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ON THE SECURE EQUITABLE DOMINATION IN GRAPHS

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Abstract. A secure equitable dominating set S of a graph G is a dominating set in which for any vertex $v \in V(G) \setminus S$ there exists at least one vertex $u \in S$ such that $u \in N_e(v)$, where $N_e(v)$ indicate the equitable neighbourhood of v, and if we swap the vertex u with v, the equitable domination property of the graph will be unharmed. $\gamma^e_{sec}(G)$ represents the secure equitable domination number of G, which is the cardinality of the minimum secure equitable dominating set in G. The improved bounds of the secure equitable domination number of some fundamental kinds of graphs are established in this study. Furthermore, we incorporate specific results based on the diameter, girth, and degree. Additionally, we determine the bounds of the secure equitable domination number of specific special classes of graphs.

Keywords: Secure equitable domination, secure equitable domination number, wheel graphs,double-wheel graph, helm graph, flower graph, sunflower graph.

AMS Subject Classification: 05C05, 05C69, 05C76

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The number of edges incident on a vertex v is called the *degree* of v in G and is denoted by $\deg_G(v)$. The difference between two sets, U and V is denoted by $U \setminus V$. The *neighbourhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the collection of all immediate neighbour vertices of v in G. The *equitable neighbourhood* $N_e(v)$ of a vertex v is the collection of all immediate neighbour vertices of v whose degrees differ by at most 1. The length of the shortest cycle in a graph G is called the *girth* of G and is denoted by girth(G). The length of the longest path in a graph G is called its *diameter*, denoted by diam(G). A *wheel graph* $W_{1,n}$ is formed by joining every vertex of a cycle graph C_n to a common vertex using an edge. The vertices of the cycle constitute the perimeter of $W_{1,n}$. A *gear graph* G_n on $n+1$ vertices is obtained by subdividing every edge on the perimeter of the wheel graph and placing a vertex at each subdivision. A *helm graph* H_n is a wheel graph adjoined with a leaf at each vertex of its perimeter. A *closed helm graph* CH_n is obtained from a helm graph by adding an edge between each pair of consecutive leaves. A *flower graph*

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Fl_n is formed by joining each leaf of a helm graph to the central vertex of its wheel. A *sunflower graph* Sf_n is derived from a wheel graph in which every incident vertices of the edges on the perimeter of the wheel graph are connected to a common vertex using an edge. A *double wheel graph* DW_n is formed by joining every vertex of the two concentric cycle graphs to a common vertex using an edge. For every basic definitions and concepts on graph theory, we refer to [4].

A set $D \subseteq V(G)$ is called a *dominating set* if for each $v \in V(G) \setminus D$, $N(v) \cap D \neq \phi$. The *domination number* $\gamma(G)$ of G is the minimum number of vertices that can dominate the graph G[5]. A dominating set D in which $|\deg_G(u) - \deg_G(v)| \leq 1$ for every $u \in V(G)$, $v \in V(G) \setminus D$ and $uv \in E(G)$ is called an *equitable dominating set.* The minimum cardinality of an equitable dominating set is called an *equitable domination number* and is denoted by $\gamma^e(G)$[8]. A set S is said to be a *secure dominating set* of a graph G if for each vertex $v \in S$ there exists $u \in V(G) \setminus S$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of a secure dominating set is called the *secure domination number* and is represented by $\gamma_s(G)$[2]. The secure and equitable domination concepts prompted the revelation of a new parameter called secure equitable domination number in 2022. We have the following definition for a secure equitable domination number of a graph: An equitable dominating set D of a simple graph G is said to be a *secure equitable dominating set* (se-dominating set) if for every $u \in V(G) \setminus D$ there exists at least one vertex $v \in D$ such that u and v are adjacent in G and $(D \setminus \{v\}) \cup \{u\}$ is an equitable dominating set of G. The minimum number of vertices that are needed to construct a secure equitable dominating set of a graph G is called the *secure equitable domination number* (se-domination number) of G, abbreviated by $\gamma^e_{sec}(G)$[7]. The secure equitable domination in a helm graph H_5 on 11 vertices is illustrated by Fig 1.

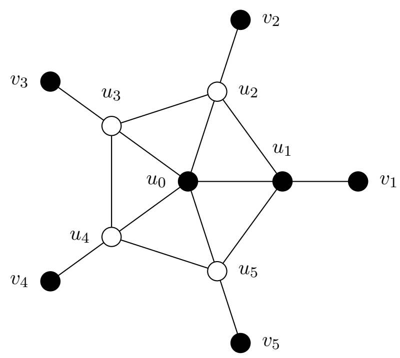


FIGURE 1. $\gamma^e_{sec}(H_5) = 7$.

