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EXISTENCE OF SOLUTIONS FOR NAVIER PROBLEM INVOLVING $(p(.), q(.))$-LAPLACIAN AND $(p(.), q(.))$-BIHARMONIC OPERATORS

A. EL KATIT[1*], A.R. EL AMROUSS[2], F. KISSI[1], §

Abstract. In the present paper, we are interested in the study of nonlinear problem driven by $(p(.), q(.))$-Laplacian and $(p(.), q(.))$-Biharmonic operators subject to Navier boundary conditions. By means of variational method and critical point theory, we establish the existence of at least one solution and infinitely many solutions under some suitable assumptions.

Keywords: $(p(.), q(.))$-Laplacian operator, $(p(.), q(.))$-Biharmonic operator, Navier boundary problem, variational method.

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1. Introduction

In this paper, we consider the following nonlinear boundary value problem

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u + |\Delta u|^{q(x)-2}\Delta u\right) - \mathcal{L}u(x) = \lambda k(x)|u|^{r(x)-2}u + \mu w(x)|u|^{s(x)-2}u \quad in\ \Omega, \\ u = \Delta u = 0 \quad on\ \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain with a Lipschitz boundary $\partial\Omega$; the variable exponents $p, q, r, s : \bar{\Omega} \to (1, \infty)$ are continuous functions with $p(x) < q(x) < N/2$ for all $x \in \bar{\Omega}$; k, $\omega \in P(\Omega)$ the set of all measurable functions on Ω; λ, μ are real parameters and

$$\mathcal{L}(u) := div\left(|\nabla u|^{p(x)-2}\nabla u + |\nabla u|^{q(x)-2}\nabla u\right).$$

In the last decades, the study of differential equations has been an interesting topic, they provide a wide area of research employing diverse mathematical techniques and tools including variational approach, topological methods, fixed point theory and existence and uniqueness results.

[1] Mohamed First University, Faculty of Sciences, Department of Mathematics, El qods, Oujda-Morocco.
e-mail: elkatit96@gmail.com; ORCID: https://orcid.org/0000-0002-7385-8513.
e-mail: kissifouad@hotmail.com; ORCID: https://orcid.org/0000-0002-0017-265X.
[2] Mohamed First University, Faculty of Sciences, Department of Mathematics, El qods, Oujda-Morocco.
e-mail: elamrouss@hotmail.com; ORCID: https://orcid.org/0000-0003-3536-398X.
[*] Corresponding author.
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Nonlinear elliptic equations with variable exponent are more general and add more complexity compared to classical elliptic equations (we refer to [14, 15, 16]).

In [17], the authors studied a nonlocal elliptic system in the case of a reaction term by means of topological degree for a class of demicontinuous operators.

Employing the concept of a Fredholm-type results for a pair of nonlinear operators, the authors in [18] obtain the existence of weak solutions. We also refer to ([19, 20, 21, 22, 23]).

Elliptic equations with variable exponent are of significant interest in various areas deriving from elastic mechanics, electrorheological fluid dynamics, image processing, etc (see [7, 13, 8, 11]).

Recall that weak solutions of problem (1) are the critical points of the associated double phase functional energy E, defined by

$$E(u) = \int_\Omega \left(\frac{|\Delta u|^{p(x)}}{p(x)} + \frac{|\Delta u|^{q(x)}}{q(x)} \right) + \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx \quad -\lambda \int_\Omega \frac{k(x)}{r(x)} |u|^{r(x)} dx$$
$$-\mu \int_\Omega \frac{\omega(x)}{s(x)} |u|^{s(x)} dx,$$

on the generalized Sobolev space $X = W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega)$. It is well known that $E \in C^1(X, \mathbb{R})$ and its derivative at $u \in X$ is given by

$$< E'(u), \varphi >= \int_\Omega (|\Delta u|^{p(x)-2}\Delta u + |\Delta u|^{q(x)-2}\Delta u)\Delta\varphi + (|\nabla u|^{p(x)-2}\nabla u + |\nabla u|^{q(x)-2}\nabla u).\nabla\varphi$$
$$-\lambda \int_\Omega k(x)|u|^{r(x)-2}u\varphi dx - \mu \int_\Omega \omega(x)|u|^{s(x)-2}u\varphi dx, \qquad \forall \varphi \in X.$$

