 1 \leq p \leq \infty\ ). \end{cases} \tag{2}$$

In (2), we identify $L_{\infty,\nu}$ with $C_0 \equiv C_0(\mathbb{R}^n_{N,+})$, the space of continuous functions vanishing at infinity.

The relevant Fourier-Bessel transform and its inverse are defined on $S(\mathbb{R}^n_{N,+})$ by

$$F_\nu(f)(x) = \int\limits_{\mathbb{R}^n_{N,+}} f(y)\, e^{-i\langle x'', y''\rangle} \prod_{k=1}^{N} j_{\nu_k}(x_k y_k) \left(y'\right)^{2\nu+1} dy,$$

$$F_\nu^{-1}(f)(x) = c_{\nu,n,N} \left(F_\nu f\right)\left(x', -x''\right)$$

where $\langle x'', y''\rangle = x_{N+1} y_{N+1} + \cdots + x_n y_n$. Also

$$c_{\nu,n,N} = \left[(2\pi)^{n-N}\, 2^{2|\nu|} \prod_{k=1}^{N} \Gamma^2(\nu_{k+1}) \right]^{-1} \tag{3}$$

and

$$j_p(t) = 2^p \Gamma(p+1)\, t^{-p} J_p(t), \ j_p(0) = 1,\ p > -1/2,\ 0 < t < \infty \tag{4}$$

is the spherical Bessel function.

The generalized convolution operator is defined on $S\left(\mathbb{R}^n_{N,+}\right)$ by

$$(f \otimes g)(x) = \int\limits_{\mathbb{R}^n_{N,+}} f(y)\, (T^y g)(x) \left(y'\right)^{2\nu+1} dy,\ x \in \mathbb{R}^n_{N,+}$$

which satisfies the following Young's inequality:

$$\|\varphi \otimes \psi\|_{r,\nu} \leq \|\varphi\|_{p,\nu} \|\psi\|_{q,\nu},\ 1 \leq p, q, r \leq \infty,\ \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

and $F_\nu\left(f\otimes g\right)=\left(F_\nu f\right)\left(F_\nu g\right).$

We will now derive the Abel-Poisson and Gauss-Weierstrass kernel generated by the generalized translation operator associated with the Laplace-Bessel differential operator Δ_B, as defined in (1). These kernels are defined as the Fourier-Bessel transformation of the functions $e^{-t|y|}$ and $e^{-t|y|^2}$, where $y\in\mathbb{R}^n_{N,+}$ respectively. Namely, by considering the formula

$$e^{-\beta}=\frac{1}{\sqrt{\pi}}\int\limits_0^\infty\frac{e^{-z}}{\sqrt{z}}e^{-\beta^2/4z}dz \text{ (see [20], p.6)}$$

and Fubini's theorem, for $x\in\mathbb{R}^n_{N,+}$ we have

$$F_\nu\left(e^{-|y|}\right)(x)=\int\limits_{\mathbb{R}^n_{N,+}}e^{-|y|}e^{-i\langle x'',y''\rangle}\prod_{k=1}^N j_{\nu_k}\left(x_ky_k\right)\left(y'\right)^{2\nu+1}dy$$

$$=\frac{1}{\sqrt{\pi}}\int\limits_{\mathbb{R}^n_{N,+}}\left[\int\limits_0^\infty\frac{e^{-z}}{\sqrt{z}}e^{-|y|^2/4z}dz\right]e^{-i\langle x'',y''\rangle}\prod_{k=1}^N j_{\nu_k}\left(x_ky_k\right)\left(y'\right)^{2\nu+1}dy$$

$$=\frac{1}{\sqrt{\pi}}\int\limits_0^\infty\frac{e^{-z}}{\sqrt{z}}\left(\prod_{k=1}^N\int\limits_0^\infty e^{-y_k^2/4z}j_{\nu_k}\left(x_ky_k\right)y_k^{2\nu_k+1}dy_k\right)\times$$

$$\times\left(\prod_{k=N+1}^n\int\limits_{\mathbb{R}}e^{-y_k^2/4z}e^{-ix_ky_k}dy_k\right)dz.$$

Taking into account (4) and the following formulas (see [9]):

$$\int\limits_{\mathbb{R}}e^{-y^2/4z}e^{-ixy}dy=2\sqrt{\pi z}e^{-zx^2}$$

and

$$\int\limits_0^\infty y^{\nu+1}e^{-ty^2}J_\nu\left(\beta y\right)dy=\frac{\beta^\nu}{(2t)^{\nu+1}}e^{-\beta^2/4t};\ Re t>0,\ Re\nu>-1,$$

we get

$$F_\nu\left(e^{-|y|}\right)(x)=\left(\sqrt{c_{\nu,n,N}}\right)^{-1}2^{|\nu|+\frac{n+N+1}{2}}\frac{1}{\sqrt{2\pi}}\frac{\Gamma\left(|\nu|+\frac{N+n+1}{2}\right)}{\left(1+|x|^2\right)^{|\nu|+\frac{N+n+1}{2}}}$$

where $c_{\nu,n,N}$ is defined by (3). Also, by using the equality

$$F_\nu\left(f\left(\lambda y\right)\right)(x)=\lambda^{-2|\nu|-N-n}F_\nu\left(f\left(y\right)\right)\left(\frac{x}{\lambda}\right),\ \lambda>0,$$

we have

$$F_\nu(e^{-t|y|})(x)=\left(\sqrt{c_{\nu,n,N}}\right)^{-1}\frac{2^{|\nu|+\frac{n+N+1}{2}}}{\sqrt{2\pi}}\Gamma\left(|\nu|+\frac{N+n+1}{2}\right)\frac{t}{\left(t^2+|x|^2\right)^{|\nu|+\frac{N+n+1}{2}}}.$$

