TWMS J. App. and Eng. Math. V.15, N.9, 2025, pp. 2166-2180

THE SZEGED INDEX OF POWER GRAPH OF FINITE GROUPS

S. BANERJEE[1*], §

Abstract. The Szeged index of a graph is an invariant with several applications in chemistry. The power graph of a finite group G is a graph having vertex set as G in which two vertices u and v are adjacent if $v = u^m$ or $u = v^n$ for some $m, n \in \mathbb{N}$. In this paper, we first obtain a formula for the Szeged index of the generalized join of graphs. As an application, we obtain the Szeged index of the power graph of the finite cyclic group $\mathbb{Z}_n$ for any $n > 2$. We further obtain a relation between the Szeged index of the power graph of $\mathbb{Z}_n$ and the Szeged index of the power graph of the dihedral group D_n. We also provide SAGE codes for evaluating the Szeged index of the power graph of $\mathbb{Z}_n$ and D_n at the end of this paper.

Keywords: Szeged index, generalized join, power graph, finite cyclic group, dihedral group.

AMS Subject Classification: 05C09, 05C25

1. Introduction

Throughout the paper, G will denote either a finite graph or a finite group, whichever is applicable, and it will be clear from the context. Also, given a set A, the number of elements in A is denoted by $|A|$.

In theoretical chemistry, topological indices are widely used to understand the physical and chemical properties of chemical compounds. There exist many different types of topological indices that aim to capture various aspects of the molecular graphs associated with the molecules considered. The oldest and most thoroughly examined topological index is the Wiener index (see [30, 24, 13] for some references). Another topological index, the Szeged index, was introduced much later and extensively studied. We discuss the Szeged index and its close association with the Wiener index in short below. Suppose G is a finite, simple, and connected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of G is denoted by $|G|$. If two vertices a and b are adjacent to each other, we denote it by $a \sim b$. Moreover, $E(G) = \{(a, b) : a, b \in V(G) \text{ with } a \sim b\}$. The distance between two vertices $a, b \in G$, denoted by $d(a, b)$, is defined to be the length of

[1*] Department of Mathematics, JIS University, 81, Nilgunj Rd, Agarpara, Kol-700109, India.
e-mail: subarshabnrj@gmail.com/subarsha.banerjee@jisuniversity.ac.in; 0000-0002-7029-7650.
[*] Corresponding author.

§ Manuscript received: August 15, 2024; accepted:December 07, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.9;

the shortest path from a to b. The *Wiener index* [26] of a connected graph G is defined as follows:

$$W(G) = \frac{1}{2} \sum_{a,b \in V(G)} d(a,b). \tag{1}$$

Suppose $e \in E(G)$ is an edge between the vertices a and b of G. We first state the following definitions: We define

$$\begin{aligned} n_1(ab|G) &= |\{x \in G : d(x,a) < d(x,b)\}| \\ n_2(ab|G) &= |\{x \in G : d(x,b) < d(x,a)\}|. \end{aligned} \tag{2}$$

The two quantities described in Equation (2) were mentioned for the first time in [34]. For a long time, it was known that the formula

$$W(G) = \sum_{e(=ab) \in E(G)} n_1(ab|G) n_2(ab|G) \tag{3}$$

holds for molecular graphs of alkanes. In [17], it was proved that Equation (3) holds for all trees. Furthermore, in [28] it was shown that Equation (3) does not hold in general (in particular for graphs containing cycles), and only holds for graphs whose each block is a complete graph. Although the attempts to change the right-hand side of Equation (3) to make it applicable to all connected graphs have been successfully made in [31, 32], the resulting expressions were very confusing. In [29], it was suggested that the complications arising with the generalization of Equation (3) to all connected graphs containing cycles could be overcome by using the right–hand side of Equation (3) as the definition of a new graph invariant. Consequently, the formula was extended to all graphs and it came to be known as the Szeged index of a graph. The *Szeged index* of a connected graph G is defined as follows:

$$Sz(G) = \sum_{e(=ab) \in E(G)} n_1(ab|G) n_2(ab|G). \tag{4}$$

The Szeged index has been considered from multiple viewpoints, see for example [25, 27] and the references therein for some literature on the same. The Szeged index is closely related to the Wiener index, see for example the references [11, 16]. In [26], the authors deduced the Szeged index of the Cartesian product of graphs. In [22], the authors determined the Szeged index of join and composition of graphs. The Szeged index of bridge graphs was determined in [21]. Some variants of the Szeged index namely the edge PI index, edge Szeged index, edge-vertex Szeged index, vertex-edge Szeged index, and revised edge Szeged index under the rooted product of graphs have been studied in [12]. Recently, the Szeged index of unicyclic graphs was studied in [6]. Motivated by the above works, in this paper, we deduce the Szeged index of the generalized join of graphs.

The *power graph* [20] of a finite group G has vertex set as G, and two distinct vertices $u, v \in G$ are adjacent if $u = v^m$ or $v = u^n$ for some $m, n \in \mathbb{N}$. Topological indices of graphs associated with algebraic structures have been studied in [4, 8, 10, 5, 1] recently. Here, in this paper, as an application of our results, we have determined the Szeged index of the power graph of certain finite groups, viz. the finite cyclic group $\mathbb{Z}_n$ and the dihedral group D_n.

The paper has been organized as follows: In Section 2, we describe the Szeged index of the generalized join of graphs in terms of the Szeged index of the constituent graphs. In Section 3.1, we find the Szeged index of the power graph of the finite cyclic group $\mathbb{Z}_n$. In Section 3.2, we find the Szeged index of the power graph of the dihedral group D_n and provide a relationship between the Szeged index of the power graph of $\mathbb{Z}_n$ and D_n. In

Section 4, we provide the SAGE codes for evaluating the Szeged index of the power graph of the finite cyclic group $\mathbb{Z}_n$, and the dihedral group D_n.

2. Szeged Index of Generalized Join of Graphs

In this section, we deduce the Szeged index of the generalized join of graphs in terms of the constituent graphs. Consider a family of n connected graphs G_i such that $\mid G_i \mid= n_i$ for $1 \leq i \leq n$. Assume that $\mathcal{G}$ is a graph such that $\mid \mathcal{G} \mid= n$. The *generalized join* or the $\mathcal{G}$-join of graphs G_i is defined as the following:

Definition 2.1. [33, p. 15] *Suppose $\mathcal{G}$ is a graph with vertex set $V(\mathcal{G}) = \{1, 2, \ldots, n\}$. Suppose G_i be disjoint connected graphs of order n_i with vertex sets $V(G_i)$ for $1 \leq i \leq n$. The $\mathcal{G}$-join of graphs $G_1, G_2, \ldots, G_n$ denoted by $G = \mathcal{G}[G_1, G_2, \ldots, G_n]$ is formed by taking the graphs G_i and any two vertices $v_i \in G_i$ and $v_j \in G_j$ are adjacent if i is adjacent to j in $\mathcal{G}$.*

We now state the main result of this section.