We can observe that every vertex v_i, where $1 \leq i \leq 5$ belongs to the secure equitable dominating set of H_5, since $|\deg_{H_5}(v_i) - \deg_{H_5}(u_i)| > 1$ for every $1 \leq i \leq 5$. Further, $|\deg_{H_5}(u_0) - \deg_{H_5}(u_i)| \leq 1$ for every $1 \leq i \leq 5$. Hence, every u_i, where $1 \leq i \leq 5$ belongs to the equitable neighbourhood of u_0. If we swap the vertex u_2 with u_1, then the vertex $u_3 \in N_e(u_0)$ and hence, it is protected. Similarly, if we swap u_2 with u_3 also, the vertex $u_1 \in N_e(u_0)$ and is protected. The remaining vertices u_4 and u_5 can be swapped with u_0. Thus, the set $S = \{u_0, u_2, v_1, v_2, v_3, v_4, v_5\}$ is a se-dominating set of H_5. Now, $S \setminus \{v_i\}$ where $1 \leq i \leq 5$ is not a se-dominating set, since $N_e(v_i) \cap V(H_5) = \phi$ for every $1 \leq i \leq 5$. Assume we remove the vertex u_2 from S. Then, the vertices u_1 and u_2 will not be protected as the vertex u_4 can be only swapped with u_0. Also, the set $S \setminus \{u_0\}$ does not satisfy the se-domination property, since $N_e(u_4) \cap S = \phi$. Hence, $\gamma^e_{sec}(H_5) = 7$.

In this paper, we establish the refined bounds of the se-domination number of some basic graph classes. In addition, we include certain results of the parameter grounded on the diameter, girth and degree. Moreover, we find the bounds of the se-domination number of certain special classes of graphs. We say a vertex u protects another vertex v means that the vertex v is secure equitably dominated by the vertex u. That is, $v \in N_e(u)$ and the swapping (interchanging) of vertex u with v is possible. We use the abbreviations γ^s-set, γ^e-set and γ^{se}-set for a secure dominating set, an equitable dominating set and a secure equitable dominating set, respectively. For every concepts on domination in graphs, we refer to [5].

We use the following Theorems and Propositions to prove the results in this paper.

Theorem 1.1. [1] *For any regular or $(k, k+1)$ biregular connected graph G, $\gamma^e_{sec}(G) = \gamma_s(G)$.*

Theorem 1.2. [2] *The secure domination number for any path P_n and any cycle C_n is $\lceil \frac{3n}{7} \rceil$.*

Theorem 1.3. [6] *For $n \geq 2$,*

$$\gamma_s(P_2 \Box C_n) = \begin{cases} \frac{3n}{4} & if\, n \equiv 0 (\text{mod } 8), \\ \lceil \frac{3n+1}{4} \rceil & \text{otherwise.} \end{cases}$$

Theorem 1.4. [7] *For any path P_n,*

$$\gamma^e_{sec}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & if\, n \neq 7, \\ 3 & if\, n = 7. \end{cases}$$

Theorem 1.5. [7] *For any cycle C_n,*

$$\gamma^e_{sec}(C_n) = \begin{cases} \frac{n}{3} & if\, n \equiv 0 (\text{mod } 3), \\ \lceil \frac{n}{3} \rceil + 1 & if\, n \equiv 1, 2 (\text{mod } 3) \text{ and } n \neq 4, 7, \\ \lceil \frac{n}{3} \rceil & if\, n = 4, 7. \end{cases}$$

Theorem 1.6. [7] *For any wheel graph $W_{1,n}$ on n vertices where $n \geq 4$,*

$$\gamma^e_{sec}(W_{1,n-1}) = \begin{cases} \frac{n}{3} + 1 & if\, n+1 \equiv 1 (\text{mod } 3), \\ \lceil \frac{n}{3} \rceil + 1 & if\, n+1 \equiv 0, 2 (\text{mod } 3). \end{cases}$$

Proposition 1.1. [3] *The only 3-self-centered graph on six vertices is C_6.*

Proposition 1.2. [3] *Let G be a 3-self-centered graph with girth seven and $\delta(G) = 2$. Then $G \cong C_7$.*

Theorem 1.7. [3] *Let G be a 3-self-centered graph and* girth$(G) = 7$. *Then, the graph G is regular.*

2. Secure equitable domination in graphs

At the beginning of this section, we introduce the refined bounds for the se-domination number of certain basic class graphs.