By taking into account the last equality, we define the Abel-Poisson kernel generated by the generalized translation operator:

$$p_\nu(x;t)=\sqrt{c_{\nu,n,N}}\frac{2^{|\nu|+\frac{n+N+1}{2}}}{\sqrt{2\pi}}\Gamma\left(|\nu|+\frac{N+n+1}{2}\right)\frac{t}{(t^2+|x|^2)^{|\nu|+\frac{n+N+1}{2}}}. \tag{5}$$

It is easy to see that the following properties holds:

i) $F_\nu\left(p_\nu\left(\cdot;t\right)\right)(x)=e^{-t|x|},\ x\in\mathbb{R}^n_{N,+},\ t>0;$
ii) $\left\|p_\nu\left(\cdot;t\right)\right\|_{1,\nu}=1;$
iii) $p_\nu\left(x;t+s\right)=p_\nu\left(x;t\right)\otimes p_\nu\left(x;s\right)$ (semigroup property).

Similarly the Gauss-Weierstrass kernel generated by the generalized translation operator is defined by

$$g_\nu\left(x;t\right)=\sqrt{c_{\nu,n,N}}2^{-\frac{n+N+2|\nu|}{2}}t^{-\frac{n-N}{2}}e^{\frac{-|x|^2}{4t}},t>0,\ x\in\mathbb{R}^n_{N,+}. \tag{6}$$

Furthermore, $F_\nu\left(g_\nu\left(\cdot;t\right)\right)(x)=e^{-t|x|^2}$, $\left\|g_\nu\left(x;t\right)\right\|_{1,\nu}=1$ and semigroup property obtained easily.

Definition 2.1. *Bi-parametric kernels $w_\nu^{(\beta)}\left(x;t\right),\ x\in\mathbb{R}^n_{N,+},\ 0<t<\infty,\ 0<\beta<\infty$ generated by generalized translation operator are defined by*

$$w_\nu^{(\beta)}\left(x;t\right)=F_\nu^{-1}(e^{-t|y|^\beta})\left(x\right)=c_{n,\nu,N}\int\limits_{\mathbb{R}^n_{N,+}}e^{-t|y|^\beta}e^{i\langle x'',y''\rangle}\prod_{k=1}^{N}j_{\nu_k}\left(x_ky_k\right)\left(y'\right)^{2\nu+1}dy.$$

It can be seen that $w_\nu^{(1)}\left(|x|;t\right)=p_\nu\left(|x|;t\right)$ is the Abel-Poisson kernel for $\beta=1$ defined in (5) and $w_\nu^{(2)}\left(|x|;t\right)=g_\nu\left(|x|;t\right)$ is the Gauss-Weierstrass kernel for $\beta=2$ defined in (6). The main properties of the bi-parametric kernels are given by the following theorem.

Theorem 2.1. ***a)*** *Let $x\in\mathbb{R}^n_{N,+},\ 0<t<\infty,0<\beta<\infty$. Then*

$$w_\nu^{(\beta)}\left(\lambda^{1/\beta}x;\lambda t\right)=\lambda^{-(2|\nu|+n+N)/\beta}w_\nu^{(\beta)}\left(x;t\right)$$

and for $\lambda=1/t$

$$w_\nu^{(\beta)}\left(x;t\right)=t^{-(2|\nu|+n+N)/\beta}w_\nu^{(\beta)}\left(t^{-1/\beta}x;1\right). \tag{7}$$

b)*For* $0<\beta\leq 2$

$$w_\nu^{(\beta)}\left(x;t\right)>0,\ x\in\mathbb{R}^n_{N,+}.$$

c) *If $\beta=2k,\ (k\in\mathbb{N})$ then*

$$w_\nu^{(\beta)}\left(x;t\right)\in S\left(\mathbb{R}^n_{N,+}\right).$$

d)

$$\left\|w_\nu^{(\beta)}\left(\cdot;t\right)\right\|_{1,\nu}=1. \tag{8}$$

provided that $0<\beta\leq 2$ or $\beta=2k,\ (k\in\mathbb{N})$.

Proof. **a)** By changing of the variable: $y=\lambda^{1/\beta}z,\ dy=\lambda^{n/\beta}dz$ we have

$$\begin{aligned}w_\nu^{(\beta)}\left(\lambda^{1/\beta}x,\lambda t\right)&=c_{\nu,n,N}\int\limits_{\mathbb{R}^n_{N,+}}e^{-t\lambda^\beta|y|^\beta}e^{i\langle\lambda x'',y''\rangle}\prod_{k=1}^{N}j_{\nu_k}\left(\lambda x_ky_k\right)\left(y'\right)^{2\nu+1}dy\\&=\lambda^{-(2|\nu|+n+N)/\beta}w_\nu^{(\beta)}\left(x;t\right).\end{aligned}$$

b)For the cases $\beta = 1$ and $\beta = 2$, the positivity of $w_\nu^{(\beta)}(x;t)$ follows immediately from (5) and (6). Let now $0 < \beta \leq 2$. According to the Bernstein's theorem (see [7], chapter18, see also [8], p.223) there is a non-negative finite measure μ_β on $[0,\infty)$ so that $\mu_\beta([0,\infty)) = 1$ and

$$e^{-z^{\beta/2}} = \int_0^\infty e^{-\xi z} d\mu_\beta(\xi), \ z \in [0,\infty).$$

Let z be replace by $t^{2/\beta}|y|^2$ in order to derive

$$e^{-t|y|^\beta} = \int_0^\infty e^{-t^{2/\beta}\xi|y|^2} d\mu_\beta(\xi). \tag{9}$$