Theorem 2.1. *Consider $\mathcal{G}$ to be a graph with vertex set $V(\mathcal{G}) = \{1, 2, \ldots, n\}$. Assume that G_i's are disjoint connected graphs of order n_i with vertex sets $V(G_i)$ for $1 \leq i \leq n$. The Szeged index of $G = \mathcal{G}[G_1, G_2, \ldots, G_n]$ is given as follows:*

$$Sz(G) = \sum_{i=1}^{n} Sz(G_i) + \sum_{\substack{i,j=1 \\ i<j, i\sim j}}^{n} \left[\left(\sum_{\substack{k=1 \\ k\sim i, k\nsim j}}^{n} |G_k| + 1\right)\left(\sum_{\substack{k=1 \\ k\nsim i, k\sim j}}^{n} |G_k| + 1\right)\right] \mid G_i \mid\mid G_j \mid .$$

Proof. Suppose $ab \in E(G)$. Since ab is an edge in G, two possible cases may arise:

1. $a \in G_i$ and $b \in G_i$ for some i where $1 \leq i \leq n$, or
2. $a \in G_i$ and $b \in G_j$ for some $1 \leq i, j \leq n$ such that $i \sim j$ in $\mathcal{G}$.

Hence,

$$\begin{aligned} Sz(G) &= \sum_{e(=ab)\in E(G)} n(ab|G) \\ &= \sum_{i=1}^{n}\left(\sum_{\substack{e(=ab)\in E(G) \\ a,b\in G_i}} n(ab|G)\right) + \sum_{\substack{e(=ab)\in E(G) \\ a\in G_i, b\in G_j \\ i\sim j}} n(ab|G) \\ &= \sum_{i=1}^{n}\left(\sum_{\substack{e(=ab)\in E(G) \\ a,b\in G_i}} n_1(ab|G)n_2(ab|G)\right) + \sum_{\substack{e(=ab)\in E(G) \\ a\in G_i, b\in G_j \\ i\sim j}} n(ab|G) \end{aligned} \quad (\mathbf{A})$$

We shall now evaluate each term of the summation given in Equation ($\mathbf{A}$) in what follows.

We have,

$$\begin{aligned} n_1(ab|G) &= |\{x \in G : d(x, a) < d(x, b)\}| \\ &= |\Big(\{x \in G_i : d(x, a) < d(x, b)\} \cup \{x \in G \setminus G_i : d(x, a) < d(x, b)\}\Big)| \\ &= |\{x \in G_i : d(x, a) < d(x, b)\}| + |\{x \in G \setminus G_i : d(x, a) < d(x, b)\}| \\ &= n_1(ab|G_i) + |\{x \in G \setminus G_i : d(x, a) < d(x, b)\}|. \end{aligned} \quad (*)$$

We note that since $a, b \in G_i$, then $d(x, a) = d(x, b)$ for all $x \in G \setminus G_i$. Hence we find that

$$\{x \in G \setminus G_i : d(x, a) < d(x, b)\} = \emptyset$$

Thus, we have,

$$|\{x \in G \setminus G_i : d(x,a) < d(x,b)\}| = 0. \tag{**}$$

Hence, using Equations (*) and (**), we find that for all $1 \leq i \leq n$,

$$n_1(ab|G) = n_1(ab|G_i). \tag{5}$$

Using similar arguments as above, we find that for all $1 \leq i \leq n$,

$$n_2(ab|G) = n_2(ab|G_i). \tag{6}$$

Hence we have,

$$\begin{aligned} \sum_{i=1}^{n}\Bigg(\sum_{\substack{e(=ab)\in E(G)\\ a,b\in G_i}} n_1(ab|G)n_2(ab|G)\Bigg) &= \sum_{i=1}^{n} n_1(ab|G_i)n_2(ab|G_i) \\ &= \sum_{i=1}^{n} Sz(G_i). \end{aligned} \tag{7}$$

We now evaluate the second term of the summation given in Equation (A).
We have,

$$\sum_{\substack{e(=ab)\in E(G)\\ a\in G_i, b\in G_j\\ i\sim j}} n(ab|G) = \sum_{\substack{e(=ab)\in E(G)\\ a\in G_i, b\in G_j\\ i\sim j}} n_1(ab|G)n_2(ab|G)$$

Here $a \in G_i, b \in G_j$ for some i, j with $i \sim j$. Consider the set $\{x \in G : d(x,a) < d(x,b)\}$. Suppose $x \in V(G)$ satisfies the relation $d(x,a) < d(x,b)$. Then one possibility is $x = a$. Moreover, $x \notin G_i$, as otherwise, since $i \sim j$, x will be adjacent to b which would imply $d(x,b) = 1$, which is false. Thus, x must belong to all those G_k such that $i \sim k$, but $j \nsim k$ in $\mathcal{G}$.

Thus,

$$\begin{aligned} n_1(ab|G) &= |\{x \in G : d(x,a) < d(x,b)\}| \\ &= \sum_{\substack{k=1\\ k\sim i, k\nsim j}}^{n} \mid G_k \mid +1. \end{aligned} \tag{8}$$

Using similar arguments we obtain,

$$\begin{aligned} n_2(ab|G) &= |\{x \in G : d(x,b) < d(x,a)\}| \\ &= \sum_{\substack{k=1\\ k\nsim i, k\sim j}}^{n} \mid G_k \mid +1. \end{aligned} \tag{9}$$

Using Equations (8) and (9) we obtain,

$$\sum_{\substack{e(=ab)\in E(G)\\ a\in G_i, b\in G_j\\ i\sim j}} n(ab|G) = \sum_{\substack{i,j=1\\ i<j, i\sim j}}^{n} \Bigg[\Bigg(\sum_{\substack{k=1\\ k\sim i, k\nsim j}}^{n} |G_k| + 1\Bigg)\Bigg(\sum_{\substack{k=1\\ k\nsim i, k\sim j}}^{n} |G_k| + 1\Bigg)\Bigg] \mid G_i \mid\mid G_j \mid . \tag{10}$$

Putting the values obtained from Equations (7) and (10) in Equation (**A**), we obtain

$$Sz(G) = \sum_{i=1}^{n} Sz(G_i) + \sum_{\substack{i,j=1 \\ i<j, i\sim j}}^{n} \left[\left(\sum_{\substack{k=1 \\ k\sim i, k\nsim j}}^{n} |G_k| + 1 \right) \left(\sum_{\substack{k=1 \\ k\nsim i, k\sim j}}^{n} |G_k| + 1 \right) \right] \mid G_i \mid\mid G_j \mid .$$

This completes the proof. □

In the next section, as an application to Theorem 2.1, we calculate the Szeged index of the power graph of the finite cyclic group $\mathbb{Z}_n$ for $n > 2$. We also determine the Szeged index of the power graph of the dihedral group D_n for $n > 2$.

