TWMS J. App. and Eng. Math. V.15, N.9, 2025, pp. 2181-2191

INTRODUCING A NOVEL SUBCLASS OF HARMONIC FUNCTIONS WITH CLOSE-TO-CONVEX PROPERTIES

S. ÇAKMAK[1*], S. YALÇIN[2], §

Abstract. In this paper, we introduce a new subclass of close-to-convex harmonic functions. We present a sufficient coefficient condition for a function to be a member of this class. Furthermore, we establish a distortion theorem. These results lay the groundwork for extending the findings to function classes involving higher-order derivatives.

Keywords: Close-to-convex functions, Harmonic functions, Starlike functions, Distortion

AMS Subject Classification: 30C45, 30C50

1. Introduction

In the realm of harmonic functions, every function $\mathfrak{f}$ belonging to the class $\mathcal{SH}^0$ can be expressed as $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where

$$\mathfrak{u}(z) = z + \sum_{m=2}^{\infty} u_m z^m, \quad \mathfrak{v}(z) = \sum_{m=2}^{\infty} v_m z^m \tag{1}$$

with both functions being analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. Provided that $|\mathfrak{u}'(z)| > |\mathfrak{v}'(z)|$ in $\mathbb{E}$, $\mathfrak{f}$ is locally univalent and sense-preserving in $\mathbb{E}$. It's noteworthy that when $\mathfrak{v}(z)$ is identically zero, $\mathcal{SH}^0$ contracts to class $\mathcal{S}$.

The subclasses of $\mathcal{S}$ that map $\mathbb{E}$ onto starlike and close-to-convex domains, respectively, are denoted by $\mathcal{S}^*$ and $\mathcal{K}$. Similarly, $\mathcal{SH}^{0,*}$ and $\mathcal{KH}^0$ represent subclasses of $\mathcal{SH}^0$ that map $\mathbb{E}$ onto their respective domains. (For further details, refer to [1, 3])

In 2005, Gao and Zhou [4] introduced the class

$$\mathcal{K}_s = \left\{ \mathfrak{u} \in \mathcal{S} : \operatorname{Re}\left\{ \frac{z^2 \mathfrak{u}'(z)}{-\phi(z)\phi(-z)} \right\} > 0 \text{ for } z \in \mathbb{E} \right\}.$$

[1] Istanbul Gelisim University, Faculty of Economics, Administrative and Social Sciences, Department of Management Information Systems, Istanbul, Türkiye.
e-mail: secakmak@gelisim.edu.tr; ORCID: https://orcid.org/0000-0003-0368-7672.
[2] Bursa Uludag University, Faculty of Arts and Sciences, Department of Mathematics, Görükle, Bursa, Türkiye.
e-mail: syalcin@uludag.edu.tr; ORCID: https://orcid.org/0000-0002-0243-8263.
[*] Corresponding author.
§ Manuscript received: August 21, 2024; accepted:January 18, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.9;

In 2011, Şeker [5] introduced a class

$$\mathcal{K}(k,\gamma)=\left\{\mathfrak{u}\in\mathcal{S}:\mathrm{Re}\left\{\frac{z^k\mathfrak{u}'(z)}{\phi_k(z)}\right\}>\gamma,\quad 0\le\gamma<1 \text{ and } z\in\mathbb{E}\right\}$$

where $k\in\mathbb{Z}^+$, $\phi(z)=z+\sum_{m=2}^{\infty}c_m z^m\in S^*\left(\frac{k-1}{k}\right)$ and the definition of $\phi_k(z)$ is as follows:

$$\phi_k(z)=\prod_{v=0}^{k-1}\mu^{-v}\phi(\mu^v z)\quad(\mu^k=1) \tag{2}$$

with $\mu=e^{2\pi i/k}$.

The classes $\mathcal{K}_s$ and $\mathcal{K}(k,\gamma)$, introduced by Gao and Zhou [4] and Şeker [5], respectively, focus solely on analytic functions and are independent of the variable $\overline{z}$. This restriction prevents the study of properties of harmonic functions that depend on the variable $\overline{z}$. This leads us to define a new function class that includes the co-analytic part of harmonic functions.

Observe that if $k=2$ in the class $\mathcal{K}(k,\gamma)$, the class $\mathcal{K}(\gamma)$ studied by Kowalczyk et al. [2] is obtained. For $k=2$ and $\gamma=0$, the class $\mathcal{K}_s$ studied by Gao and Zhou [4] is obtained. It is clear that the class $\mathcal{K}(k,\gamma)$ encompasses both classes. The class $\mathcal{KH}^0(k,\gamma)$, which we will define shortly, covers the class $\mathcal{K}(k,\gamma)$ since it is defined using harmonic conjugates of analytic functions belonging to the classes $\mathcal{K}(k,\gamma)$. Therefore, the class $\mathcal{KH}^0(k,\gamma)$ is a generalization of the $\mathcal{K}(k,\gamma)$, $\mathcal{K}_s$, and $\mathcal{K}(\gamma)$ classes.

This generalization allows for a broader investigation of geometric properties, such as distortion bounds and close-to-convexity, for harmonic functions that depend on $\overline{z}$.

Definition 1.1. *The class $\mathcal{KH}^0(k,\gamma)$ is defined as the collection of functions $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{SH}^0$ that adhere to the following inequality:*

$$\mathrm{Re}\left\{\frac{z^k\mathfrak{u}'(z)}{\phi_k(z)}-\gamma\right\}>\left|\frac{z^k\mathfrak{v}'(z)}{\phi_k(z)}\right|, \tag{3}$$

where $0\le\gamma<1$ and $\phi_k(z)$ is given by (2).