In the constant case, V. Bobkov and M. Tanaka in ([5]) were concerned with the existence and non-existence of positive solutions for the (p,q)-Laplace problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \alpha|u|^{p-2}u + \beta|u|^{q-2}u \text{ in } \Omega, \\ u = 0 \quad on\ \partial\Omega, \end{cases} \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$ and $1 < q < p < \infty$. Moreover, for weight w satisfying the Muckenhoupt condition, the author in ([6]) has shown the existence of unique solution for the fourth order elliptic equation

$$\begin{cases} \Delta\left(\omega(|\Delta u|^{p-2}\Delta u + |\Delta u|^{q-2}\Delta u)\right) \\ -\operatorname{div}\left(\omega(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u)\right) = f(x) - \operatorname{div}(G(x)) \text{ in } \Omega, \\ u = \Delta u = 0 \quad on\ \partial\Omega, \end{cases} \tag{3}$$

where $f \in L^{p'}(\Omega, \omega^{-1/(p-1)})$ and $G \in \left[L^{p'}(\Omega, \omega^{-1/(q-1)})\right]^N$.

In the variable case, based on a three critical points theorem of B. Ricceri, the authors in ([2]) have established the existence of at least three solutions for the (p, q)-biharmonic systems

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) = \lambda F_u(x, u, v) \quad \text{in } \Omega, \\ \Delta\left(|\Delta u|^{q(x)-2}\Delta u\right) = \lambda F_v(x, u, v) \quad \text{in } \Omega, \\ u = \Delta u = v = \Delta v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in [0,\infty)$; $p, q \in C(\bar{\Omega})$ with $\frac{N}{2} < p^- := \inf_{x\in\bar{\Omega}} p(x) \leq p^+ := \sup_{x\in\bar{\Omega}} p(x) < \infty$, $\frac{N}{2} < q^- := \inf_{x\in\bar{\Omega}} q(x) \leq q^+ := \sup_{x\in\bar{\Omega}} q(x) < \infty$; $F : \bar{\Omega}\times\mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(.,s,t)$ is continuous on $\bar{\Omega}$ for each $(s,t)\in\mathbb{R}^2$ and $F(x,.,.)$ is C^1 in $\mathbb{R}^2$ for all $x \in \Omega$, and $\sup_{\{|t|\leq\theta,|s|\leq\theta\}} (|F_u(.,s,t)| + |F_v(.,s,t)|) \in L^1(\Omega)$ for all $\theta > 0$, with F_u, F_v denote the partial derivatives of F , with respect to u and v respectively.

A function $u \in X$ is called a weak solution of problem (1) if and only if for every $\varphi \in X$,

$$\int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta\varphi + |\Delta u|^{q(x)-2}\Delta u\Delta\varphi) + \int_\Omega (|\nabla u|^{p(x)-2}\nabla u.\nabla\varphi + |\nabla u|^{q(x)-2}\nabla u.\nabla\varphi)$$
$$= \int_\Omega \left(k(x)|u|^{r(x)-2}u + \omega(x)|u|^{s(x)-2}u\right)\varphi.$$

In order to present our main results, we assume that $k(x)$ and $\omega(x)$ satisfy the following hypotheses:

(A1) $k(x) \in L^{\frac{\alpha(x)}{\alpha(x)-r(x)}}(\Omega)$ where $\alpha \in C_+(\bar{\Omega})$ and $r(x) < \alpha(x) < q_2^*(x)$ for all $x \in \bar{\Omega}$ with $q_2^*(x) := Nq(x)/N - 2q(x)$ is the critical Sobolev exponent.

(A2) $w(x) \in L^{\frac{\beta(x)}{\beta(x)-s(x)}}(\Omega)$ where $\beta \in C_+(\bar{\Omega})$ and $s(x) < \beta(x) < q_2^*(x)$ for all $x \in \bar{\Omega}$.

The first main result of this paper is the following theorem.

Theorem 1.1. *Assume that (A1) and (A2) hold in which $r^+ < q^- < q^+ < s^-$ and $\omega(x) > 0$ a.e $x \in \Omega$, then for any $\mu > 0$, there exists $\lambda^* = \lambda^*(\mu) > 0$ such that for each $\lambda \in (0,\lambda^*)$, the problem (1) has at least one nontrivial weak solution.*

The second result of this work is the next theorem.