Hence, owing to (6), we obtain

$$\begin{aligned} w_\nu^{(\beta)}(x;t) &= F_\nu^{-1}\left(e^{-t|y|^\beta}\right)(x) = F_\nu^{-1}\left(\int_0^\infty e^{-t^{2/\beta}\xi|y|^2} d\mu_\beta(\xi)\right)(x) \\ &= \int_0^\infty F_\nu^{-1}\left(e^{-t^{2/\beta}\xi|y|^2}\right)(x)\, d\mu_\beta(\xi) \\ &= \sqrt{c_{\nu,n,N}}\, 2^{-\frac{n+N+2|\nu|}{2}} t^{-\frac{n-N}{\beta}} \int_0^\infty \xi^{-\frac{n-N}{2}} e^{\frac{-|x|^2}{4\xi}t^{-2/\beta}} d\mu_\beta(\xi) > 0. \end{aligned}$$

c)Since F_ν is an automorphizm of the space $S\left(\mathbb{R}^n_{N,+}\right)$ and $e^{-|x|^{2k}} \in S\left(\mathbb{R}^n_{N,+}\right)$ then its follows that $w_\nu^{(2k)}(x;t) \in S\left(\mathbb{R}^n_{N,+}\right)$.

d) Let $\beta = 2k,\ k \in \mathbb{N}$. Since $e^{-|y|^{2k}} \in S\left(\mathbb{R}^n_{N,+}\right)$ that is, $F_\nu^{-1}\left(e^{-t|y|^{2k}}\right)(x) = w_\nu^{(2k)}(|x|;t) \in L_{1,\nu}\left(\mathbb{R}^n_{N,+}\right)$, then $w_\nu^{(2k)}(|x|;t)$ is infinitely smooth and rapidly decreasing on $\mathbb{R}^n_{N,+}$. So

$$F_\nu\left(w_\nu^{(2k)}(|x|;t)\right) = e^{-t|y|^{2k}}.$$

Setting $x = (0,\cdots,0)$, we have

$$\int_{\mathbb{R}^n_{N,+}} w_\nu^{(2k)}(|x|;t)\left(x'\right)^{2\nu+1} dx = 1.$$

Now, let $0 < \beta < 2$. By applying (9) and Fubini's theorem, we have

$$\int_{\mathbb{R}^n_{N,+}} w_\nu^{(\beta)}(|x|;t)\,(x')^{2\nu+1} dx =$$

$$= \int\limits_{\mathbb{R}^n_{N,+}} \left(\int\limits_{\mathbb{R}^n_{N,+}} e^{-t|y|^{\beta}} e^{i\langle x'',y''\rangle} \prod_{k=1}^{N} j_{\nu_k}(x_k y_k) \left(y'\right)^{2\nu+1} dy \right) (x')^{2\nu+1} dx$$

$$= \int\limits_{\mathbb{R}^n_{N,+}} \left[\int\limits_{\mathbb{R}^n_{N,+}} \left(\int\limits_0^{\infty} e^{-t^{2/\beta}\xi|y|^2} d\mu_\beta(\xi) \right) e^{i\langle x'',y''\rangle} \prod_{k=1}^{N} j_{\nu_k}(x_k y_k) \left(y'\right)^{2\nu+1} dy \right] (x')^{2\nu+1} dx$$

$$= \int\limits_0^{\infty} \left[\int\limits_{\mathbb{R}^n_{N,+}} \left(\int\limits_{\mathbb{R}^n_{N,+}} e^{-t^{2/\beta}\xi|y|^2} e^{i\langle x'',y''\rangle} \prod_{k=1}^{N} j_{\nu_k}(x_k y_k) \left(y'\right)^{2\nu+1} dy \right) (x')^{2\nu+1} dx \right] d\mu_\beta(\xi)$$

$$= \int\limits_0^{\infty} \left(\int\limits_{\mathbb{R}^n_{N,+}} w_\nu^{(2)}\left(|x|; t^{2/\beta}\xi\right) (x')^{2\nu+1} dx \right) d\mu_\beta(\xi) = \int\limits_0^{\infty} d\mu_\beta(\xi) = 1.$$

□

Definition 2.2. *The bi-parametric semigroups (integral) generated by the generalized translation operator are define by*

$$W_{\nu,t}^{(\beta)} f(x) = \left(w_\nu^{(\beta)}(\cdot;t) \otimes f \right) = \int\limits_{\mathbb{R}^n_{N,+}} w_\nu^{(\beta)}(|y|;t) T^y f(x) \left(y'\right)^{2\nu+1} dy. \tag{10}$$

It is not difficult to verify that this convolution-type integral satisfies the semigroup property by using the Fourier-Bessel transform:

$$W_{\nu,r+s}^{(\beta)} f = W_{\nu,r}^{(\beta)} W_{\nu,s}^{(\beta)}. \tag{11}$$

The following theorem presents some properties of the bi-parametric semigroups defined in (10).

Theorem 2.2. *Let* $f \in L_{p,\nu}\left(\mathbb{R}^n_{N,+}\right)$, $1 \le p \le \infty$, $\beta = 2k$, $k \in \mathbb{N}$ *or* $0 < \beta \le 2$. *Then*

a)

$$\left\| W_{\nu,t}^{(\beta)} f \right\|_{p,\nu} \le c(\beta) \|f\|_{p,\nu} \tag{12}$$

where $c(\beta) = \int\limits_{\mathbb{R}^n_{N,+}} \left| w_\nu^{(\beta)}(x,1) \right| (x')^{2\nu+1} dx.$

b)

$$\lim_{t\to 0^+} \left(W_{\nu,t}^{(\beta)} f \right)(x) = f(x)$$

where the limit is understood in the L_p*-norm or pointwise for almost all* $x \in \mathbb{R}^n_{N,+}$ *. In case of* $f \in C_0$, *the convergence uniform.*

c)

$$\sup_{t>0} \left| \left(W_{\nu,t}^{(\beta)} f \right)(x) \right| \le c (M_\nu f)(x) \tag{13}$$