Specifically, when $\mathfrak{v}(z)\equiv 0$, the class $\mathcal{KH}^0(k,\gamma)$ reduces to the class $\mathcal{K}(k,\gamma)$. Also, by setting $\mathfrak{v}(z)\equiv 0$, $k=2$ and $\gamma=0$, we obtain $\mathcal{KH}^0(2,0)=\mathcal{K}_s$. The inclusion of the co-analytic term $\overline{\mathfrak{v}(z)}$ extends these classes, providing a more general framework that accommodates both analytic and co-analytic components. Additionally, when $\phi(z)=z$, and by appropriately selecting the parameters, the class $\mathcal{KH}^0(k,\gamma)$ can be reduced to several well-known subclasses of harmonic functions, as outlined below.

i $\mathcal{KH}^0(k,0)=\mathcal{PH}^0$ [6].
ii $\mathcal{KH}^0(k,\gamma)=\mathcal{PH}^0(\gamma)$ [7, 8].
ii $\mathcal{KH}^0(k,0)=\mathcal{WH}^0(0)$ ([9]).
iv $\mathcal{KH}^0(k,\gamma)=\mathcal{WH}^0(0,\gamma)$ ([10]).
v $\mathcal{KH}^0(k,\gamma)=\mathcal{AH}^0(1,0,\gamma)$ ([11]).
vi $\mathcal{KH}^0(k,0)=\mathcal{RH}^0(0,0)$ ([12]).

For further details on harmonic function classes defined by differential inequality, see [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

In this work, we investigate the distortion theorems and coefficient bounds for functions in the class $\mathcal{KH}^0(k,\gamma)$ and demonstrate that functions within this class exhibit close-to-convex behavior.

2. Examples of Functions in the Class $\mathcal{KH}^0(k,\gamma)$

Example 2.1. *Let* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} = z + \frac{1-\gamma}{m}\overline{z}^m$ *and* $\phi(z) = z$. *For* $0 \le \gamma < 1$ *and* $|z| < 1$, *we have*

$$\operatorname{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)} - \gamma\right\} = 1 - \gamma > (1-\gamma)\,|z|^{m-1} = \left|\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right|.$$

Hence, $\mathfrak{f} \in \mathcal{KH}^0(k,\gamma)$.

The following examples can be given for the special case of the parameters in Example 2.1.

Example 2.2. *Let* $\mathfrak{f} = z + \frac{99}{200}\overline{z}^2$, $\gamma = \frac{1}{100}$ *and* $\phi(z) = z$. *Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{33}{100})$. *The unit disk is mapped to a starlike region by the function* f. *The depiction in Figure 1 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f}(z) = z + \frac{99}{200}\overline{z}^2$.

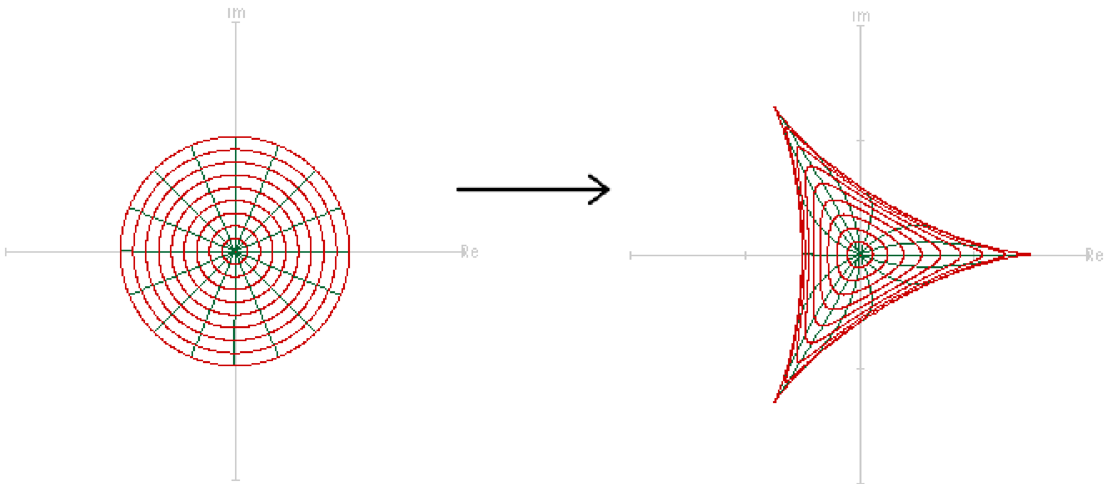

Figure 1. Under the map $\mathfrak{f} = z + \frac{99}{200}\overline{z}^2$, the image of the unit disk.

Example 2.3. *Let* $\mathfrak{f} = z + \frac{1}{10}\overline{z}^2$, $\gamma = \frac{4}{5}$, *and* $\phi(z) = z$. *Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{4}{5})$. *The unit disk is mapped to a convex region by the function* f. *The depiction in Figure 2 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{1}{10}\overline{z}^2$.