3. Szeged Index of Power Graph of Finite Groups

The notion of *directed power graph* of a semigroup S was introduced by Kelarav and Quinn in [23] as the digraph $\mathcal{P}(S)$ with vertex set S in which there is an arc from one vertex u to another vertex v if $v = u^m$ for some $m \in \mathbb{N}$. Motivated by this, Chakrabarty et al. in [20] defined the undirected *power graph* $\mathcal{P}(S)$ of a semi-group S as an undirected graph whose vertex set is S and two distinct vertices $u, v \in S$ are adjacent if $u = v^m$ or $v = u^n$ for some $m, n \in \mathbb{N}$. In [20] it was further shown that $\mathcal{P}(G)$ is connected for any finite group G, and it is complete if and only if G is a cyclic group of order 1 or p^m where p is a prime and $m \in \mathbb{N}$. The power graph has received widespread attention from researchers over the recent years. Cameron and Ghosh have shown in [18] that if two finite abelian groups G_1, G_2 have isomorphic power graphs, then G_1 and G_2 are isomorphic to each other. In [19] Cameron has proved that if two finite groups have isomorphic power graphs, then they have the same number of elements of each order. The spectral properties of power graphs of finite groups have been studied in [2, 3, 7, 9]. For more interesting results on power graphs, the readers may refer to [15].

3.1. Szeged Index of Power Graph of $\mathbb{Z}_n$.

In this section, we derive a formula for the Szeged index of the power graph of $\mathbb{Z}_n$ for any $n > 2$. Before getting into the details, we first throw some light on the structure of $\mathcal{P}(\mathbb{Z}_n)$.

We shall denote the prime decomposition of n by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where $p_1 < p_2 < \cdots < p_m$ are primes arranged in increasing order and $\alpha_i' s$ are positive integers. An element d is said to be a *proper divisor* of n if $d \mid n$ and $d \notin \{1, n\}$. Given $n \in \mathbb{N}$, the total number of positive divisors of n is given as follows:

$$D' = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_m + 1).$$

The number of *positive proper divisors* of n is given as follows:

$$D = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_m + 1) - 2. \quad (11)$$

Suppose $d_1 < d_2 < \cdots < d_D$ be the set of all positive proper divisors of n arranged in increasing order.

Assume that $\mathcal{G}$ is a graph with vertex set $V(\mathcal{G}) = \{1, d_1, d_2, \ldots, d_D\}$. Moreover, the adjacency criterion in $\mathcal{G}$ is given as follows:
$1 \sim d_i$ for all $1 \le i \le D$, and $d_i \sim d_j$ if and only if $d_i \mid d_j$. Using the above information, we have the following result about the structure of $\mathcal{P}(\mathbb{Z}_n)$.

Theorem 3.1. [14, Theorem 2.2] *If $n > 2$, then $\mathcal{P}(\mathbb{Z}_n) = \mathcal{G}[K_\ell, K_{\varphi(d_1)}, K_{\varphi(d_2)}, \ldots, K_{\varphi(d_D)}]$, where $d_1, d_2, \ldots, d_D$ form the set of all positive proper divisors of n, D is given by Equation* (11) *and $\ell = \varphi(n) + 1$. Here, $\varphi(n)$ denotes the Euler's totient function, which counts the positive integers up to a given integer n that are relatively prime to n.*

We now use Theorems 2.1 and 3.1 to compute the Szeged index of $\mathcal{P}(\mathbb{Z}_n)$.

Theorem 3.2. *If $n > 2$, then the Szeged index of $\mathcal{P}(\mathbb{Z}_n)$ is given as follows:*

$$Sz(\mathcal{P}(\mathbb{Z}_n)) = \binom{\ell}{2} + \sum_{i=1}^{D} \binom{\varphi(d_i)}{2} + \sum_{i=1}^{D} \Big(n - \ell - \varphi(d_i) + 1\Big)\varphi(d_i) \times \ell$$
$$+ \sum_{\substack{i,j=1 \\ i<j, d_i \sim d_j}}^{D} \Bigg[\Big(\sum_{\substack{k=1 \\ d_k \sim d_i, d_k \nsim d_j}}^{D} \varphi(d_k) + 1\Big)\Big(\sum_{\substack{k=1 \\ d_k \nsim d_i, d_k \sim d_j}}^{D} \varphi(d_k) + 1\Big)\Bigg]\varphi(d_i)\varphi(d_j)$$

Proof. We note that $Sz(K_\ell) = \binom{\ell}{2}$. Moreover, if $d_i > 2$, then $Sz(K_{\varphi(d_i)}) = \binom{\varphi(d_i)}{2}$ for $1 \le i \le D$.

We define $S = \{x \in \mathbb{Z}_n : \gcd(x, n) = 1\} \cup \{0\}$. Consider an edge $e = ab$ of $\mathcal{P}(\mathbb{Z}_n)$ such that $a \in S$ and $b \notin S$. Since the vertices of S are adjacent to every other vertex of $\mathcal{P}(\mathbb{Z}_n)$, we observe that $d(x, a) = 1$ for all $x \neq a$. Thus, $d(x, a) > d(x, b)$ is true only when $x = b$.

Hence,

$$n_2(ab|\mathcal{P}(\mathbb{Z}_n)) = |\{x : d(x, a) > d(x, b)\}| = 1.$$

It follows that

$$\begin{aligned} \sum_{\substack{e(=ab)\in E(\mathcal{P}(\mathbb{Z}_n)) \\ a\in S, b\notin S}} n(ab|\mathcal{P}(\mathbb{Z}_n)) &= \sum_{\substack{e(=ab)\in E(\mathcal{P}(\mathbb{Z}_n)) \\ a\in S, b\notin S}} n_1(ab|\mathcal{P}(\mathbb{Z}_n)) \\ &= \sum_{\substack{e(=ab)\in E(\mathcal{P}(\mathbb{Z}_n)) \\ a\in S, b\notin S}} |\{x : d(x, a) < d(x, b)\}|. \end{aligned} \tag{12}$$

Since $b \notin S$, so $b \in K_{\varphi(d_i)}$ for some $1 \le i \le D$. In order for x to satisfy $d(x, a) < d(x, b)$, x must be a or x must be in $K_{\phi(d_k)}$ such that $d_k \nsim d_i$.