where $M_\nu f$ *is the modified Hardy-Littlewood maximal operator*

$$(M_\nu f)(x) = \sup_{r>0} \frac{1}{r^{n+N+2|\nu|}\omega(n,\nu,N)} \int\limits_{E(0,r)} T^y f(x) \left(x'\right)^{2\nu+1} dx$$

which is strong-$(L_{p,\nu}, L_{p,\nu})$, $(1 < p \leq \infty)$ *and weak-*$(L_{1,\nu}, L_{1,\nu})$, *(see* [10]*).*

d)

$$\sup_{x\in\mathbb{R}^n_{N,+}} \left|\left(W^{(\beta)}_{\nu,t} f\right)(x)\right| \leq t^{-\frac{n+N+2|\nu|}{p\beta}} c \left\|f\right\|_{p,\nu}, \ 1 \leq p < \infty.$$

Proof. **a)** By using generalized Minkowski inequality, and taking into account (2) we have

$$\begin{aligned} \left\|W^{(\beta)}_{\nu,t} f\right\|_{p,\nu} &\leq \int\limits_{\mathbb{R}^n_{N,+}} \left|w^{(\beta)}_{\nu}(|y|\,;t)\right| \left(\int\limits_{\mathbb{R}^n_{N,+}} |T^y f(x)|^p (x')^{2\nu+1} dx\right)^{\frac{1}{p}} (y')^{2\nu+1} dy \\ &= \sup_{y\in R^n_{N,+}} \|T^y f\|_{p,\nu} \int\limits_{\mathbb{R}^n_{N,+}} \left|w^{(\beta)}_{\nu}(|y|\,;t)\right| (y')^{2\nu+1} dy, \ (\text{set} \ \ y = t^{\frac{1}{\beta}} z, \ dy = t^{\frac{1}{\beta}} dz) \\ &= \sup_{y\in R^n_{N,+}} \|T^y f\|_{p,\nu} t^{-\frac{n+N+2|\nu|}{\beta}} t^{\frac{n}{\beta}} t^{\frac{2|\nu|+N}{\beta}} \int\limits_{\mathbb{R}^n_{N,+}} \left|w^{(\beta)}_{\nu}(|z|\,;1)\right| (z')^{2\nu+1} dz \\ &\leq c(\beta) \|f\|_{p,\nu}. \end{aligned}$$

b) By applying the generalized Minkowski inequality and using the equality (8), we obtain for $f \in L_{p,\nu}$, $1 \leq p \leq \infty$ that

$$\begin{aligned} \left\|W^{(\beta)}_{\nu,t} f - f\right\|_{p,\nu} &\leq \int\limits_{\mathbb{R}^n_{N,+}} \left|w^{(\beta)}_{\nu}(|y|\,,t)\right| \|T^y f - f\|_{p,\nu} (y')^{2\nu+1} dy \\ &= \int\limits_{\mathbb{R}^n_{N,+}} \left|w^{(\beta)}_{\nu}(|z|\,,1)\right| \left\|T^{t^{-1/\beta} z} f - f\right\|_{p,\nu} (z')^{2\nu+1} dy. \end{aligned}$$

Now, by taking into account (2) we have $\left\|T^{t^{-1/\beta} z} f - f\right\|_{p,\nu} \leq 2\|f\|_{p,\nu}$ and $\lim\limits_{\alpha\to 0^+} \left\|T^{t^{-1/\beta} z} f - f\right\|_{p,\nu} = 0$, $(1 \leq p \leq \infty)$ ([15]). Then, Lebesgue-dominanted convergence theorem yields

$$\lim_{\alpha\to 0^+} \left\|W^{(\beta)}_{\nu,t} f - f\right\|_{p,\nu} = 0, \ 1 \leq p \leq \infty.$$

Here $L_{\infty,\nu} \equiv C_0$ and in this case convergence is uniform.

c) In the article by Aliev and Bayrakci [1], utilizing Theorem 2.1, if $\varphi \in L_{1,\nu}$ has a decreasing, positive, and radial majorant $\psi(|x|)$ that satisfies

$$\int\limits_{\mathbb{R}^n_{N,+}} \psi(|x|)(x')^{2\nu+1} dx < \infty,$$

then for every $f \in L_{p,\nu}\left(\mathbb{R}^n_{N,+}\right)$, $(1 \leq p \leq \infty)$ and $\varphi_\varepsilon(x) = \varepsilon^{-(n+N+2|\nu|)}\varphi(\frac{x}{\varepsilon})$ we obtain

$$\sup_{\varepsilon>0} |(\varphi_\varepsilon \otimes f)(x)| \leq \|\psi\|_{1,\nu} (M_\nu f)(x). \tag{14}$$

By setting $\psi(|x|) = w^{(\beta)}_{\nu}(|x|;1)$, $\varepsilon = t^{1/\beta}$ in the last equation and taking into account the equations (7), (14) we derive

$$\sup_{t>0} \left|\left(W^{(\beta)}_{\nu,t} f\right)(x)\right| \leq c (M_\nu f)(x)$$

where

$$c = \int\limits_{\mathbb{R}^n_{N,+}} \left| w_\nu^{(\beta)}(|x|;1) \right| (x')^{2\nu+1} dx < \infty.$$

d) By using the Hölder inequality, we obtain

$$\begin{aligned} \sup_{x\in\mathbb{R}^n_{N,+}} \left| \left(W_{\nu,t}^{(\beta)} f \right)(x) \right| &= \left\| w_\nu^{(\beta)}(\cdot;t) \otimes f \right\|_{\infty,\nu} \le \|f\|_{p,\nu} \left\| w_\nu^{(\beta)}(|x|;t) \right\|_{q,\nu} \;;\; \frac{1}{p}+\frac{1}{q}=1 \\ &\overset{(7)}{=} t^{-\frac{n+N+2|\nu|}{p\beta}} \|f\|_{p,\nu} \left\| w_\nu^{(\beta)}(|\cdot|;1) \right\|_{q,\nu} = c\, t^{-\frac{n+N+2|\nu|}{p\beta}} \|f\|_{p,\nu} . \end{aligned}$$