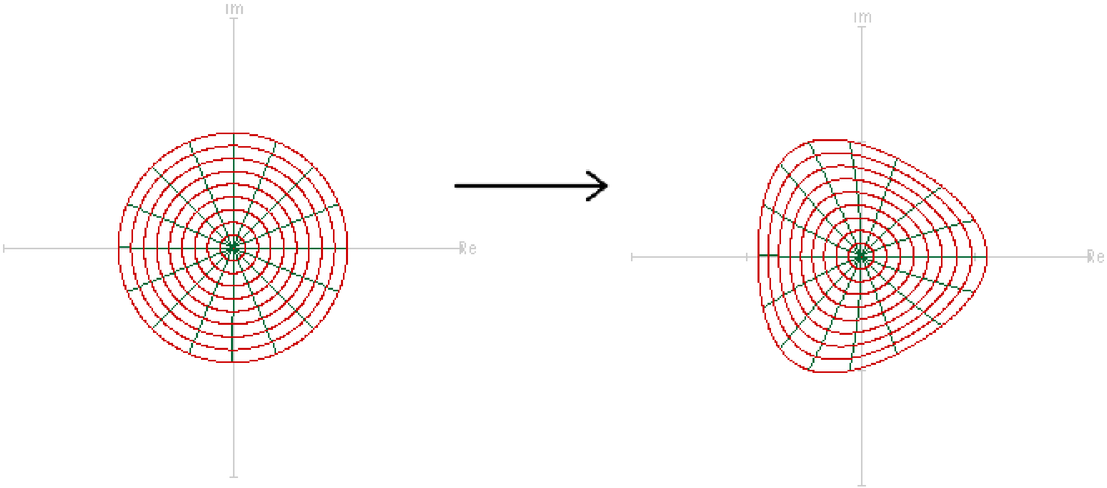

FIGURE 2. Under the map $\mathfrak{f} = z + \frac{1}{10}\overline{z}^2$, the image of the unit disk.

Example 2.4. *Let* $\mathfrak{f} = z + \frac{33}{100}\overline{z}^3$, $\gamma = \frac{1}{100}$ *and* $\phi(z) = z$. *Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{33}{100})$. *The unit disk is mapped to a starlike region by the function* f. *The depiction in Figure 3 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{33}{100}\overline{z}^3$.

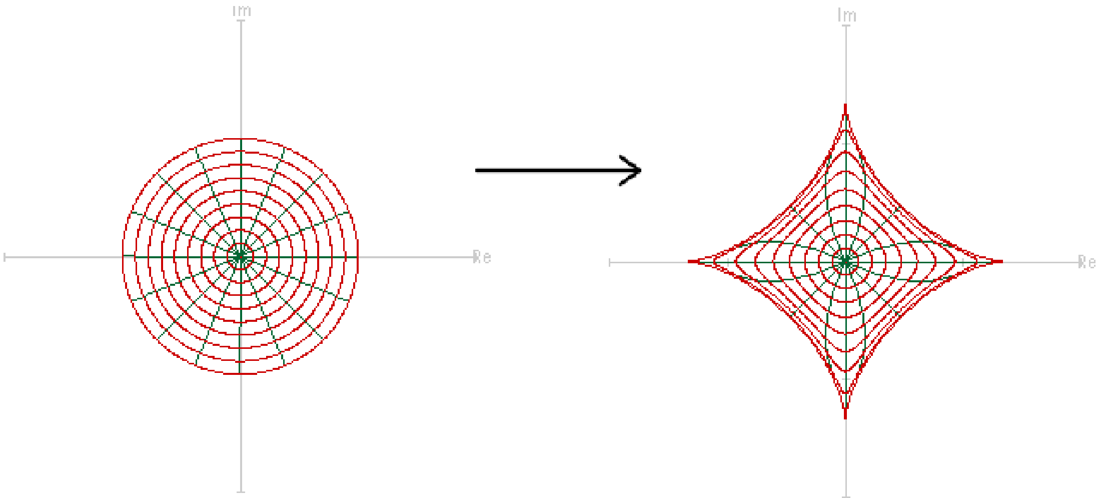

FIGURE 3. Under the map $\mathfrak{f} = z + \frac{33}{100}\overline{z}^3$, the image of the unit disk.

Example 2.5. *Let* $\mathfrak{f} = z + \frac{1}{15}\overline{z}^3$, $\gamma = \frac{4}{5}$ *and* $\phi(z) = z$. *Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{4}{5})$. *The unit disk is mapped to a convex region by the function* f. *The depiction in Figure 4 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{1}{15}\overline{z}^3$.

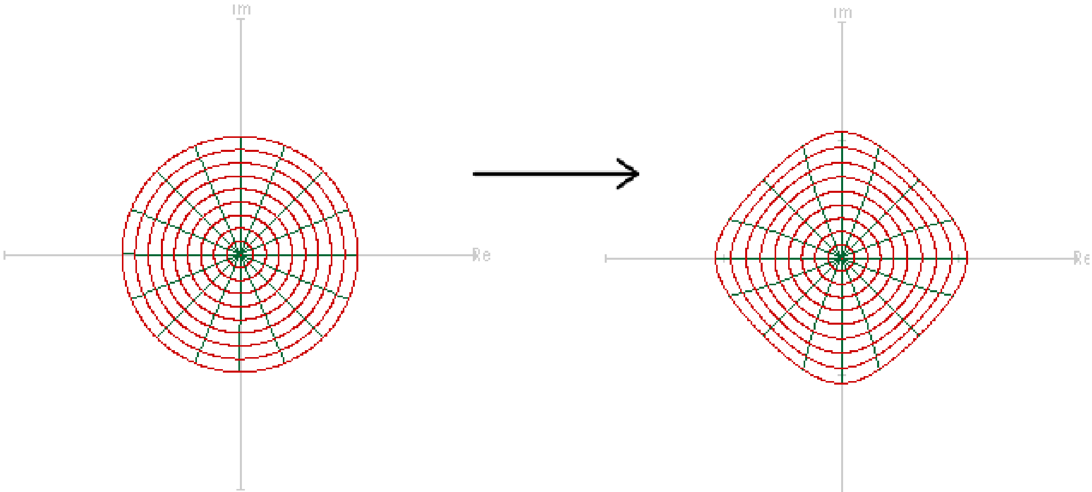

FIGURE 4. Under the map $\mathfrak{f} = z + \frac{1}{15}\overline{z}^3$, the image of the unit disk.