Thus using Equation (12) we get,

$$\sum_{\substack{e(=ab)\in E(\mathcal{P}(\mathbb{Z}_n)) \\ a\in S, b\notin S}} n(ab|\mathcal{P}(\mathbb{Z}_n)) = \sum_{i=1}^{D} \Bigg[\Big(\sum_{\substack{k=1 \\ d_k \nsim d_i}}^{D} \varphi(d_k) + 1\Big)\Bigg] \times \ell \times \varphi(d_i). \tag{13}$$

Using Theorem 2.1 and Equation (13) we have

$$\begin{aligned} Sz(\mathcal{P}(\mathbb{Z}_n)) &= \binom{\ell}{2} + \sum_{i=1}^{D} \binom{\varphi(d_i)}{2} + \sum_{i=1}^{D} \Bigg[\Big(\sum_{\substack{k=1 \\ d_k \nsim d_i}}^{D} \varphi(d_k) + 1\Big)\Bigg] \times \ell \times \varphi(d_i) \\ &+ \sum_{\substack{i,j=1 \\ i<j, d_i \sim d_j}}^{D} \Bigg[\Big(\sum_{\substack{k=1 \\ d_k \sim d_i \\ d_k \nsim d_j}}^{D} |K_{\varphi(d_k)}| + 1\Big)\Big(\sum_{\substack{k=1 \\ d_k \nsim d_i \\ d_k \sim d_j}}^{D} |K_{\varphi(d_k)}| + 1\Big)\Bigg] |K_{\varphi(d_i)}||K_{\varphi(d_j)}| \\ &= \binom{\ell}{2} + \sum_{i=1}^{D} \binom{\varphi(d_i)}{2} + \sum_{i=1}^{D} \Bigg[\Big(\sum_{\substack{k=1 \\ d_k \nsim d_i}}^{D} \varphi(d_k) + 1\Big)\Bigg] \times \ell \times \varphi(d_i) \\ &+ \sum_{\substack{i,j=1 \\ i<j, d_i \sim d_j}}^{n} \Bigg[\Big(\sum_{\substack{k=1 \\ d_k \sim d_i, d_k \nsim d_j}}^{n} \varphi(d_k) + 1\Big)\Big(\sum_{\substack{k=1 \\ d_k \nsim d_i, d_k \sim d_j}}^{n} \varphi(d_k) + 1\Big)\Bigg]\varphi(d_i)\varphi(d_j). \end{aligned}$$

□

We now use Theorem 3.2 to determine the Szeged index of $\mathcal{P}(\mathbb{Z}_n)$ for some specific forms of n.

Proposition 3.1. *If $n = p^m$ where p is a prime and $m \in \mathbb{N}$, then $Sz(\mathcal{P}(\mathbb{Z}_n)) = \binom{n}{2}$.*

Proof. We know that if $n = p^m$, then $\mathcal{P}(\mathbb{Z}_n)$ is a complete graph ([20]). Hence the result follows. □

Proposition 3.2. *If $n = pq$, where p, q are prime numbers with $p < q$, then*

$$Sz(\mathcal{P}(\mathbb{Z}_n)) = \frac{\ell(\ell-1)}{2} + \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + \ell(2pq - (p+q)).$$

Proof. If $n = pq$, then the positive proper divisors of n are p, q. Thus, $d_1 = p, d_2 = q$, $D = 2$ and $\ell = \varphi(pq) + 1$. Since $p \nmid q$, we note that $d_1 \nsim d_2$. Using Theorem 3.1 we find that $\mathcal{P}(\mathbb{Z}_n)$ is the $\mathcal{G}$-join of $K_\ell, K_{\phi(p)}, K_{\phi(q)}$ where $\mathcal{G}$ is shown in Figure 1.

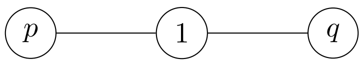

FIGURE 1. $\mathcal{G} = P_3$ for $\mathcal{P}(\mathbb{Z}_{pq})$

Using Theorem 3.2 we have

$$\begin{aligned} Sz(\mathcal{P}(\mathbb{Z}_{pq})) &= \binom{\ell}{2} + \binom{\varphi(p)}{2} + \binom{\varphi(q)}{2} + \Big((\varphi(q)+1) \times \ell \times \varphi(p)\Big) \\ &\quad + \Big((\varphi(p)+1) \times \ell \times \varphi(q)\Big) \\ &= \frac{\ell(\ell-1)}{2} + \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + q\ell(p-1) + p\ell(q-1) \\ &= \frac{\ell(\ell-1)}{2} + \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + \ell(2pq - (p+q)). \end{aligned}$$

□

Proposition 3.3. *If $n = pq^2$, then the Szeged index of $\mathcal{P}(\mathbb{Z}_n)$ is given by*

$$\begin{aligned} Sz(\mathcal{P}(\mathbb{Z}_n)) &= \binom{\ell}{2} + \binom{\varphi(p)}{2} + \binom{\varphi(q)}{2} + \binom{\varphi(q^2)}{2} + \binom{\varphi(pq)}{2} \\ &\quad + \ell\Big(2pq^3 - 2pq^2 + 3pq - 2p - 2q^3 + 3q^2 - 3q + 1\Big) \\ &\quad + p^2q^4 - 3p^2q^3 + 5p^2q^2 - 4p^2q + p^2 - 3pq^2 + 4pq - p - q^4 + 4q^3 - 4q^2 + q. \end{aligned}$$

Proof. If $n = pq^2$, then the positive proper divisors of n are p, q, q^2, pq. Here $d_1 = p, d_2 = q, d_3 = q^2, d_4 = pq$ and $D = 4$. Also $\ell = \varphi(n) + 1 = q(p-1)(q-1) + 1$. Using Theorem 3.1 we find that $\mathcal{P}(\mathbb{Z}_n)$ is the $\mathcal{G}$-join of $K_\ell, K_{\phi(p)}, K_{\phi(q)}, K_{\phi(pq)}, K_{\phi(q^2)}$ where $\mathcal{G}$ is shown in Figure 2.