□

3. Main Definitions and Theorems

The main definitions and corresponding results are presented in this section

Definition 3.1. *The bi-parametric potentials generated by the generalized translation operator associated with Laplace-Bessel differential operator Δ_B are defined by*

$$\mathcal{B}_{\nu,\beta}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha/\beta)} \int\limits_0^\infty t^{\alpha/\beta} e^{-t} W_{\nu,t}^{(\beta)} f(x) \frac{dt}{t} \tag{15}$$

where the operators $\left\{ W_{\nu,t}^{(\beta)} f \right\}_{t\ge 0}$ are bi-parametric semigroups, defined in (10).

Bi-parametric potentials $\mathcal{B}_{\nu,\beta}^{\alpha}$ are interpreted as the fractional powers of order $(-\alpha/\beta)$ of the fractional differential operator $\left(I + (-\Delta_B)^{\beta/2} \right)$, i.e. formally,

$$\mathcal{B}_{\nu,\beta}^{\alpha} f = \left(I + (-\Delta_B)^{\beta/2} \right)^{-\alpha/\beta} f, \;\; f \in S\left(\mathbb{R}^n_{N,+}\right).$$

Note that these potentials coincide with the Bessel potentials for $\beta = 1$ and the modified Bessel potentials $\beta = 2$ respectively, generated by the generalized translation operator, (see [2]). The following theorem gives some basic properties of the bi-parametric potentials $\mathcal{B}_{\nu,\beta}^{\alpha} f$ defined in (15).

Theorem 3.1. *Let $1 \le p \le \infty$ and $f \in L_{p,\nu}$, ($L_{\infty,\nu} \equiv C_0$), $0 < \alpha, \beta < \infty$. Then*
a)

$$\left\| \mathcal{B}_{\nu,\beta}^{\alpha} f \right\|_{p,\nu} \le c(\beta) \|f\|_{p,\nu}$$

where $c(\beta) = 1$ for $0 < \beta \le 2$.
b) *Bi-parametric potentials $\mathcal{B}_{\nu,\beta}^{\alpha}$ are an convolution-type operators. Namely,*

$$F_\nu \left(\mathcal{B}_{\nu,\beta}^{\alpha} f \right)(x) = \left(1 + |x|^\beta \right)^{-\alpha/\beta} F_\nu(f), \qquad f \in S\left(\mathbb{R}^n_{N,+}\right). \tag{16}$$

c) *The operator $\mathcal{B}_{\nu,\beta}^{\alpha}$ are an automorphism in $S\left(\mathbb{R}^n_{N,+}\right)$.*
d)*For a fixed $\beta > 0$, the family $\left\{ B_{\nu,\beta}^{\alpha} \right\}_{\alpha \ge 0}$ have the following semigroup property:*

$$\mathcal{B}_{\nu,\beta}^{\alpha_1+\alpha_2} = \mathcal{B}_{\nu,\beta}^{\alpha_1} \mathcal{B}_{\nu,\beta}^{\alpha_2} \;,$$

where $B_{\nu,\beta}^{0} = E$ is the identity operator and $f \in L_{p,\nu}, 1 \le p \le \infty$, $0 \le \alpha_1, \alpha_2 < \infty$.

Proof. **a)** By applying generalized Minkowski inequality and taking into account (12) we have

$$\left\|\mathcal{B}^{\alpha}_{\nu,\beta} f\right\|_{p,\nu} \le \left\|W^{(\beta)}_{\nu,t} f(x)\right\|_{p,\nu} \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\alpha/\beta-1} e^{-t} dt = \left\|W^{(\beta)}_{\nu,t} f(x)\right\|_{p,\nu} \le c(\beta) \|f\|_{p,\nu}.$$

b)By using Fubini's theorem for $f \in S\left(\mathbb{R}^n_{N,+}\right)$ we obtain

$$\begin{aligned} F_\nu\left(\mathcal{B}^{\alpha}_{\nu,\beta} f\right)(x) &= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\alpha/\beta-1} e^{-t} F_\nu\left(W^{(\beta)}_{\nu,t} f(y)\right)(x)\, dt \\ &= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\alpha/\beta-1} e^{-t} F_\nu\left(w^{(\beta)}_{\nu}(|y|;t) \otimes f(y)\right)(x)\, dt \\ &= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\alpha/\beta-1} e^{-t} e^{-t|x|^\beta} (F_\nu f)(x)\, dt \\ &= \left(1+|x|^\beta\right)^{-\alpha/\beta} (F_\nu f)(x). \end{aligned}$$

c) Since $F_\nu : S\left(\mathbb{R}^n_{N,+}\right) \to S\left(\mathbb{R}^n_{N,+}\right)$ is an automorphism, then the statement easily follow from (16).

d) The identity is obvious in Fourier-Bessel terms for functions $f \in S\left(\mathbb{R}^n_{N,+}\right)$. The general $L_{p,\nu}$-case is the consequence of the density of Schwartz space $S\left(\mathbb{R}^n_{N,+}\right)$. □

Lemma 3.1. *Let $1 \le p \le \infty$ and $f \in L_{p,\nu}, (L_{\infty,\nu} \equiv C_0)$, $0 < \alpha, \beta < \infty$. The operators $\mathcal{B}^{\alpha}_{\nu,\beta}$ and $W^{(\beta)}_{\nu,t}$ are commutative:*

$$\mathcal{B}^{\alpha}_{\nu,\beta} W^{(\beta)}_{\nu,t} f = W^{(\beta)}_{\nu,t} \mathcal{B}^{\alpha}_{\nu,\beta} f.$$