Example 2.6. *Let* $\mathfrak{f} = z + \frac{99}{500}\overline{z}^5$, $\gamma = \frac{1}{100}$ *and* $\phi(z) = z$. *Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{99}{500})$. *The unit disk is mapped to a starlike region by the function* f. *The depiction in Figure 5 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{99}{500}\overline{z}^5$.

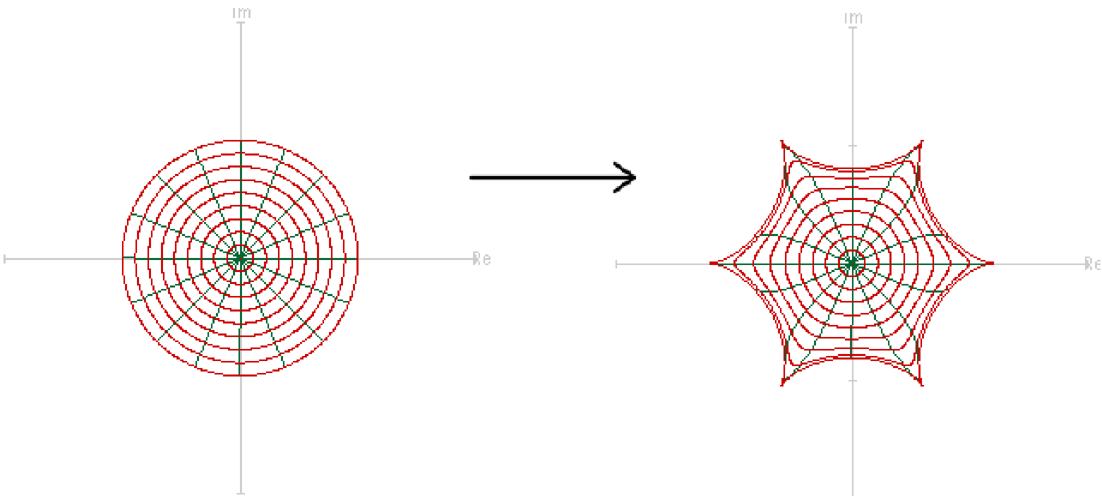

FIGURE 5. Under the map $\mathfrak{f} = z + \frac{99}{500}\overline{z}^5$, the image of the unit disk.

Example 2.7. *Let* $\mathfrak{f} = z + \frac{1}{25}\overline{z}^5$, $\gamma = \frac{4}{5}$ *and* $\phi(z) = z$. *Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{4}{5})$. *The unit disk is mapped to a convex region by the function* f. *The depiction in Figure 6 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{1}{25}\overline{z}^5$.

3. Geometric Properties of the class $\mathcal{KH}^0(k, \gamma)$

First, we give a result that establishes a sufficient condition for $\mathfrak{f} \in \mathcal{SH}^0$ to be close-to-convex, which comes from Clunie and Sheil-Small [1].

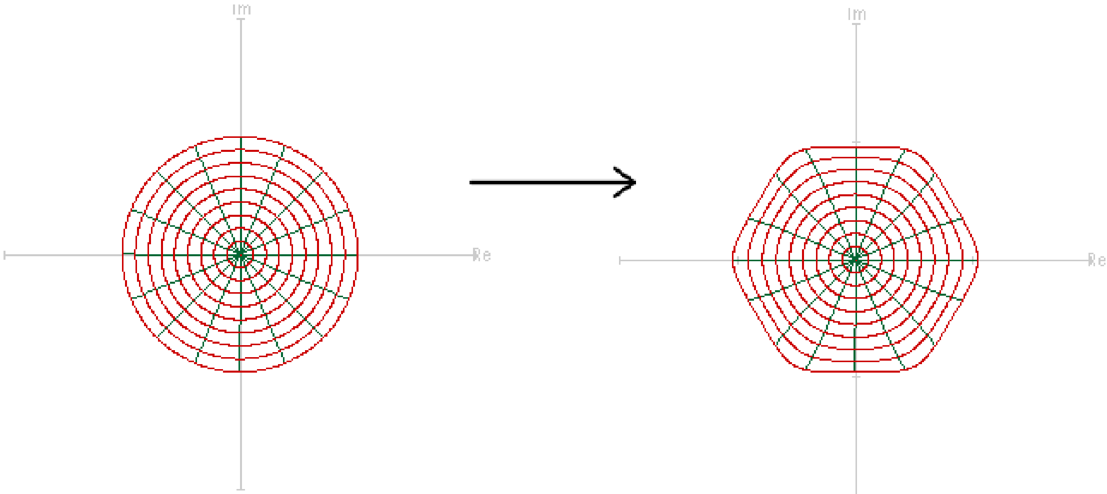

FIGURE 6. Under the map $\mathfrak{f} = z + \frac{1}{25}\overline{z}^5$, the image of the unit disk.

Lemma 3.1. *Let* $\mathfrak{u}$ *and* $\mathfrak{v}$ *be analytic functions in* $\mathbb{E}$*, such that* $|\mathfrak{v}'(0)| < |\mathfrak{u}'(0)|$*, and for each* ε $(|\varepsilon| = 1)$*,* $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v}$ *is close-to-convex. Then,* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ *is close-to-convex in* $\mathbb{E}$*.*

The result we will present now establishes a connection between the $\mathcal{KH}^0(k,\gamma)$ harmonic function class and the $\mathcal{K}(k,\gamma)$ analytic function class.