Theorem 1.2. *Suppose that (A1) and (A2) are satisfied in which $r^+ < q^- < q^+ < s^-$.*

(i) For each given $\mu, \lambda \in \mathbb{R}$ such that $\mu\omega(x) > 0$ a.e $x \in \Omega$, the problem (1) admits a sequence of weak solutions $\{\pm u_k\}_{k\in\mathbb{N}}$ such that $E(\pm u_k) \to \infty$ as $k \to \infty$.

(ii) For each given $\mu, \lambda \in \mathbb{R}$ such that $\lambda k(x) > 0$ a.e $x \in \Omega$ and $\mu\omega(x)$ keeps a constant sign, the problem (1) has a sequence of weak solutions $\{\pm v_k\}_{k\in\mathbb{N}}$ such that $E(\pm v_k) \to 0$.

This paper is organized as follows. In section 2, we recall some preliminaries about the variable exponent Lebesgue–Sobolev spaces. In section 3 and 4, we give the proof of Theorem 1.1 and Theorem 1.2 respectively.

2. Abstract framework

For the suitability of readers, we remind some backgrounds about the variable exponent Lebesgue Sobolev spaces. Set

$C_+(\bar{\Omega}) = \{\varrho \in C(\bar{\Omega}) : \ \varrho(x) > 1 \text{ for all } x \in \bar{\Omega}\}$,

$\varrho^+ = \max_{\bar{\Omega}} \varrho(x), \quad \varrho^- = \min_{\bar{\Omega}} \varrho(x), \quad$ *for* $\varrho \in C_+(\Omega)$.

For a measurable exponent $q(.)$ in $C_+(\Omega)$, we introduce the variable exponent Lebesgue space $L^{q(x)}(\Omega)$ composed of measurable real-valued functions u such that

$$\int_\Omega |u(x)|^{q(x)}dx < \infty,$$

equipped with the norm

$$|u|_{L^{q(x)}(\Omega)} = \inf\left\{\nu > 0 : \ \int_\Omega \left|\frac{u(x)}{\nu}\right|^{q(x)} dx \leq 1\right\},$$

then the space $L^{q(x)}(\Omega)$ endowed with the above norm is reflexive and Banach space (see [9]).

For positive integer k, the variable exponent Sobolev space $W^{k,q(x)}(\Omega,\omega)$ is defined by

$$W^{k,q(x)}(\Omega)=\left\{u\in L^{q(x)}(\Omega)\ :\ D^\alpha u\in L^{q(x)}(\Omega), |\alpha|\le k\right\},$$

with $\alpha\in\mathbb{N}^*$, $|\alpha|=\sum_{i=1}^n\alpha_i$ and $D^\alpha u=\partial^{|\alpha|}u/\partial^{\alpha_1}x_1\cdots\partial^{\alpha_N}x_N$. We can define on $W^{k,q(x)}(\Omega)$ the norm

$$\|u\|_{W^{k,q(x)}(\Omega)}=\sum_{|\alpha|\le k}|D^\alpha u|_{L^{q(x)}(\Omega)}.$$

Proposition 2.1. *([9]) The space $(L^{q(x)}(\Omega), |u|_{q(x)})$ is separable, reflexive and uniformly convex Banach, and its conjugate space is $L^{p(x)}(\Omega)$, where $\frac{1}{q(x)}+\frac{1}{p(x)}=1, \forall x\in\bar{\Omega}$. For any $v\in L^{q(x)}(\Omega)$ and $w\in L^{p(x)}(\Omega)$, we have*

$$\left|\int_\Omega vwdx\right|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|v|_{L^{q(x)}(\Omega)}|w|_{L^{p(x)}(\Omega)}\le 2|v|_{L^{q(x)}(\Omega)}|w|_{L^{p(x)}(\Omega)}.$$

Denote $\rho : L^{q(x)}(\Omega)\to\mathbb{R}$, the modular of the $L^{q(x)}(\Omega,\omega)$ space, defined by $\rho(u)=\int_\Omega|u(x)|^{q(x)}$. In view of [[9], Theorem 1.3], we have the following Lemma.

Lemma 2.1. *For each $u_n, u\in L^{q(x)}(\Omega)$, we have*

(1) $|u|_{L^{q(x)}(\Omega)}>1$ *then* $|u|^{q^-}_{L^{q(x)}(\Omega)}\le\rho(u)\le|u|^{q^+}_{L^{q(x)}(\Omega)}$;
(2) $|u|_{L^{q(x)}(\Omega)}<1$ *then* $|u|^{q^+}_{L^{q(x)}(\Omega)}\le\rho(u)\le|u|^{q^-}_{L^{q(x)}(\Omega}$;
(3) $\lim_{n\to+\infty}|u_n-u|_{L^{q(x)}(\Omega)}=0$ *if and only if* $\lim_{n\to+\infty}\rho(u_n-u)=0$.