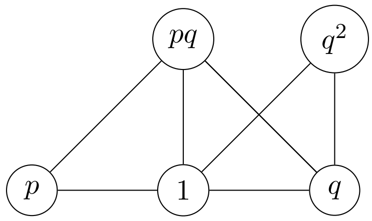

FIGURE 2. $\mathcal{G}$ for $\mathcal{P}(\mathbb{Z}_{pq^2})$

Using Theorem 3.2 we obtain

$$Sz(\mathcal{P}(\mathbb{Z}_n)) = \binom{\ell}{2} + \binom{\varphi(p)}{2} + \binom{\varphi(q)}{2} + \binom{\varphi(q^2)}{2} + \binom{\varphi(pq)}{2} + \sum_{i=1}^{D}\Bigg[\bigg(\sum_{\substack{k=1\\ d_k \nsim d_i}}^{D} \varphi(d_k)+1\bigg)\Bigg] \times \ell \times \varphi(d_i) + X \tag{14}$$

where,

$$X = \sum_{\substack{i,j=1\\ i<j, d_i \sim d_j}}^{D} \Bigg[\bigg(\sum_{\substack{k=1\\ d_k \sim d_i, d_k \nsim d_j}}^{D} \varphi(d_k)+1\bigg)\bigg(\sum_{\substack{k=1\\ d_k \nsim d_i, d_k \sim d_j}}^{D} \varphi(d_k)+1\bigg)\Bigg]\varphi(d_i)\varphi(d_j).$$

Now,

$$\begin{aligned}
&\sum_{i=1}^{D}\Bigg[\bigg(\sum_{\substack{k=1\\ d_k \nsim d_i}}^{D} \varphi(d_k)+1\bigg)\Bigg] \times \ell \times \varphi(d_i)\\
&= \ell\Bigg[\bigg(\varphi(q)+\varphi(q^2)+1\bigg) \times \varphi(p) + \bigg(\varphi(p)+1\bigg)\varphi(q)\\
&+ \bigg(\varphi(p)+\varphi(pq)+1\bigg) \times \varphi(q^2) + \bigg(\varphi(q^2)+1\bigg) \times \varphi(pq)\Bigg]\\
&= \ell\Bigg[q^2(p-1)+p(q-1)+q(q-1)\bigg(p+(p-1)(q-1)\bigg)+(p-1)(q-1)\bigg(q(q-1)+1\bigg)\Bigg]\\
&= \ell\Bigg[2pq^3-2pq^2+3pq-2p-2q^3+3q^2-3q+1\Bigg].
\end{aligned} \tag{15}$$

Moreover, $p \sim pq, q \sim q^2, q \sim pq$. Thus we have,

$$
\begin{aligned}
X = & \sum_{\substack{i,j=1 \\ i<j, d_i \sim d_j}}^{D} \Bigg[\Bigg(\sum_{\substack{k=1 \\ d_k \sim d_i, d_k \nsim d_j}}^{D} \varphi(d_k) + 1\Bigg)\Bigg(\sum_{\substack{k=1 \\ d_k \nsim d_i, d_k \sim d_j}}^{D} \varphi(d_k) + 1\Bigg)\Bigg] \varphi(d_i)\varphi(d_j) \\
= & \Bigg[\Bigg((\varphi(q)+1)\varphi(p)\varphi(pq)\Bigg) + \Bigg((\varphi(pq)+1)\varphi(q)\varphi(q^2)\Bigg) \\
& + \Bigg((\varphi(q^2)+1)(\varphi(p)+1)\varphi(q)\varphi(pq)\Bigg)\Bigg] \\
= & \, q(p-1)^2(q-1) + q(q-1)^2\Bigg((p-1)(q-1)+1\Bigg) + p(p-1)(q-1)^2\Bigg(q(q-1)+1\Bigg) \\
= & \, p^2q^4 - 3p^2q^3 + 5p^2q^2 - 4p^2q + p^2 - 3pq^2 + 4pq - p - q^4 + 4q^3 - 4q^2 + q.
\end{aligned} \tag{16}
$$

Substituting the values of Equations (15) and (16) in Equation (14) we have,

$$
\begin{aligned}
Sz(\mathcal{P}(\mathbb{Z}_n)) = & \binom{\ell}{2} + \binom{\varphi(p)}{2} + \binom{\varphi(q)}{2} + \binom{\varphi(q^2)}{2} + \binom{\varphi(pq)}{2} \\
& + \ell\Bigg(2pq^3 - 2pq^2 + 3pq - 2p - 2q^3 + 3q^2 - 3q + 1\Bigg) \\
& + p^2q^4 - 3p^2q^3 + 5p^2q^2 - 4p^2q + p^2 - 3pq^2 + 4pq - p - q^4 + 4q^3 - 4q^2 + q.
\end{aligned}
$$

Thus the result follows. □

3.2. **Szeged Index of Power Graph of Dihedral Group.** In this section, we calculate the Szeged index of the power graph of the dihedral group D_n for $n > 2$. We establish a relation between the Szeged index of the power graph of dihedral group D_n, and the Szeged index of the power graph of finite cyclic group $\mathbb{Z}_n$.

Definition 3.1. *The dihedral group* D_n *of order* $2n$ *is given by the following presentation:*

$$\mathrm{D}_n = \langle r, s : r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

If $n = 6$, $\mathcal{P}(\mathrm{D}_n)$ takes the following form(Figure 3). Clearly for all $n > 2$, $\mathcal{P}(\mathrm{D}_n)$ contains $\mathcal{P}(\mathbb{Z}_n)$ as a subgraph.

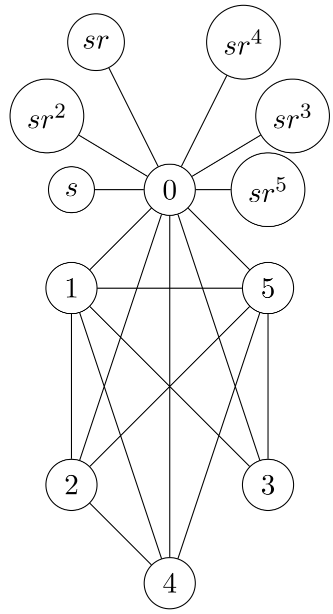

FIGURE 3. $\mathcal{P}(\mathrm{D}_6)$

We now express the Szeged index of $\mathcal{P}(\mathrm{D}_n)$ in terms of the Szeged index of $\mathcal{P}(\mathbb{Z}_n)$.