Theorem 3.1. $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$ *if and only if* $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v} \in \mathcal{K}(k,\gamma)$ *for each* ε $(|\varepsilon| = 1)$*.*

Proof. Assume $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$. For each ε $(|\varepsilon| = 1)$, we have

$$\begin{aligned} &\operatorname{Re}\left\{\frac{z^k F_\varepsilon{}'(z)}{\phi_k(z)}\right\} \\ &= \operatorname{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)}\right\} + \varepsilon \operatorname{Re}\left\{\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right\} \\ &> \operatorname{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)}\right\} - \left|\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right| > \gamma \ \ (z \in \mathbb{E}). \end{aligned}$$

Hence, $F_\varepsilon \in \mathcal{K}(k,\gamma)$ for each ε $(|\varepsilon| = 1)$.

Conversely, suppose $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v} \in \mathcal{K}(k,\gamma)$. Then,

$$\operatorname{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)}\right\} > \operatorname{Re}\left[-\varepsilon\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right] + \gamma \ \ (z \in \mathbb{E}).$$

Choosing an appropriate ε $(|\varepsilon| = 1)$ yields

$$\operatorname{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)} - \gamma\right\} > \left|\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right| \ \ (z \in \mathbb{E}),$$

and thus $\mathfrak{f} \in \mathcal{KH}^0(k,\gamma)$. □

We will now demonstrate that harmonic functions belonging to the class $\mathcal{KH}^0(k,\gamma)$ map the open unit disk onto a close-to-convex region. To this end, we first state the following lemma and subsequently establish that functions in the class $\mathcal{K}(k,\gamma)$ are close-to-convex within the open unit disk.

Lemma 3.2. [4] *Let* $\phi(z) = z + \sum_{m=2}^{\infty} c_m z^m \in \mathcal{S}^*\left(\frac{k-1}{k}\right)$ *for* $k \geq 1$. *Then,*

$$\Phi_k(z) = \frac{\phi_k(z)}{z^{k-1}} = z + \sum_{m=2}^{\infty} C_m z^m \in S^* \tag{4}$$

where $\phi_k(z)$ *is given by* (2).

Theorem 3.2. *If* F *is a function in the class* $\mathcal{K}(k,\gamma)$, *then* F *is close-to-convex of order* γ *in the region* $\mathbb{E}$.

Proof. Let $F \in \mathcal{K}(k,\gamma)$. We have

$$\mathrm{Re}\left\{\frac{z^k F'(z)}{\phi_k(z)}\right\} = \mathrm{Re}\left\{\frac{zF'(z)}{\Phi_k(z)}\right\} > \gamma,$$

where $\Phi_k(z)$ is given by (4). Therefore, the function F is close-to-convex of order γ since $\Phi_k(z) \in \mathcal{S}^*$. □

Theorem 3.3. *Every function in the class* $\mathcal{KH}^0(k,\gamma)$ *is close-to-convex within the region* $\mathbb{E}$.

Proof. Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ belong to class $\mathcal{KH}^0(k,\gamma)$. Then by Theorem 3.1 the function $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v}$ belongs to class $\mathcal{K}(k,\gamma)$ and by Theorem 3.2 also close to convex in $\mathbb{E}$. Therefore, by Lemma 3.1, $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$ is also close to convex in $\mathbb{E}$. □

In the following result, we derive a coefficient bound for functions belonging to the class $\mathcal{KH}^0(k,\gamma)$.

Theorem 3.4. *Let* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$. *For* $m \geq 2$, *the following inequalities hold:*

$$|u_m| + |v_m| \leq \gamma + m(1-\gamma).$$

For the function $\mathfrak{f}(z) = z + [\gamma + m(1-\gamma)]z^m$, *every outcome is sharp and every equality is holds.*

Proof. Suppose that $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$. $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v} \in \mathcal{K}(k,\gamma)$ for ε ($|\varepsilon| = 1$), according to Theorem 3.1. With respect to every ε ($|\varepsilon| = 1$), we possess

$$\mathrm{Re}\left\{\frac{z^k F_\varepsilon{}'(z)}{\phi_k(z)}\right\} = \mathrm{Re}\left\{\frac{zF_\varepsilon{}'(z)}{\Phi_k(z)}\right\} = \mathrm{Re}\left\{\frac{z\left(\mathfrak{u}'(z) + \varepsilon\mathfrak{v}'(z)\right)}{\Phi_k(z)}\right\} > \gamma.$$

for $z \in \mathbb{E}$. On the other hand, there is an analytic function $\mathfrak{P}(z) = 1 + \sum\limits_{m=1}^{\infty} \mathfrak{p}_m z^m$ in $\mathbb{E}$ whose real part is positive, satisfying

$$\frac{z\left(\mathfrak{u}'(z) + \varepsilon\mathfrak{v}'(z)\right)}{\Phi_k(z)} = \gamma + (1-\gamma)\,\mathfrak{P}(z). \tag{5}$$

or

$$z\left(\mathfrak{u}'(z) + \varepsilon\mathfrak{v}'(z)\right) = \left[\gamma + (1-\gamma)\,\mathfrak{P}(z)\right]\Phi_k(z). \tag{6}$$

Upon comparing the coefficients in (6), it can be observed that

$$m(u_m + \varepsilon v_m) = C_m + (1-\gamma)\mathfrak{p}_{m-1} + (1-\gamma)\mathfrak{p}_1 C_{m-1} + \cdots + (1-\gamma)\mathfrak{p}_{m-2}C_2. \tag{7}$$