Let $V(P_4) = \{u_1, u_2, u_3, u_4\}$ and $S = \{u_2, u_4\}$. We can observe that $|\deg_{P_4}(u) - \deg_{P_4}(v)| \leq 1$ for any $u \in V(P_4)$ and $v \in S$. Also, the vertex u_2 can be swapped with u_1 and the vertex u_4 can be swapped with u_3, respectively. Therefore, S is a se-dominating set of P_4. If $S = \{u_i\}$, where $1 \leq i \leq 4$, then S is not a se-dominating set since a single vertex is insufficient to defend every other vertex of P_4. Thus, $S = \{u_2, u_4\}$ is a γ^{se}-set of P_4. Since $|S| = 2$, $\gamma^e_{sec}(P_4) = 2$. However, by Theorem 1.4, $\gamma^e_{sec}(P_4) = \lceil \frac{4}{3} \rceil + 1 = 2 + 1 = 3 > 2$. The secure equitable domination of P_4 is given in Fig 2.

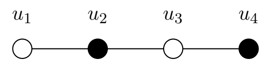


FIGURE 2. $\gamma^e_{sec}(P_4) = 2$.

Therefore, we obtained a tight bound for the secure equitable domination number of paths as follows:

Theorem 2.1. *The secure equitable domination number of a path P_n is $\lceil \frac{3n}{7} \rceil$.*

Proof. Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Since $|\deg_{P_n}(u_i) - \deg_{P_n}(u_j)| \leq 1$, where $1 \leq i, j \leq n$, $\gamma^e_{sec}(P_n) = \gamma_s(P_n)$ by Theorem 1.1. Then, by Theorem 1.2 we have $\gamma^e_{sec}(P_n)= \min\{|S| : S$ is a secure dominating set of $P_n\}= \lceil \frac{3n}{7} \rceil \leq \lceil \frac{n}{3} \rceil + 1$. Hence, the result is true. □

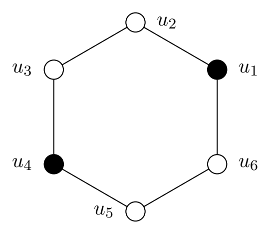


FIGURE 3. $\gamma^e_{sec}(C_6) \neq 2$

Let $V(C_6) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and let $S = \{u_1, u_4\}$. In Fig 3, we can observe that the vertices $u_2, u_6 \in N_e(u_1)$ and the vertices $u_3, u_5 \in N_e(u_4)$. But, if we swap u_1 with u_2, then the vertex u_6 will not be defended by any vertex of S. Thus, two vertices are insufficient to protect the vertices of C_6 secure equitably. At the same time, $\gamma^e_{sec}(C_6) = \frac{6}{3} = 2$ by Theorem 1.5. Now, suppose $S' = \{u_1, u_3, u_5\}$. Then, every vertex $u \in V(C_6) \setminus S'$ belong to the equitable neighbourhood of a vertex of S', and the safe swapping of every vertex of $V(C_6) \setminus S$ with a vertex of S is also possible. Hence, S' is a γ^{se}-set of C_6. Since $|S'| = 3$, $\gamma^e_{sec}(C_6) = 3$. The se-domination in C_6 can be seen in Fig 4.

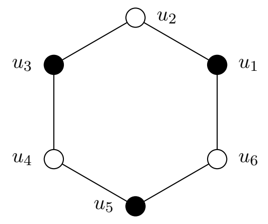


FIGURE 4. $\gamma^e_{sec}(C_6) = 3$

That being so, we obtained a tight bound for the se-domination number of cycles as follows:

Theorem 2.2. *The secure equitable domination number of a cycle C_n on the n vertices is $\lceil \frac{3n}{7} \rceil$, where $n \geq 4$.*

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. Since $|\deg_{C_n}(u) - \deg_{C_n}(v)| = 0$ for every $u, v \in V(C_n)$, $\gamma^e_{sec}(C_n) = \gamma_s(C_n)$ by Theorem 1.1. Then, by Theorem 1.2, at least $\lceil \frac{3n}{7} \rceil$ vertices are needed to protect the cycle C_n. Hence, $\gamma^e_{sec}(C_n)=$ min$\{|S|$: S is a secure dominating set of $C_n\}= \lceil \frac{3n}{7} \rceil$. Hence, the bound is true. □

Consider a wheel graph $W_{1,5}$ on 6 vertices. Let $V(W_{1,5}) = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ and $S = \{u_0,u_1,u_3,u_5\}$. Since $|\deg_{W_{1,5}}(u_0) - \deg_{W_{1,5}}(u_i)| > 1$ where $1 \leq i \leq 5$, $N_e(u_0) = \phi$. Thus, the vertex u_0 belongs to every se-dominating set of $W_{1,5}$. The remaining vertices of $W_{1,5}$ induce a cycle of order 5. Then, at least 3 vertices of the cycle must belong to the se-dominating set of $W_{1,5}$ by Theorem 2.2. Consequently, $S = \{u_0,u_1,u_3,u_5\}$ is a γ^{se}-set and $\gamma^e_{sec}(W_{1,5}) = |S| = 4$ (see Fig 5). But, $\gamma^e_{sec}(W_{1,5}) = \lceil \frac{5}{3} \rceil + 1 = 2 + 1 = 3 < 4$ by Theorem 1.6. However, it is clear that 3 vertices are insufficient to protect $W_{1,5}$. That being so, we obtained a tight bound for the se-domination number of wheel graphs and the following theorem carries the same.