Theorem 3.3. *If $n > 2$, then the Szeged index of $\mathcal{P}(\mathrm{D}_n)$ is given as follows:*

$$Sz(\mathcal{P}(\mathrm{D}_n)) = Sz(\mathcal{P}(\mathbb{Z}_n^*)) + \sum_{i=1}^{D}\Bigg(n + 1 + \sum_{\substack{k=1\\ d_k \nsim d_i}}^{D} \varphi(d_k)\Bigg)\varphi(d_i) + n(2n - 1 + \varphi(n)) + \varphi(n).$$

Here, $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}$, and $d_k \sim d_i$ only when $d_k \mid d_i$.

Proof. We shall denote $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}$. We can obtain $\mathcal{P}(\mathbb{Z}_n^*)$ from $\mathcal{P}(\mathbb{Z}_n)$ by deleting the vertex 0 and all edges incident on it from $\mathcal{P}(\mathbb{Z}_n)$. Now,

$$\begin{aligned} Sz(\mathcal{P}(\mathrm{D}_n)) &= \sum_{e(=ab)\in E(\mathcal{P}(\mathrm{D}_n))} n(ab|\mathcal{P}(\mathrm{D}_n)) \\ &= \sum_{e(=ab)\in E(\mathcal{P}(\mathbb{Z}_n^*))} n(ab|\mathcal{P}(\mathrm{D}_n)) + \sum_{e\in\tau} n(ab|\mathcal{P}(\mathrm{D}_n)) + \sum_{e\in\sigma} n(ab|\mathcal{P}(\mathrm{D}_n)) \end{aligned} \tag{17}$$

where $\tau = \{(0,t) : t \in V(\mathcal{P}(\mathbb{Z}_n))\} \subseteq E(\mathcal{P}(\mathbb{Z}_n))$ and $\sigma = \{(0, sr^i) : 0 \le i \le n-1\} \subseteq E(\mathcal{P}(\mathrm{D}_n))$.

Now, we shall evaluate each term of the summation given in Equation (17) one by one in what follows.

$$\sum_{e(=ab)\in E(\mathcal{P}(\mathbb{Z}_n^*))} n(ab|\mathcal{P}(\mathrm{D}_n)) = \sum_{e(=ab)\in E(\mathcal{P}(\mathbb{Z}_n^*))} n_1(ab|\mathcal{P}(\mathrm{D}_n))n_2(ab|\mathcal{P}(\mathrm{D}_n))$$

We observe that the vertex 0 is adjacent to every other vertex of $\mathcal{P}(\mathrm{D}_n)$. Moreover, every vertex of $\mathcal{P}(\mathbb{Z}_n^*)$ has a distance 2 from all the vertices in the set $\{sr^i : 0 \le i \le n-1\}$.

Thus, we have,

$$\begin{aligned} n_1(ab|\mathcal{P}(\mathrm{D}_n)) &= |\{x \in V(\mathcal{P}(\mathrm{D}_n)) : d(x,a) < d(x,b)\}| \\ &= |\{x \in V(\mathcal{P}(\mathbb{Z}_n^*)) : d(x,a) < d(x,b)\}| \\ &= n_1(ab|\mathcal{P}(\mathbb{Z}_n^*)) \end{aligned} \tag{18}$$

Similarly,

$$n_2(ab|\mathcal{P}(\mathrm{D}_n)) = n_2(ab|\mathcal{P}(\mathbb{Z}_n^*)) \tag{19}$$

Using Equations (18) and (19) we have,

$$n(ab|\mathcal{P}(\mathrm{D}_n)) = n(ab|\mathcal{P}(\mathbb{Z}_n^*)) \tag{20}$$

Again,

$$\sum_{e\in\tau} n(ab|\mathcal{P}(\mathrm{D}_n)) = \sum_{e\in\tau} n_1(ab|\mathcal{P}(\mathrm{D}_n)) \sum_{e\in\tau} n_2(ab|\mathcal{P}(\mathrm{D}_n))$$

Assume that $e = (a, b) \in \tau$. Then

$$\begin{aligned} n_1(ab|\mathcal{P}(\mathrm{D}_n)) &= |\{x \in \mathcal{P}(\mathrm{D}_n) : d(x, a) < d(x, b)\}| \\ &= |\{x \in \mathcal{P}(\mathrm{D}_n) : d(x, 0) < d(x, t)\}| \text{ where } t \in V(\mathcal{P}(\mathbb{Z}_n)). \end{aligned}$$

We consider the following two mutually exclusive and exhaustive cases:

Case 1: t is a generator of $\mathbb{Z}_n$, i.e. $\gcd(t, n) = 1$.
In this case x can be 0, or x can be one of the elements of the set $\{sr^i : 0 \le i \le n-1\}$. Thus, we get $n + 1$ choices for x.

Case 2: t is not a generator of $\mathbb{Z}_n$, i.e. $\gcd(t, n) \neq 1$.
Suppose $x \in K_{\varphi(d_i)}$ for some positive proper divisor d_i of n where $1 \le i \le D$. Then, clearly x can be 0, or x can be one of the elements of the set $\{sr^i : 0 \le i \le n-1\}$, or x must be in those $K_{\varphi(d_k)}$ such that $d_i \nsim d_k$ where $1 \le k \le D$. Thus, we have $n + 1 + \sum_{\substack{k=1 \\ d_k \nsim d_i}}^{D} \varphi(d_k)$ choices for x.

We further note that $n_2(ab|\mathcal{P}(\mathrm{D}_n)) = 1$ for all $(a, b) \in \tau$. Thus we obtain,

$$\sum_{e\in\tau} n(ab|\mathcal{P}(\mathrm{D}_n)) = \sum_{i=1}^{D} \Bigg(n + 1 + \sum_{\substack{k=1 \\ d_k \nsim d_i}}^{D} \varphi(d_k) \Bigg) \varphi(d_i). \tag{21}$$

Again,

$$\sum_{e\in\sigma} n(ab|\mathcal{P}(\mathrm{D}_n)) = \sum_{e\in\sigma} n_1(ab|\mathcal{P}(\mathrm{D}_n)) n_2(ab|\mathcal{P}(\mathrm{D}_n))$$

We note that

$$\begin{aligned} n_1(ab|\mathcal{P}(\mathrm{D}_n)) &= |\{x : d(x, a) < d(x, b)\}| \\ &= |\{x : d(x, 0) < d(x, sr^i)\}| \\ &= |(\mathrm{D}_n \setminus \{sr^i\})| \\ &= 2n - 1. \end{aligned}$$

We further note that $n_2(ab|\mathcal{P}(\mathrm{D}_n)) = 1$ for all $(a, b) \in \sigma$.
Thus we have

$$\sum_{e\in\sigma} n(ab|\mathcal{P}(\mathrm{D}_n)) = \sum_{i=1}^{n} 2n - 1 = n(2n - 1). \tag{22}$$