Since $\Phi_k(z)$ is starlike, we have $|C_m| \leq m$, and since $Re\{\mathfrak{P}(z)\} > 0$, we have $|\mathfrak{p}_m| \leq 2$ for $m \geq 1$. Hence, by equation 7, we have

$$m\left|u_m + \varepsilon v_m\right| \leq m\left[\gamma + m(1-\gamma)\right]. \tag{8}$$

Since ε ($|\varepsilon| = 1$) is arbitrary, it follows that the proof is concluded. The function $\mathfrak{f}(z) = z + [\gamma + m(1-\gamma)]z^m$, demonstrates the sharpness of inequality. □

Now, we provide the necessary coefficient condition for a harmonic function to belong to the class $\mathcal{KH}^0(k,\gamma)$.

Theorem 3.5. *Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{SH}^0$ with the series expansions given by (1). If the following inequality holds:*

$$\sum_{m=2}^{\infty}2m\left(|u_m|+|v_m|\right)+\sum_{m=2}^{\infty}\left(|1-2\gamma|+1\right)|C_m|\leq 2(1-\gamma), \tag{9}$$

then $\mathfrak{f}\in\mathcal{KH}^0(k,\gamma)$.

Proof. Consider $\mathfrak{u}$ and $\mathfrak{v}$ as functions with series expansions given by (1). Let $F_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ for each ε ($|\varepsilon|=1$). Define the functions $\phi_k(z)$ and $\Phi_k(z)$ as given by (2) and (4), respectively. Then we can express A as

$$\begin{aligned}
A &= \left|zF_\varepsilon'-\frac{\phi_k(z)}{z^{k-1}}\right|-\left|zF_\varepsilon'+\frac{(1-2\gamma)\phi_k(z)}{z^{k-1}}\right| \\
&= \left|z(\mathfrak{u}+\varepsilon\mathfrak{v})'-\Phi_k(z)\right|-\left|z(\mathfrak{u}+\varepsilon\mathfrak{v})'+(1-2\gamma)\Phi_k(z)\right| \\
&= \left|\sum_{m=2}^{\infty}m(u_m+\varepsilon v_m)z^m-\sum_{m=2}^{\infty}C_mz^m\right| \\
&- \left|(2-2\gamma)z+\sum_{m=2}^{\infty}m(u_m+\varepsilon v_m)z^m+(1-2\gamma)\sum_{m=2}^{\infty}C_mz^m\right| \\
&\leq \sum_{m=2}^{\infty}m\left|u_m+\varepsilon v_m\right||z|^m+\sum_{m=2}^{\infty}|C_m||z|^m \\
&- \left((2-2\gamma)|z|-\sum_{m=2}^{\infty}m\left|u_m+\varepsilon v_m\right||z|^m-|1-2\gamma|\sum_{m=2}^{\infty}|C_m||z|^m\right) \\
&< \left\{-2(1-\gamma)+\sum_{m=2}^{\infty}2m\left|u_m+\varepsilon v_m\right|+\left(|1-2\gamma|+1\right)\sum_{m=2}^{\infty}|C_m|\right\}|z|
\end{aligned}$$

Since ε ($|\varepsilon|=1$) is arbitrary, and from the inequality (9), we obtain that $A<0$. This implies that $F_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ belongs to the class $\mathcal{K}(k,\gamma)$, and consequently, according to Theorem 2, it shows that $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ belongs to the class $\mathcal{KH}^0(k,\gamma)$. □

The result we present now provides the distortion bounds for functions in the class $\mathcal{KH}^0(k,\gamma)$.

Theorem 3.6. *Assuming $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{KH}^0(k,\gamma)$, the following inequalities hold for all z:*

$$|z|+\sum_{m=2}^{\infty}(-1)^{m-1}[m(1-\gamma)+\gamma]\,|z|^m\leq|\mathfrak{f}(z)|\leq|z|+\sum_{m=2}^{\infty}[m(1-\gamma)+\gamma]\,|z|^m.$$

These are sharp inequality for the function $\mathfrak{f}(z)=z+\sum\limits_{m=2}^{\infty}[m(1-\gamma)+\gamma]z^m$.

Proof. Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{KH}^0(k,\gamma)$. Then using Theorem 3.1, $F_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}\in\mathcal{K}(k,\gamma)$ for each ε ($|\varepsilon|=1$). Additionally, from Theorem 3.4 in [5], we obtain

$$\frac{1-(1-2\gamma)|z|}{(1+|z|)^3}\leq\left|F_\varepsilon'(z)\right|\leq\frac{1+(1-2\gamma)|z|}{(1-|z|)^3} \tag{10}$$

Since

$$\begin{aligned}\left|F_\varepsilon'(z)\right| &= \left|\mathfrak{u}'(z)+\varepsilon\mathfrak{v}'(z)\right| \\ &\leq 1+\sum_{m=2}^{\infty} m[m(1-\gamma)+\gamma]\,|z|^{m-1}\end{aligned}$$

and

$$\begin{aligned}\left|F_\varepsilon'(z)\right| &= \left|\mathfrak{u}'(z)+\varepsilon\mathfrak{v}'(z)\right| \\ &\geq 1+\sum_{m=2}^{\infty} (-1)^{m-1} m[m(1-\gamma)+\gamma]\,|z|^{m-1},\end{aligned}$$

in particular, we get

$$\left|\mathfrak{u}'(z)\right|+\left|\mathfrak{v}'(z)\right| \leq 1+\sum_{m=2}^{\infty} m[m(1-\gamma)+\gamma]\,|z|^{m-1}$$

and

$$\left|\mathfrak{u}'(z)\right|-\left|\mathfrak{v}'(z)\right| \geq 1+\sum_{m=2}^{\infty} (-1)^{m-1} m[m(1-\gamma)+\gamma]\,|z|^{m-1}.$$