Proof. The equality $\mathcal{B}^{\alpha}_{\nu,\beta} W^{(\beta)}_{\nu,t} \varphi = W^{(\beta)}_{\nu,t} \mathcal{B}^{\alpha}_{\nu,\beta} \varphi$ is straightforward for $\varphi \in S\left(\mathbb{R}^n_{N,+}\right)$ and follows from using the Fourier-Bessel transform. The general case follows from the density of the class $S\left(\mathbb{R}^n_{N,+}\right)$ in $L_{p,\nu}$. □

We now define a wavelet-like transform generated by bi-parametric semigroups defined in (10). This transform will be used for inversion of the bi-parametric potentials. The wavelet-like transforms are a class of continuous wavelet transforms generated by two components, namely, a kernel function and a wavelet. Both are in our disposal. These transforms are known composite wavelet transform in literature and introduced by Aliev, Rubin, [3].

Definition 3.2. *Let μ be a wavelet measure on $[0,\infty)$, that is a finite Borel measure on $[0,\infty)$ and $\mu\{[0,\infty)\} = 0$. A wavelet transform generated by wavelet measure μ and bi-parametric semigroups is defined by*

$$\left(\mathcal{A}^{(\beta)}_{\mu} \varphi\right)(x,t) = \mu(\{0\})\, \varphi(x) + \int_0^\infty e^{-st} \left(W^{(\beta)}_{\nu,st} \varphi\right)(x)\, d\mu(s) \tag{17}$$

where $W_{\nu,t}^{(\beta)}\varphi$, *are the bi-parametric semigroups and* $x \in \mathbb{R}_{N,+}^{n}$, $0 < t < \infty$ *and*

$$\int_a^b (\cdots)\, d\mu(s) = \int_{[a,b)} (\cdots)\, d\mu(s).$$

It is easy to see that the wavelet transform $\mathcal{A}_{\mu}^{(\beta)}$ is well defined for $\varphi \in L_{p,\nu}, 1 \le p \le \infty$. That is, by the generalized Minkowski inequality we have

$$\left\|\mathcal{A}_{\mu}^{(\beta)}\varphi(\cdot,s)\right\|_{p,\nu} \le \int_0^\infty e^{-st}\left\|W_{\nu,st}^{(\beta)}\varphi\right\|_{p,\nu} d|\mu|(t) \le c(\beta)\|\mu\|\,\|\varphi\|_{p,\nu}$$

where $\|\mu\| = \int_0^\infty d|\mu|(t) < \infty$. The following Lemma is of great importance for us which is a special case of the Rubin Lemma in [17].

Lemma 3.2. *(cf. Lemma 1.3 from* [17]*) Let* μ *be a finite signed Borel measure on* $[0,\infty)$ *and*

$$K_\theta(s) = \frac{1}{s}\left(I^{\theta+1}\mu\right)(s), \tag{18}$$

where

$$\left(I^{\theta+1}\mu\right)(s) = \frac{1}{\Gamma(\theta+1)}\int_0^s (s-t)^\theta\, d\mu(t),\ (s>0,\ \theta>0)$$

is the Riemann-Liouville fractional integral of order $(\theta+1)$ *of the measure* μ*. Suppose that* μ *satisfies the following conditions:*

$$\int_1^\infty t^\gamma d|\mu|(t) < \infty \text{ for some } \gamma > \theta, \tag{19}$$

$$\int_0^\infty t^j d\mu(t) = 0;\ j = 0,1,2,3,\cdots,[\theta],\ \text{(the integral part } \theta\text{)}. \tag{20}$$

Then $K_\theta(s)$ *has decreasing integrable majorant and*

$$C_{\theta,\mu} \equiv \int_0^\infty K_\theta(s)\, ds = \left\{\begin{array}{ll} \Gamma(-\theta)\int_0^\infty z^\theta d\mu(z), & \text{if}\quad \theta \ne 1,2,3,\cdots \\ (-1)^{\theta+1}\frac{1}{\theta!}\int_0^\infty z^\theta \ln z\, d\mu(z), & \text{if}\ \ \theta = 1,2,3,\cdots \end{array}\right\}. \tag{21}$$

In addition, if $\widetilde{\mu} = \int_0^\infty e^{-tz} d\mu(z)$ *is the Laplace transform of* μ*, then*

$$C_{\theta,\mu} \equiv \int_0^\infty t^{-1-\theta}\widetilde{\mu}(t)\, dt. \tag{22}$$

Remark 3.1. *In particular case, when* $0 < \theta < 1$*, the conditions (19), (20) and (21) have the following simple form respectively:*

$$\int_1^\infty t^\gamma d|\mu|(t) < \infty\ ;\int_0^\infty d\mu(t) = 0;\quad C_{\theta,\mu} = \int_0^\infty K_\theta(s)\, ds = \Gamma(-\theta)\int_0^\infty s^\theta d\mu(s).$$

Lemma 3.3. *(see* [9], *No:3.238(3)) Let* $\gamma > 1$, $0 < \alpha, \beta < \infty$. *Then*

$$\int_1^\gamma t^{-\alpha/\beta-1} (\gamma - t)^{\alpha/\beta-1} dt = \frac{\Gamma(\alpha/\beta)}{\Gamma(1+\alpha/\beta)} \frac{1}{\gamma} (\gamma-1)^{\alpha/\beta}.$$

The main result of the paper is the following theorem, where the inversion formula for the bi-parametric potentials $\mathcal{B}^\alpha_{\nu,\beta}$ generated by the generalized translation operator are obtained by using the wavelet transform $\mathcal{A}^\beta_\mu$ defined as in (17). It should be noted that the proof of the theorem is based on general technique developed by Aliev and Rubin [3].