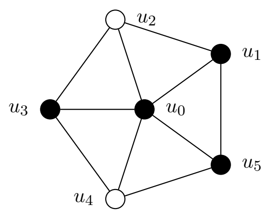


FIGURE 5. $\gamma^e_{sec}(W_{1,5}) = 4$.

Theorem 2.3. *For any wheel graph $W_{1,n}$,*

$$\gamma^e_{sec}(W_{1,n}) = \begin{cases} 1 & if\, n = 3, \\ 2 & if\, n = 4, \\ \lceil \frac{3n}{7} \rceil + 1 & \text{otherwise.} \end{cases} \quad (1)$$

Proof. Let $V(W_{1,n}) = \{u_0, u_1, \ldots, u_n\}$, where u_0 is the universal vertex of the wheel graph $W_{1,n}$ and $u_1, u_2, \ldots, u_n$ denote the vertices of its perimeter.

Case(i) Let $n = 3$. If we take any vertex $u \in V(W_{1,3})$, then $v \in N_e(u)$ for every $v \in V(W_{1,3})$. Let $S = \{u_1\}$. Subsequently, the vertices u_1,u_2,$u_3 \in N_e(u_1)$. The vertices u_0 and u_1 can be swapped as u_2,$u_3 \in N_e(u_0)$. Similarly, the vertices u_2 and u_3 can be swapped with u_1 without losing the domination property. Consequently, only one vertex is enough to protect $W_{1,3}$ and the safe swapping of the same vertex with every other vertex of $W_{1,3}$ is also possible. Hence, S is a γ^{se}-set of the wheel graph $W_{1,3}$ and $\gamma^e_{sec}(W_{1,3}) = 1$.

Case(ii) Let $n = 4$. The set $S = \{u_1\}$ is not a se-dominating set of $W_{1,4}$, since $N(u_3) \cap S = \phi$. So, we add the vertex u_3 to the set S. Therefore, $S = \{u_1, u_3\}$. Now, we can observe that for every vertex $u \in V(W_{1,4}) \setminus S$, $N_e(u) \cap S \neq \phi$. Also, any of the vertices u_2,u_4 and u_0 can be swapped with the vertex u_1 without losing the domination property. Thus, $S = \{u_1, u_3\}$ is a se-dominating set of $W_{1,4}$. Hence, $\gamma^e_{sec}(W_{1,4}) = \min\{|S| :$ S is a se $-$ dominating set of $W_{1,4}\} = 2$.

Case(iii) Let $n \geq 5$. Then, $|\deg_{W_{1,n}}(u_0) - \deg_{W_{1,n}}(u_i)| > 1$, where $1 \leq i \leq n$. Thus, $N_e(u_0) = \phi$. Consequently, u_0 belongs to every se-dominating set of $W_{1,n}$. The remaining vertices of $W_{1,n}$ induce a cycle of order n. Therefore, at least $\lceil \frac{3n}{7} \rceil$ vertices are needed to protect the vertices of $< u_i : 1 \leq i \leq n >$ by Theorem 2.2. Hence, $\gamma^e_{sec}(W_{1,n}) = \min\{|S| :$ S is a se − dominating set of $W_{1,n}\} = \lceil \frac{3n}{7} \rceil + 1$. Hence, the proof. □

Next, we include results concerning the se-domination number grounded on the diameter, girth and degree.

Theorem 2.4. *For any 3-self-centred graph G with* girth$(G) = 7$, $\gamma^e_{sec}(G) = \gamma_s(G)$.

Proof. Let G be a 3-self-centered graph of girth 7. Then, G is a regular graph by Theorem 1.7. That is, we have $|deg_G(u) - deg_G(v)| = 0$ for every $u, v \in V(G)$. Consequently, every secure dominating set is a se-dominating of G by Theorem 1.1. Hence, $\gamma^e_{sec}(G) = \gamma_s(G)$. □

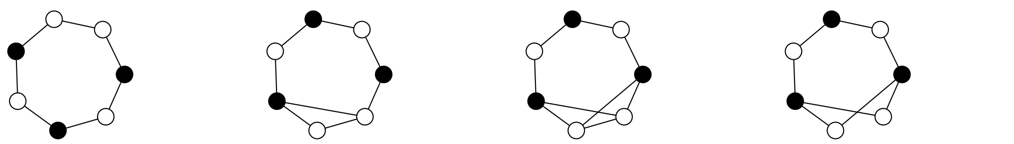
Corollary 2.1. *The secure equitable domination number of a 3-self-centred graph with 6 vertices is 3.*

Proof. By Proposition 1.1, the only 3-self-centered graph G with 6 vertices is a cycle of order 6. Then, $\gamma^e_{sec}(G) = 3$ by Theorem 2.2. □

Corollary 2.2. *For any 3-self-centred graph G with* girth$(G) = 7$ *and* $\delta(G) = 2$, $\gamma^e_{sec}(G) = 3$.