Assume Γ is the radial segment extending from 0 to z, so

$$\begin{aligned}|\mathfrak{f}(z)| &= \left|\int_\Gamma \frac{\partial\mathfrak{f}}{\partial\mathfrak{z}}d\mathfrak{z}+\frac{\partial\mathfrak{f}}{\partial\bar{\mathfrak{z}}}d\bar{\mathfrak{z}}\right| \leq \int_\Gamma \left(\left|\mathfrak{u}'(\mathfrak{z})\right|+\left|\mathfrak{v}'(\mathfrak{z})\right|\right)|d\mathfrak{z}| \\ &\leq \int_0^{|z|}\left(1+\sum_{m=2}^{\infty} m[m(1-\gamma)+\gamma]\,|\rho|^{m-1}\right)d\rho \\ &= |z|+\sum_{m=2}^{\infty}[m(1-\gamma)+\gamma]\,|z|^{m},\end{aligned}$$

and

$$\begin{aligned}|\mathfrak{f}(z)| &\geq \int_\Gamma \left(\left|\mathfrak{u}'(\mathfrak{z})\right|-\left|\mathfrak{v}'(\mathfrak{z})\right|\right)|d\mathfrak{z}| \\ &\geq \int_0^{|z|}\left(1+\sum_{m=2}^{\infty} (-1)^{m-1} m[m(1-\gamma)+\gamma]\,|\rho|^{m-1}\right)d\rho \\ &= |z|+\sum_{m=2}^{\infty}(-1)^{m-1}[m(1-\gamma)+\gamma]\,|z|^{m}.\end{aligned}$$

□

Theorem 3.7. *The class $\mathcal{KH}^0(k,\gamma)$ is closed under convex combinations.*

Proof. Suppose $\mathfrak{f}_\alpha=\mathfrak{u}_\alpha+\overline{\mathfrak{v}_\alpha}\in\mathcal{KH}^0(k,\gamma)$ for $\alpha=1,2,...,p$ and $\sum_{\alpha=1}^{p} s_\alpha=1$ $(0\leq s_\alpha\leq 1)$. The convex combination of functions $\mathfrak{f}_\alpha$ $(\alpha=1,2,...,p)$ can be expressed as:

$$\mathfrak{f}(z)=\sum_{\alpha=1}^{p} s_\alpha\mathfrak{f}_\alpha(z)=\mathfrak{u}(z)+\overline{\mathfrak{v}(z)},$$

where

$$\mathfrak{u}(z) = \sum_{\alpha=1}^{p} s_\alpha \mathfrak{u}_\alpha(z) \text{ and } \mathfrak{v}(z) = \sum_{\alpha=1}^{p} s_\alpha \mathfrak{v}_\alpha(z).$$

Both $\mathfrak{u}$ and $\mathfrak{v}$ are analytic within the open unit disk $\mathbb{E}$, satisfying initial conditions $\mathfrak{u}(0) = \mathfrak{v}(0) = \mathfrak{u}'(0) - 1 = \mathfrak{v}'(0) = 0$ and

$$\operatorname{Re}\left[\frac{z^k\mathfrak{u}'(z)}{\phi_k(z)} - \gamma\right] = \operatorname{Re}\left[\sum_{\alpha=1}^{p} s_\alpha \left(\frac{z^k\mathfrak{u}'_\alpha(z)}{\phi_k(z)} - \gamma\right)\right] > \sum_{\alpha=1}^{p} s_\alpha \left|\frac{z^k\mathfrak{v}'_\alpha(z)}{\phi_k(z)}\right| = \left|\frac{z^k\mathfrak{v}'(z)}{\phi_k(z)}\right|$$

showing that $\mathfrak{f} \in \mathcal{KH}^0(k,\gamma)$. □

4. Conclusions

In this paper, we introduced a new class of harmonic functions denoted by $\mathcal{KH}^0(k,\gamma)$. We established a relationship between $\mathcal{KH}^0(k,\gamma)$ and $\mathcal{K}(k,\gamma)$. We demonstrated that $\mathcal{KH}^0(k,\gamma)$ is close-to-convex. For functions in the $\mathcal{KH}^0(k,\gamma)$ class, we derived coefficient bounds and distortion theorems. Finally, we proved that $\mathcal{KH}^0(k,\gamma)$ is closed under convolution.

References

[1] Clunie, J., Sheil-Small, T., (1984), Harmonic univalent functions, Annales Fennici Mathematici, 9(1), 3-25. DOI: https://doi.org/10.5186/aasfm.1984.0905.

[2] Kowalczyk, J., Leś-Bomba, E., (2010), On a subclass of close-to-convex functions, Applied Mathematics Letters, 23(10), 1147-1151. DOI: https://doi.org/10.1016/j.aml.2010.03.004.

[3] Owa, S., Nunokawa, M., Saitoh, H., Srivastava, H. M., (2002), Close-to-convexity, starlikeness, and convexity of certain analytic functions, Applied Mathematics Letters, 15(1), 63-69. DOI: https://doi.org/10.1016/S0893-9659(01)00094-5.

[4] Gao, C., Zhou, S., (2005), On a class of analytic functions related to the starlike functions, Kyungpook Mathematical Journal, 45(1), 123-130.