Theorem 3.2. *Let* $\mathcal{A}^\beta_\mu$, $\beta > 0$ *be the wavelet transform and* $\mathcal{B}^\alpha_{\nu,\beta}$, $\alpha > 0$ *bi-parametric potentials of the function* $f \in L_{p,\nu}\left(\mathbb{R}^n_{N,+}\right)$, $(1 \le p \le \infty)$. *Suppose that* μ *is a finite Borel measure on* $[0,\infty)$ *satisfying the conditions* (19) *and* (20). *Then*

$$\int_0^\infty t^{-\alpha/\beta} \left(\mathcal{A}^{(\beta)}_\mu \mathcal{B}^\alpha_{\nu,\beta} f\right)(x,t) \frac{dt}{t} \equiv \lim_{h\to 0} \int_h^\infty t^{-\alpha/\beta} (\mathcal{A}^{(\beta)}_\mu \mathcal{B}^\alpha_{\nu,\beta} f)(x,t) \frac{dt}{t} = Cf \qquad (23)$$

where $C \equiv C_{\frac{\alpha}{\beta},\mu}$ *is defined as* (21)-(22). *The limit is to be understood in the* $L_{p,\nu}$, $(1 \le p < \infty)$ *norm or pointwise a.e. on* $\mathbb{R}^n_{N,+}$. *If* $f \in C_0$, *then the convergence is uniform.*

Proof. Let $f \in L_{p,\nu}$. By using Lemma 3.1 we have

$$\begin{aligned}
\left(\mathcal{A}^{(\beta)}_\mu \mathcal{B}^\alpha_{\nu,\beta} f\right)(x,t) &= \int_0^\infty e^{-st} W^{(\beta)}_{\nu,st} \mathcal{B}^\alpha_{\nu,\beta} f(x)\, d\mu(s) = \int_0^\infty e^{-st} \mathcal{B}^\alpha_{\nu,\beta} W^{(\beta)}_{\nu,st} f(x)\, d\mu(s) \\
&\overset{(15)}{=} \int_0^\infty e^{-st} \left(\frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty h^{\alpha/\beta} e^{-h} W^{(\beta)}_{\nu,h} W^{(\beta)}_{\nu,st} f(x) \frac{dh}{h}\right) d\mu(s) \\
&\overset{(11)}{=} \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty e^{-st} \left(\int_0^\infty h^{\alpha/\beta-1} e^{-h} W^{(\beta)}_{\nu,h+st} f(x)\, dh\right) d\mu(s).
\end{aligned}$$

By substituting h with $h - st$ in the last equation, we get

$$\left(\mathcal{A}^{(\beta)}_\mu \mathcal{B}^\alpha_{\nu,\beta} f\right)(x,t) =$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty e^{-st} \left(\int_{st}^\infty (h-st)^{\alpha/\beta-1} e^{-h+st} W^{(\beta)}_{\nu,h-st+st} f(x)\, dh\right) d\mu(s) \\
&= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty \left(\int_0^\infty (h-st)_+^{\alpha/\beta-1} e^{-h} W^{(\beta)}_{\nu,h} f(x)\, dh\right) d\mu(s)
\end{aligned}$$

where

$$(h-st)_+^{\alpha/\beta-1} = \left\{ \begin{array}{ccc} (h-st)^{\alpha/\beta-1} & , & h-st > 0 \\ 0 & , & h-st \le 0 \end{array} \right\}. \qquad (24)$$

Now, considering Fubini's theorem, the definition in (24) for a given $\delta > 0$, and then taking into account Lemma 3.3 we have

$$\int_{\delta}^{\infty} t^{-\alpha/\beta-1} \left(\mathcal{A}_{\mu}^{(\beta)} \mathcal{B}_{\nu,\beta}^{\alpha} f\right)(x,t)\, dt$$

$$= \frac{1}{\Gamma(\alpha/\beta)} \int_0^{\infty} \left(\int_0^{\infty} e^{-h} W_{\nu,h}^{(\beta)} f(x) \left(\int_{\delta}^{\infty} t^{-\alpha/\beta-1} (h-st)_+^{\alpha/\beta-1} dt \right) dh \right) d\mu(s)$$

$$= \frac{1}{\Gamma(\alpha/\beta)} \int_0^{\infty} e^{-h} W_{\nu,h}^{(\beta)} f(x) \left(\int_0^{\infty} s^{\alpha/\beta-1} \left(\int_{\delta}^{\infty} t^{-\alpha/\beta-1} \left(\frac{h}{s}-t\right)_+^{\alpha/\beta-1} dt \right) d\mu(s) \right) dh$$

$$= \frac{1}{\Gamma(\alpha/\beta)} \int_0^{\infty} e^{-h} W_{\nu,h}^{(\beta)} f(x) \left(\int_0^{\frac{h}{\delta}} s^{\alpha/\beta-1} \left(\int_{\delta}^{\frac{h}{s}} t^{-\frac{\alpha}{\beta}-1} \left(\frac{h}{s}-t\right)^{\alpha/\beta-1} dt \right) d\mu(s) \right) dh$$

$$= \frac{1}{\Gamma(\alpha/\beta)} \int_0^{\infty} e^{-\delta h} W_{\nu,\delta h}^{(\beta)} f(x) \left(\int_0^{h} s^{\alpha/\beta-1} \left(\int_{1}^{\frac{h}{s}} t^{-\frac{\alpha}{\beta}-1} \left(\frac{h}{s}-t\right)^{\alpha/\beta-1} dt \right) d\mu(s) \right) dh$$