Proof. Let G be a 3-self-centered graph with a girth of seven and $\delta(G) = 2$. Then, by Proposition 1.2 we have $G \cong C_7$. Consequently, $\gamma^e_{sec}(G) = 3$ by Theorem 2.2. □

Remark 2.1. *The only 3-self-centered graphs on seven vertices are C_7, H_1, H_2 and H_3 (see Fig 6)*[3]*. Then, for any 3-self-centered graph G of order 7, $\gamma^e_{sec}(G) = 3$.*



(A) $\gamma^e_{sec}(C_7) = 3$. (B) $\gamma^e_{sec}(H_1) = 3$. (C) $\gamma^e_{sec}(H_2) = 3$. (D) $\gamma^e_{sec}(H_3) = 3$.

FIGURE 6

3. SE-DOMINATION IN SPECIAL CLASSES OF GRAPHS

In this section, we establish the bounds for the secure equitable domination number of certain wheel-related graph classes like gear graph, helm graph, closed helm graph, double wheel graph, sunflower graph and flower graph. We have the succeeding theorem for a universal vertex of a graph.

Theorem 3.1. *Every universal vertex v of a non-complete graph G with $N_e(v) = \phi$ belongs to the γ^{se}-set of G.*

Proof. Suppose G is a non-complete graph and $v \in V(G)$ is a universal vertex of G with $N_e(v) = \phi$. Then, no vertex in the neighbourhood can protect the vertex v, since $|\deg_G(v) - \deg_G(u)| \geq 1$ for every $u \in N(v)$. Thus, to make v secured, it has to be in the γ^{se}-set of G. Hence, the result. □

Proposition 3.1. *For any gear graph G_n,*

$$\gamma^e_{sec}(G_n) = \begin{cases} \frac{n}{2} & if \;\; n = 6, 8, \\ \lceil \frac{3n}{7} \rceil + 1 & \text{otherwise.} \end{cases} \tag{2}$$

Proof. Let $V(G_n) = \{u_0, u_1, \dots, u_n\}$. The vertices u_i, where $1 \le i \le n$ and $i \equiv 1(\text{mod}3)$, represent the vertices on the perimeter of a wheel graph which are connected to its universal vertex u_0 and the vertices u_j, where $1 \le j \le n$ and $j \equiv 0(\text{mod}3)$, denote those vertices added after edge sub-division of the wheel graph. Let S be a se-dominating set of G_n.

Case(i) Let $n = 6, 8$. Consider the gear graph G_6 on 7 vertices. We have $|\deg_{G_6}(u) - \deg_{G_6}(v)| \le 1$ for every $u, v \in V(G_6)$. By Theorem 2.2, at least 3 secure equitable dominating vertices must be there to protect a cycle of order 6. That being so, we define $S = \{u_1, u_3, u_5\}$. Suppose we swap the vertices u_1 and u_0, the vertices $u_2, u_4 \in N_e(u_3)$ and $u_6 \in N_e(u_5)$. If we swap a vertex $v \in V(G_n) \setminus S$, where $v \neq u_0$, with a vertex of S, then also the domination property holds. Thus, S is a γ^{se}-set of G_6. Thus , $\gamma^e_{sec}(G_6) = \min\{|S| : \text{S is a se} - \text{dominating set of } G_6\} = 3$. In the same way, we can define a se-dominating set $S = \{u_1, u_3, u_5, u_7\}$ of G_8 by Theorem 2.2. However, the vertices u_j, where $j \equiv 0(\text{mod}3)$, can be replaced with an adjacent u_i, where $i \equiv 1(\text{mod}3)$, and the vertex u_0 can be swapped with any u_i, where $i \equiv 1(\text{mod}3)$, without losing domination property.