[5] Şeker, B., (2011), On certain new subclass of close-to-convex functions, Applied Mathematics and Computation, 218(3), 1041-1045. DOI: https://doi.org/10.1016/j.amc.2011.03.018.

[6] Ponnusamy, S., Yamamoto, H., Yanagihara, H., (2013), Variability regions for certain families of harmonic univalent mappings, Complex Variables and Elliptic Equations, 58(1), 23-34. DOI: https://doi.org/10.1080/17476933.2010.551200.

[7] Li, L., Ponnusamy, S., (2013), Disk of convexity of sections of univalent harmonic functions, Journal of Mathematical Analysis and Applications, 408, 589-596. DOI: https://doi.org/10.1016/j.jmaa.2013.06.021.

[8] Li, L., Ponnusamy, S., (2013), Injectivity of sections of univalent harmonic mappings, Nonlinear Analysis, 89, 276-283. DOI: https://doi.org/10.1016/j.na.2013.05.016.

[9] Ghosh, N., Vasudevarao, A., (2019), On a subclass of harmonic close-to-convex mappings, Monatshefte für Mathematik, 188, 247-267. DOI: https://doi.org/10.1007/s00605-017-1138-7.

[10] Rajbala, Prajapat, J. K., (2020), Certain geometric properties of close-to-convex harmonic mappings, Asian-European Journal of Mathematics. DOI: https://doi.org/10.1142/S1793557121501023.

[11] Çakmak, S., Yaşar, E., Yalçın, Y., (2024), Some basic properties of a subclass of close-to-convex harmonic mappings, TWMS Journal of Pure and Applied Mathematics, 15(2), 163-173. DOI: https://doi.org/10.30546/2219-1259.15.2.2024.01163.

[12] Yaşar, E., Yalçın, S., (2021), Close-to-convexity of a class of harmonic mappings defined by a third-order differential inequality, Turkish Journal of Mathematics, 45(2), 678-694. DOI: https://doi.org/10.3906/mat-2004-50.

[13] Bshouty, D., Lyzzaik, A., (2011), Close-to-convexity criteria for planar harmonic mappings, Complex Analysis and Operator Theory, 5, 767-774. DOI: https://doi.org/10.1007/s11785-010-0056-7.

[14] Kalaj, D., Ponnusamy, S., Vuorinen, M., (2014), Radius of close-to-convexity and full starlikeness of harmonic mappings, Complex Variables and Elliptic Equations, 59, 539-552. DOI: https://doi.org/10.1080/17476933.2012.759565.

[15] Ghosh, N., Vasudevarao, A., (2018), Some basic properties of certain subclass of harmonic univalent functions, Complex Variables and Elliptic Equations, 63, 1687-1703. DOI: https://doi.org/10.1080/17476933.2017.1403426.

[16] Ghosh, N., Vasudevarao, A., (2019), The radii of fully starlikeness and fully convexity of a harmonic operator, Monatshefte für Mathematik, 188, 653-666. DOI: https://doi.org/10.1007/s00605-018-1163-1.

[17] Ghosh, N., Vasudevarao, A., (2020), On some subclass of harmonic mappings, Bulletin of the Australian Mathematical Society, 101, 130-140. DOI: https://doi.org/10.1017/S0004972719000698.

[18] Ali, M. F., Allu, V., Ghosh, N., (2020), A convolution property of univalent harmonic right half mappings, Monatshefte für Mathematik, 193, 729-736. DOI: https://doi.org/10.1007/s00605-020-01442-3.

[19] Yalçın, S., Bayram, H., Oros, G. I., (2024), Some properties and graphical applications of a new subclass of harmonic functions defined by a differential inequality, Mathematics, 12(15), 2338. DOI: https://doi.org/10.3390/math12152338.

[20] Breaz, D., Durmuş, A., Yalçın, S., Cotirla, L.-I., Bayram, H., (2023), Certain properties of harmonic functions defined by a second-order differential inequality, Mathematics, 11(19), 4039. DOI: https://doi.org/10.3390/math11194039.

[21] Çakmak, S., Yaşar, E., Yalçın, S., (2022), Some basic geometric properties of a subclass of harmonic mappings, Boletín de la Sociedad Matemática Mexicana, 28, 54. DOI: https://doi.org/10.1007/s40590-022-00448-1.

[22] Dorff, M., Hamidi, S. G., Jahangiri, J. M., Yaşar, E., (2021), Convolutions of planar harmonic strip mappings, Complex Variables and Elliptic Equations, 66(11), 1904-1921. DOI: https://doi.org/10.1080/17476933.2020.1789864.

[23] Çakmak, S., Yaşar, E., Yalçın, S., (2022), New subclass of the class of close-to-convex harmonic mappings defined by a third-order differential inequality, Hacettepe Journal of Mathematics and Statistics, 51(1), 172-186. DOI: https://doi.org/10.15672/hujms.922981.

Serkan ÇAKMAK received his Ph.D. degree in Mathematics in 2022 from Bursa Uludag University, Turkey. Since 2022, he has been an Assistant Professor at Istanbul Gelisim University. His research interests include harmonic mappings, geometric function theory, and close-to-convex functions. He has published research articles in reputed international journals of mathematics.

Sibel YALÇIN received her Ph.D. degree in Mathematics in 2001 from the Bursa Uludag University of Bursa, Turkey. She became a full Professor in 2011. She is currently with the Department of Mathematics, Bursa Uludag University. Her research interests include harmonic mappings, geometric function theory, meromorphic functions, and convolution operators. She has published research articles in reputed international journals of mathematics.