$$= \frac{1}{\Gamma(\alpha/\beta)} \int_0^{\infty} e^{-\delta h} W_{\nu,\delta h}^{(\beta)} f(x) \left(\int_0^{h} s^{\alpha/\beta-1} \frac{\Gamma(\alpha/\beta)}{\Gamma(\alpha/\beta+1)} \frac{s}{h} \left(\frac{h}{s}-1\right)^{\alpha/\beta} d\mu(s) \right) dh$$

$$= \frac{1}{\Gamma(\alpha/\beta+1)} \int_0^{\infty} e^{-\delta h} W_{\nu,\delta h}^{(\beta)} f(x) \left(\int_0^{h} \frac{1}{h} (h-s)^{\alpha/\beta} d\mu(s) \right) dh$$

$$= \int_0^{\infty} e^{-\delta h} W_{\nu,\delta h}^{(\beta)} f(x)\, K_{\alpha/\beta}(h)\, dh \tag{25}$$

where $K_{\alpha/\beta}(h) = \frac{1}{h} \frac{1}{\Gamma(\alpha/\beta+1)} \int_0^h (h-s)^{\frac{\alpha}{\beta}} d\mu(s)$ from (18). We will continue the technique of the approximation to the identity. Namely, taking into account the notation $C \equiv C_{\alpha/\beta,\mu} = \int_0^{\infty} K_{\alpha/\beta}(h)\, dh$ (see (21), (22)) we get

$$\int_{\delta}^{\infty} t^{-\alpha/\beta-1} \left(\mathcal{A}_{\mu}^{(\beta)} \mathcal{B}_{\nu,\beta}^{\alpha} f\right)(x,t)\, dt - Cf(x) = \int_0^{\infty} \left(e^{-\delta h} W_{\nu,\delta h}^{(\beta)} f(x) - f(x) \right) K_{\alpha/\beta}(h)\, dh$$

$$= \int_0^{\infty} e^{-\delta h} \left(W_{\nu,\delta h}^{(\beta)} f(x) - f(x) \right) K_{\alpha/\beta}(h)\, dh + f(x) \int_0^{\infty} \left(1 - e^{-\delta h}\right) K_{\alpha/\beta}(h)\, dh.$$

By using the generalized Minkowski inequality, we obtain for $1 \leq p \leq \infty$

$$\left\| \int_{\delta}^{\infty} t^{-\alpha/\beta-1} \left(\mathcal{A}_{\mu}^{(\beta)} \mathcal{B}_{\nu,\beta}^{\alpha} f \right) (\cdot, t)\, dt - Cf \right\|_{p,\nu}$$
$$\leq \int_{0}^{\infty} e^{-\delta h} \left\| W_{\nu,\delta h}^{(\beta)} f - f \right\|_{p,\nu} \left| K_{\alpha/\beta}(h) \right| dh + \|f\|_{p,\nu} \int_{0}^{\infty} \left(1 - e^{-\delta h}\right) \left| K_{\alpha/\beta}(h) \right| dh.$$

Finally, the Lebesgue-domineted convergence theorem yields that

$$\lim_{\delta \to 0} \left\| \int_{\delta}^{\infty} t^{-\alpha/\beta-1} \left(\mathcal{A}_{\mu}^{(\beta)} \mathcal{B}_{\nu,\beta}^{\alpha} \right) (\cdot, t)\, dt - Cf \right\|_{p,\nu} = 0 \tag{26}$$

where for $L_{\infty,\nu} \equiv C_0$ the convergence is uniform.

Now let us prove the pointwise (a.e.) convergence in (23). From the following inequalities

$$\sup_{\delta>0} \left| \int_{\delta}^{\infty} t^{-\alpha/\beta-1} \left(\mathcal{A}_{\mu}^{(\beta)} \mathcal{B}_{\nu,\beta}^{\alpha} \right) (x, t)\, dt \right| \overset{(25)}{=} \sup_{\delta>0} \left| \int_{0}^{\infty} e^{-\delta h} W_{\nu,\delta h}^{(\beta)} f(x)\, K_{\alpha/\beta}(h)\, dh \right|$$
$$\leq \sup_{t>0} \left| W_{\nu,t}^{(\beta)} f(x) \right| \int_{0}^{\infty} \left| K_{\alpha/\beta}(h) \right| dh \overset{(13)}{\leq} c\, (M_{\nu} f)(x)$$

it follows that the maximal operator

$$\sup_{\delta>0} \left| \int_{\delta}^{\infty} t^{-\alpha/\beta-1} \left(\mathcal{A}_{\mu}^{(\beta)} \mathcal{B}_{\nu,\beta}^{\alpha} \right) (x, t)\, dt \right|, \left(x \in \mathbb{R}_{N,+}^{n} \right)$$

is weak-$(L_{1,\nu}, L_{1,\nu})$ and strong-$(L_{p,\nu}, L_{p,\nu})$, $1 \leq p \leq \infty$. Since the convergence in (26) is pointwise (in fact uniformly) for any $f \in C_0 \cap L_{p,\nu}$, $(1 < p \leq \infty)$ and this class is dense in $L_{p,\nu}$, $(1 \leq p \leq \infty)$, it follows that

$$\lim_{\delta \to 0} \int_{\delta}^{\infty} t^{-\alpha/\beta-1} \left(\mathcal{A}_{\mu}^{(\beta)} \mathcal{B}_{\nu,\beta}^{\alpha} f \right) (x, t) = Cf(x)$$

pointwise for a.e. $x \in \mathbb{R}_{N,+}^{n}$, (see [20], p.60). □

Declaration of competing interest. All authors of this article declare no competing interest.
Funding. No financial support has been used in this study.
Data availability. No data was used for the research described in the article.

Acknowledgment

We would like to express our deep thanks to the referees for carefully reading the manuscript and giving kind comments and useful suggestions.

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