Case(ii) suppose $n \ge 10$. We have $|\deg_{G_n}(u_0) - \deg_{G_n}(u_i)| > 1$ for every $1 \le i \le n$. Therefore, $N_e(u_0) \cap S = \phi$ and $u_0 \in S$. The remaining vertices of G_n induce a cycle of order n. Then, at least $\lceil \frac{3n}{7} \rceil$ vertices are needed to protect the graph G_n by Theorem 2.2. Hence, $\gamma^e_{sec}(G_n) = \min\{|S| : \text{S is a se} - \text{dominating set of } G_n\} = \lceil \frac{3n}{7} \rceil + 1$ for every $n \ge 10$. ☐

Proposition 3.2. *For any helm graph H_n on $2n + 1$ vertices,*

$$\gamma^e_{sec}(H_n) = \begin{cases} n+1 & if \;\; n = 3, \\ n+2 & if \;\; n = 4, 5, \\ \lceil \frac{3n}{7} \rceil + n + 1 & \text{otherwise.} \end{cases} \tag{3}$$

Proof. Let $V(H_n) = \{u_0, u_1, u_2 \dots, u_n, v_1, v_2, \dots, v_n\}$, where u_0 denotes the universal vertex and $u_1, u_2 \dots, u_n$ denote the vertices on the perimeter of the wheel of a helm graph H_n, $v_1, v_2, \dots, v_n$ denote the pendant vertices adjoined with the corresponding u_i, where $1 \le i \le n$. Since $|\deg_{H_n}(u_i) - \deg_{H_n}(v_i)| = 3$ for every $1 \le i \le n$, $N_e(v_i) = \phi$. Thus, $\{v_i : 1 \le i \le n\}$ is contained in every se-dominating set of H_n.

Case(i) Let $n = 3$. The vertex u_0 is enough to protect every u_i, where $1 \le i \le 3$, since every u_i belongs to the equitable neighbourhood of u_0 and any u_i, where $1 \le i \le 3$, can be swapped with u_0. Thus, $\gamma^e_{sec}(H_3) = 3 + 1 = 4$.

Case(ii) Suppose $n = 4, 5$. Subsequently, we have $|\deg_{H_n}(u_0) - \deg_{H_n}(u_i)| \le 1$. Hence, the vertex u_0 belongs to the equitable neighbourhood of any dominating vertex of the set $\{u_i : 1 \le i \le n, n = 4, 5\}$. Now, consider H_4. We define $S = \{u_0, u_1, v_1, v_2, v_3, v_4\}$. We have $|\deg_{H_4}(u_0) - \deg_{H_4}(u_i)| < 1$ for every $i = 2, 3, 4$. Moreover, the vertices u_2, u_3 and u_4 can be replaced with u_0 without altering the domination property. Thus, S is a se-dominating set of H_4. Further, $S \setminus \{u_0\}$ is not a se-dominating set, since $N_e(u_3) \cap S = \phi$. Therefore, at least two vertices of $\{u_i : 1 \le i \le 4\}$ (any two u_i can be selected) are required to construct a se-dominating set along with the vertices v_i, where $1 \le i \le 4$. Hence, $\gamma^e_{sec}(H_4) = \min\{|S| : \text{S is a se} - \text{dominating set of } H_4\} = n + 2$.

In the same way, we can prove that the set $S = \{u_0, u_1, v_1, v_2, \dots, v_n\}$ is a se-dominating set of H_5 and at least $n + 2$ vertices are needed to construct the same. Now, let $S = \{u_1, u_3, v_1, v_2, \dots, v_n\}$ is an se-dominating set of H_5. We have $|S| = n + 2$. However, if we swap the vertex u_2 with u_3, the vertex u_4 will not be protected as $N_e(v_4) \cap S = \phi$. Therefore, the vertex u_0 must be in every se-dominating set of H_5. Hence, the result is true.

case (iii) Suppose $n \geq 6$. Then, $|\deg_{H_n}(u_0) - \deg_{H_n}(u_i)| > 1$ and $|\deg_{H_n}(u_i) - \deg_{H_n}(v_i)| > 1$ for every $1 \leq i \leq n$. Consequently, every vertex v_i, where $1 \leq i \leq n$, and the vertex u_0 belong to every se-dominating set of H_n for $n \geq 6$. The remaining vertices of H_n induce a cycle of order n. Then, at least $\lceil \frac{3n}{7} \rceil$ vertices are required to protect every u_i, where $1 \leq i \leq n$. Thus, $\gamma^e_{sec}(H_n) = \min\{|S| : \text{S is a se} - \text{dominating set of } H_n\} = \lceil \frac{3n}{7} \rceil + n + 1$. □

Proposition 3.3. *For any double wheel graph DW_n,*

$$\gamma^e_{sec}(DW_n) = \begin{cases} n & if \;\; n = 3, \\ 2\lceil \frac{3n}{7} \rceil + 1 & \text{otherwise.} \end{cases} \tag{4}$$