Substituting the values obtained from Equations (20) to (22) in Equation (17) we have,

$$\begin{aligned}
Sz(\mathcal{P}(\mathrm{D}_n)) &= \sum_{e(=ab)\in E(\mathcal{P}(\mathbb{Z}_n^*))} n(ab|\mathcal{P}(\mathrm{D}_n)) + \sum_{e\in\tau} n(ab|\mathcal{P}(\mathrm{D}_n)) + \sum_{e\in\sigma} n(ab|\mathcal{P}(\mathrm{D}_n)) \\
&= Sz(\mathcal{P}(\mathbb{Z}_n^*)) + (n+1)\varphi(n) + \sum_{i=1}^{D}\Bigg(n+1+\sum_{\substack{k=1\\ d_k\nsim d_i}}^{D}\varphi(d_k)\Bigg)\varphi(d_i) + n(2n-1) \\
&= Sz(\mathcal{P}(\mathbb{Z}_n^*)) + \sum_{i=1}^{D}\Bigg(n+1+\sum_{\substack{k=1\\ d_k\nsim d_i}}^{D}\varphi(d_k)\Bigg)\varphi(d_i) + n(2n-1+\varphi(n)) + \varphi(n).
\end{aligned}$$

Thus the result follows. □

We end this section with the following result.

Proposition 3.4. *If $n = pq$ where p, q are distinct primes with $p < q$, then the Szeged index of $Sz(\mathcal{P}(\mathrm{D}_n))$ is given as follows:*

$$Sz(\mathcal{P}(\mathrm{D}_n)) = Sz(\mathcal{P}(\mathbb{Z}_n^*)) + p(q^2-1) + q(p^2-1) + pq(3pq-p-q) + (p-1)(q-1).$$

Proof. Since $n = pq$, the only positive proper divisors of n are p and q. Thus $D = 2$, $d_1 = p$, and $d_2 = q$. Moreover, since $p \nmid q$, so $d_1 \nsim d_2$. Hence, using Theorem 3.3, we have

$$\begin{aligned}
Sz(\mathcal{P}(\mathrm{D}_n)) =& Sz(\mathcal{P}(\mathbb{Z}_n^*)) + \sum_{i=1}^{2}\Bigg(n+1+\sum_{\substack{k=1\\ d_k\nsim d_i}}^{2}\varphi(d_k)\Bigg)\varphi(d_i) + n(2n-1+\varphi(n)) + \varphi(n) \\
=& Sz(\mathcal{P}(\mathbb{Z}_n^*)) + (n+1+\varphi(d_1)) + (n+1+\varphi(d_2))\varphi(d_1) \\
&+ n(2n-1+\varphi(n)) + \varphi(n) \\
=& Sz(\mathcal{P}(\mathbb{Z}_n^*)) + (pq+1+(p-1))(q-1) + (pq+1+(q-1))(p-1) \\
&+ pq(3pq-p-q) + (p-1)(q-1) \\
=& Sz(\mathcal{P}(\mathbb{Z}_n^*)) + p(q^2-1) + q(p^2-1) + pq(3pq-p-q) + (p-1)(q-1).
\end{aligned}$$

□

4. SAGE CODE

In this section, we provide the corresponding SAGE codes for calculating the Szeged index of $\mathcal{P}(\mathbb{Z}_n)$ and $\mathcal{P}(\mathrm{D}_n)$. On providing a given value of n in the codes provided below, Code 1 gives a plot of $\mathcal{P}(\mathbb{Z}_n)$, and Code 2 gives the Szeged index of $\mathcal{P}(\mathbb{Z}_n)$. Moreover, Code 3 gives a plot of $\mathcal{P}(\mathrm{D}_n)$, and Code 4 gives the Szeged index of $\mathcal{P}(\mathrm{D}_n)$.

Listing 1. Plot of $\mathcal{P}(\mathbb{Z}_n)$.

```
G=Graph()
E=[]
n=21
for i in range (n):
for j in range (n):
```

```
if(i!=j):
for k in range(n):
if(((k*i)%n==j) or ((k*j)%n==i)):
E.append((i,j))
G.add_edges(E)
G.plot()
```

Listing 2. Szeged index of $\mathcal{P}(\mathbb{Z}_n)$.

```
G=Graph()
E=[]
n=21
for i in range (n):
for j in range (n):
if(i!=j):
for k in range(n):
if(((k*i)%n==j) or ((k*j)%n==i)):
E.append((i,j))
G.add_edges(E)
S=G.szeged_index()
print("Szeged index of power graph of Zn=",S)
```

Listing 3. Plot of $\mathcal{P}(\mathrm{D}_n)$.

```
G=Graph()
E=[]
n=10
for i in range (n):
for j in range (n):
if(i!=j):
for k in range(n):
if(((k*i)%n==j) or ((k*j)%n==i)):
E.append((i,j))
G.add_edges(E)
for i in range(n,2*n):
G.add_vertex(i)
G.add_edge([0,i])
G.plot()
```

Listing 4. Szeged index of $\mathcal{P}(\mathrm{D}_n)$.

```
G=Graph()
E=[]
n=10
for i in range (n):
for j in range (n):
```

```
if(i!=j):
for k in range(n):
if(((k*i)%n==j) or ((k*j)%n==i)):
E.append((i,j))
G.add_edges(E)
for i in range(n,2*n):
G.add_vertex(i)
G.add_edge([0,i])
S=G.szeged_index()
print("Szeged index of power graph of dihedral group=",S)
```

5. Conclusion

In this paper, we have studied the Szeged index of the power graph of finite cyclic and dihedral groups. We first obtained a formula for the Szeged index of the generalized join of graphs in terms of the constituent graphs. Using it, we first determined the Szeged index of the power graph of finite cyclic groups. We further obtain a relation between the Szeged index of the power graph of dihedral groups and the Szeged index of the power graph of finite cyclic groups. Finally, we provide SAGE codes for evaluating the Szeged indices of the power graph of finite cyclic groups and dihedral groups.

6. Conflict of Interest

The author states that there is no conflict of interest.

References

[1] Kharkongor, D., Boro, L., Singh, M. M., and Dutta, S., (2023), Topological indices of total graph of the ring Zn × Zm, TWMS Journal Of Applied And Engineering Mathematics, 13 (4), pp. 1434–1445.

[2] Banerjee, S., (2023), On structural and spectral properties of reduced power graph of finite groups, Asian-European Journal of Mathematics, 16 (9), p. 2350170.