Proof. Let $V(DW_n) = \{u_0, u_1, u_2 \ldots, u_n, v_1, v_2, \ldots, v_n\}$, where u_0 is the universal vertex of the double wheel graph DW_n, $u_1, u_2 \ldots, u_n$ denote the vertices of the inner cycle and $v_1, v_2, \ldots, v_n$ denote the vertices of the outer cycle of DW_n. The vertex u_0 belongs to every se-dominating set of DW_n by Theorem 3.1. Then, $< V(DW_n) \setminus \{u_0\} >$ consists of two disjoint cycle components of order n. Consequently, by Theorem 2.2, at least $\lceil \frac{3n}{7} \rceil$ vertices are sufficient to protect the vertices of each cycle. Hence, $\gamma^e_{sec}(DW_n) = \min\{|S| : \text{S is a se} - \text{dominating set of } DW_n\} = 2\lceil \frac{3n}{7} \rceil + 1$. □

Proposition 3.4. *For any closed helm graph CH_n on $2n + 1$ vertices,*

$$\gamma^e_{sec}(CH_n) = \begin{cases} 2 & if \;\; n = 3, \\ n & if \;\; n = 4, 5, \\ \frac{3n}{4} + 1 & if \;\; n \equiv 0(mod\ 8) \text{ and } n \geq 6, \\ \lceil \frac{3n+1}{4} \rceil + 1 & \text{otherwise.} \end{cases} \tag{5}$$

Proof. Let $V(CH_n) = \{u_0, u_1, u_2 \ldots, u_n, v_1, v_2, \ldots, v_n\}$, where u_0 denotes the central vertex connected to the vertices $u_1, u_2 \ldots u_n$ on the inner cycle of CH_n and $v_1, v_2 \ldots v_n$ denote the corresponding vertices on the outer cycle.

Case(i) Let $n = 3$. Evidently, $\{u_1, v_1\}$ is a γ^{se}-set of CH_3 and $\gamma^e_{sec}(CH_3) = 2$.

Case(ii) Suppose $n = 4, 5$. We have $|\deg_{CH_n}(u_0) - \deg_{CH_n}(u_i)| \leq 1$ and $|\deg_{CH_n}(u_i) - \deg_{CH_n}(v_j)| = 1$ for every $1 \leq i, j \leq n$ and $u_i v_j \in E(CH_n)$. If we define $S = \{u_1, u_3, v_1, v_3\}$ for $n = 4$ and $S = \{u_1, u_3, v_2, v_4, v_5\}$ for $n = 5$, then S is clearly a γ^{se}-set of CH_n and the result follows.

Case(iii) Assume $n \geq 6$. Then, $|\deg_{CH_n}(u_0) - \deg_{CH_n}(u)| > 1$ for every $u \in V(CH_n)$ and $u \neq u_0$. Consequently, u_0 belongs to every se-dominating set since it can not be protected by any dominating vertex on the cycles of CH_n. The remaining dominating vertices in S are contributed by the vertices on the two cycles (inner and outer) of CH_n. Moreover, the graph induced by the vertices on the cycles of CH_n is isomorphic to the Cartesian product of the path P_2 and cycle C_n on n vertices. Furthermore, $|\deg_H(u) - \deg_H(v)| \leq 1$ for every $u, v \in V(H)$, where $H =< u_i, v_j : 1 \leq i, j \leq n >$. Consequently, every γ^s-set of H is also a γ^{se}-set of H. Hence, the bound is true by Theorem 1.3. □

Theorem 3.2. *For any flower graph Fl_n,*

$$\gamma^e_{sec}(Fl_n) = \begin{cases} n + 2 & if \;\; n = 3, \\ \lceil \frac{3n}{7} \rceil + n + 1 & \text{otherwise.} \end{cases} \tag{6}$$

Proof. Let $V(Fl_n) = \{u_0, u_1, u_2 \ldots, u_n, v_1, v_2, \ldots, v_n\}$, where $u_1, u_2 \ldots, u_n$ denote the vertices on the perimeter of the wheel of a flower graph Fl_n, u_0 denotes the universal vertex of Fl_n to which the pendant vertices $v_1, v_2, \ldots, v_n$ on the perimeter of its wheel are adjoined.

The vertex u_0 as well as the vertex v_i, where $1 \leq i \leq n$, belong to every se-dominating set of Fl_n, since $|\deg_{Fl_n}(u_0) - \deg_{Fl_n}(u_i)| > 1$ and $|\deg_{Fl_n}(v_i) - \deg_{Fl_n}(u_i)| > 1$ for every $1 \leq i \leq n$. Thus, $\{u_0, v_1, v_2, \ldots, v_n\} \subseteq S$, where S is a se-dominating set of Fl_n.

Case(i) Let $n = 3$. As $u_0, v_1, v_2, v_3 \in S$, we have to obtain the number of $u_i s$ that is sufficient to protect the remaining vertices of Fl_3. Since $< u_i : 1 \leq i \leq 3 >$ is a cycle on 3 vertices, only one vertex is sufficient to protect every u_i by Theorem 2.2. Consequently, $\gamma^e_{sec}(Fl_3) = n + 1 + 1 = n + 2$.

Case(ii) Let $n \geq 4$. We have $< u_i : 1 \leq i \leq n >$ is a cycle of order n. Then, at least $\lceil \frac{3n}{7} \rceil$ vertices are needed to protect every u_i, where $1 \leq i \leq n$. Hence, $\gamma^e_{sec}(Fl_n) = \min\{|S| : \text{S is a se} - \text{dominating set of } Fl_n\} = \lceil \frac{3n}{7} \rceil + n + 1$. □

Theorem 3.3. *For any sunflower graph Sf_n,*

$$\gamma^e_{sec}(Sf_n) = \begin{cases} n+2 & if \;\; n = 4, 5, \\ \lceil \frac{3n}{7} \rceil + n + 1 & \text{otherwise.} \end{cases} \tag{7}$$

Proof. Let $V(Sf_n) = \{u_0, u_1, u_2 \ldots, u_n, v_1, v_2, \ldots, v_n\}$, where u_0 denotes the central vertex, $u_1, u_2 \ldots u_n$ denote the vertices on the inner cycle and $v_1, v_2 \ldots\ v_n$ denote the vertices attached between each pair of adjacent u_i, where $1 \leq i \leq n$. Since the vertices $u_0, v_1, v_2, \ldots, v_n$ do not belong to the equitable neighbourhood of any vertex on the cycle of Sf_n, no vertex of the cycle can protect them. Moreover, u_0 can not defend any v_i, where $1 \leq i \leq n$, since $|\deg_{Sf_n}(u_0) - \deg_{Sf_n}(v_i)| > 1$. Thus, the vertices u_0 and v_i, where $1 \leq i \leq n$, belong to every se-dominating set of Sf_n. Let S denote a se-dominating set of Sf_n. Then, we have the following cases:

Case(i) Let $n = 4, 5$. Then, the vertex $u_0 \in N_e(u_i)$, where $1 \leq i \leq n$. Furthermore, every v_i, where $1 \leq i \leq n$, belong to each and every se-dominating set of Sf_n, since $N_e(v_i) = \phi$. We define a set $S = \{u_0, u_1, v_1, v_2, \ldots, v_n\}$. Consider the sunflower graph Sf_4. The vertices u_2 and u_4 can be swapped with u_1. Also, the vertex u_3 and u_0 are interchangeable. Thus, $S = \{u_0, u_1, v_1, v_2, v_3,\ v_4\}$ is a se-dominating set of Sf_4. Moreover, $S \setminus \{u_0\}$ is not a se-dominating set, since $N_e(u_3) = \phi$. Deleting any v_i from S also affects the equitability, since every v_i, where $1 \leq i \leq n$, is an equitable isolate. In the same way, we can prove that $S = \{u_0, u_1, v_1, v_2, v_3, v_4, v_5\}$ is a γ^{se}-set of Sf_5 and hence, $\gamma^e_{sec}(Sf_5) = |S| = 5 + 2 = 7$.

Case(ii) Let $n \geq 6$. Let $S = \{u_1, u_3, u_5, v_1, v_2, \ldots, v_6\} \subset V(Sf_6)$. The vertices u_2, u_4 and u_6 can be interchanged with the adjacent $u_i \in S$, where $i = 1, 3, 5$. Also, u_0 can be swapped with u_1, since $u_6 \in N_e(u_5)$ and $u_2 \in N_e(u_3)$. Hence, S is a se-dominating set of Sf_6. Moreover, by Theorem 2.2, at least $\lceil \frac{3n}{7} \rceil = \frac{18}{7} = 3$ vertices are required to protect every u_i, where $1 \leq i \leq 6$. Thus, $\gamma^e_{sec}(Sf_6) = \min\{|S| : \text{S is a se} - \text{dominating set of } Sf_6\} = 6 + 3 + 1 = 10$.

Now, we assume that $n \geq 7$. The graph $< u_i : 1 \leq i \leq n >$ is a cycle on n vertices. Then, at least $\lceil \frac{3n}{7} \rceil$ number of vertices must be added to construct the set S. Hence, $\gamma^e_{sec}(Sf_n) = \min\{|S| : \text{S is a se} - \text{dominating set of } Sf_n\} = \lceil \frac{3n}{7} \rceil$. □

4. Conclusion

In this paper, we obtained new bounds for the se-domination number of some basic graph classes. Moreover, we studied the se-domination number of certain special wheel-related graph classes. In addition, results were obtained based on the diameter, girth and degree. In various graph constructions and graph operations, the idea of secure equitable domination can be further examined. A comparative study can be done with other dominating parameters.

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