[3] Banerjee, S., and Adhikari, A., (2023), On spectra of power graphs of finite cyclic and dihedral groups, Rocky Mountain Journal of Mathematics, 53 (2), pp. 341-356.

[4] Rehman, M. U., Salman, M., Khan, S., Maden, A. D., and Ali, F., (2022), Mostar index of graphs associated to groups, Main Group Metal Chemistry, 45 (1), pp. 124–135.

[5] Salman, M., Noreen, T., Rehman, M. U., Cao, J., and Abbas, M. Z., (2022), Non-commuting graph of the dihedral group determined by Hosoya parameters, Alexandria Engineering Journal, 61 (5), pp. 3709–3717.

[6] Liu, H., (2022), On revised Szeged index of a class of unicyclic graphs, Discrete Mathematics, Algorithms and Applications, 14 (2), p. 2150115.

[7] Banerjee, S., and Adhikari, A., (2021), On spectra and spectral radius of Signless Laplacian of power graphs of some finite groups, Asian-European Journal of Mathematics, 14 (6), p. 2150090.

[8] Abbas, G., Rani, A., Salman, M., Noreen, T., and Ali, U., (2021), Hosoya properties of the commuting graph associated with the group of symmetries, Main Group Metal Chemistry, 44 (1), pp. 173–184.

[9] Banerjee, S., and Adhikari, A., (2020), Signless Laplacian spectrum of power graphs of finite cyclic groups, AKCE International Journal of Graphs and Combinatorics, 17 (1), pp. 356-366.

[10] Alimon, N., Sarmin, N., and Erfanian, A., (2020), The Szeged and Wiener indices for coprime graph of dihedral groups, in AIP Conference Proceedings, AIP Publishing, vol. 2266.

[11] Bonamy, M., Knor, M., Lužar, B., Pinlou, A., and Škrekovski, R., (2017), On the difference between the Szeged and the Wiener index, Applied Mathematics and Computation, 312, pp. 202–213.

[12] Azari, M., (2016), Some variants of the Szeged index under rooted product of graphs, Miskolc Mathematical Notes, 17 (2), pp. 761–775.

[13] Knor, M., Škrekovski, R., and Tepeh, A., (2016), Mathematical aspects of Wiener index, Ars Math. Contemp., 11 (2), pp. 327–352.

[14] Mehranian, Z., Gholami, A., and Ashrafi, A. R., (2016), A note on the power graph of a finite group, International Journal of Group Theory, 5 (1), pp. 1–10.
[15] Abawajy, J., Kelarev, A., and Chowdhury, M., (2013), Power graphs: A survey, Electronic Journal of Graph Theory and Applications (EJGTA), 1 (2), pp. 125–147.
[16] Chen, L., Li, X., Liu, M., and Gutman, I., (2012), On a relation between Szeged and Wiener indices of bipartite graphs, Transactions on Combinatorics, 1 (4), pp. 43–49.
[17] Gutman, I., and Polansky, O. E., (2012), Mathematical concepts in organic chemistry, Springer Science & Business Media.
[18] Cameron, P. J., and Ghosh, S., (2011), The power graph of a finite group, Discrete Mathematics, 311 (13), pp. 1220–1222.
[19] Cameron, P. J., (2010), The power graph of a finite group, II, J. Group Theory, 13 (6), pp. 779–783.
[20] Chakrabarty, I., Ghosh, S., and Sen, M., (2009), Undirected power graphs of semigroups, in Semigroup Forum, Springer, 78, pp. 410–426.
[21] Mansour, T., and Schork, M., (2009), The vertex pi index and Szeged index of bridge graphs, Discrete Applied Mathematics, 157 (7), pp. 1600–1606.
[22] Khalifeh, M., Yousefi-Azari, H., and Ashrafi, A. R., (2008), A matrix method for computing Szeged and vertex pi indices of join and composition of graphs, Linear Algebra and its Applications, 429 (11-12), pp. 2702–2709.
[23] Kelarev, A., and Quinn, S., (2002), Directed graphs and combinatorial properties of semigroups, Journal of Algebra, 251 (1), pp. 16–26.
[24] Dobrynin, A. A., Entringer, R., and Gutman, I., (2001), Wiener index of trees: Theory and applications, Acta Applicandae Mathematica, 66 (3), pp. 211–249.
[25] Gutman, I., and Dobrynin, A. A., (1998), The Szeged index–a success story, Graph Theory Notes NY, 34, pp. 37–44.
[26] Klavžar, S., Rajapakse, A., and Gutman, I., (1996), The Szeged and the Wiener index of graphs, Applied Mathematics Letters, 9 (5), pp. 45–49.
[27] Gutman, I., Khadikar, P., Rajput, P., and Karmarkar, S., (1995), The Szeged index of polyacenes, Journal of the Serbian Chemical Society, 60, pp. 759–759.
[28] Dobrynin, A., and Gutman, I., (1994), On a graph invariant related to the sum of all distances in a graph, Publ. Inst. Math. (Beograd) (N.S.), 56(70), pp. 18–22.
[29] Gutman, I., (1994), A formula for the Wiener number of trees and its extension to graphs containing cycles, Graph Theory Notes NY, 27 (9), pp. 9–15.
[30] Gutman, I., Yeh, Y.-N., Lee, S.-L., and Luo, Y.-L., (1993), Some recent results in the theory of the Wiener number, Indian Journal of Chemistry, 32 (A), pp. 651–661.
[31] Lukovits, I., (1992), Correlation between components of the Wiener index and partition coefficients of hydrocarbons, International Journal of Quantum Chemistry, 44 (S19), pp. 217–223.
[32] Lukovits, I., (1990), Wiener indices and partition coefficients of unsaturated hydrocarbons, Quantitative Structure-Activity Relationships, 9 (3), pp. 227–231.
[33] Schwenk, A. J., (1974), Computing the characteristic polynomial of a graph, in Graphs and Combinatorics, Springer, pp. 153–172.
[34] Wiener, H., (1947), Structural determination of paraffin boiling points, Journal of the American Chemical Society, 69 (1), pp. 17–20.

Dr. Subarsha Banerjee is currently serving as an Assistant Professor in the Department of Mathematics at JIS University, India. He completed his Doctor of Philosophy (Ph.D.) in 2022 from the University of Calcutta. Dr. Banerjee has also been awarded a Postdoctoral Fellowship from Banaras Hindu University. He has authored several research papers which have been published in reputed journals. His research areas include Spectral Graph Theory, Combinatorics, and Topological Graph Theory.