Let us define on X the equivalent norm

$$\|u\|:=\inf\left\{\nu>0:\ \int_\Omega\left|\frac{\Delta u(x)}{\nu}\right|^{q(x)}dx\le 1\right\}.$$

Remark 2.1. *Set $\mathcal{K}(u)=\int_\Omega|\Delta u|^{q(x)}dx$, then we also have $\|u\|^{q^-}-1\le\mathcal{K}(u)\le\|u\|^{q^+}+1$ for each $u\in X$.*

Theorem 2.1. *([3]) Let $\gamma\in C_+(\bar{\Omega})$ such that $\gamma(x)<q_2^*(x)$ for all $x\in\bar{\Omega}$. Then, there is a continuous and compact embedding from $W^{2,q(x)}(\Omega)\cap W_0^{1,q(x)}(\Omega)$ into $L^{\gamma(x)}(\Omega)$.*

Since X is reflexive and separable Banach space, there are $\{e_n,\ n\ge 1\}\subset X$ and $\{f_n,\ n\ge 1\}\subset X^*$ such that $<f_n, e_m>=\delta_{n,m}$, $X=\overline{span\{e_n:\ n\in\mathbb{N}^*\}}$ and $X^*=\overline{span\{f_n,\ n\in\mathbb{N}^*\}}^{X^*}$. For $k\ge 1$, we denote by

$$X_k=span\{e_k\}, Y_k=\bigoplus_{i=1}^k X_i \text{ and } Z_k=\overline{\bigoplus_{i=k}^\infty X_i}.$$

Proposition 2.2. *([4]) Let X be a separable space and $\psi : X\to\mathbb{R}$ weakly-strongly continuous with $\psi(0)=0$.*
For each $\gamma>0$ given, $\beta=\beta(\gamma)=\sup\{|\psi(u)|:\ u\in Z_k \text{ and } \|u\|\le\gamma\}\to 0$ as $k\to\infty$.

3. Proof of Theorem 1.1

Assume that (A1) and (A2) hold in which $r^+ < q^- < q^+ < t^-$. The following proposition shows the geometry of E.

Proposition 3.1. *Assume that $\omega(x) > 0$ a.e $x \in \Omega$. For each given $\mu > 0$, there exist $\lambda^* = \lambda^*(\mu) > 0$ and $\delta, \rho > 0$ such that for each $\lambda \in (0, \lambda_*)$, we have $E(u) \geq \rho$ if $\|u\| = \delta$.*

Proof. Invoking Theorem 2.1, there exist two constants $C_\alpha, C_\beta > 0$ such that

$$|u|_{\alpha(x)} \leq C_\alpha \|u\|, \quad |u|_{\beta(x)} \leq C_\beta \|u\|, \quad \forall u \in X. \tag{4}$$

Let $\mu, \lambda > 0$ and $\delta > 0$. Based on Proposition 2.1, Remark 2.1 and (4), for $\|u\| = \delta(< 1)$, we have

$$\begin{aligned} E(u) &\geq \frac{1}{q^+}\mathcal{K}(u) - \frac{\lambda}{r^-}\int_\Omega |k(x)||u|^{r(x)}dx - \frac{\mu}{s^-}\int_\Omega \omega(x)|u|^{s(x)}dx \\ &\geq \frac{1}{q^+}\mathcal{K}(u) - \frac{2}{r^-}\lambda|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}|u|^{\tilde{r}}_{\alpha(x)} - \frac{2}{s^-}\mu|\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}|u|^{\tilde{s}}_{\beta(x)} \\ &\geq \frac{1}{2q^+}\|u\|^{q^+} - \frac{\lambda C_\alpha}{r^-}|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}\|u\|^{\tilde{r}} + \left(\frac{1}{2q^+} - \frac{\mu C_\beta}{s^-}|\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}\|u\|^{\tilde{s}-q^+}\right)\|u\|^{q^+}, \end{aligned}$$

where $\tilde{r} \in [r^-, r^+]$ and $\tilde{s} \in [s^-, s^+]$. Take the function $h : [0, 1] \to \mathbb{R}$ defined by $h(y) = \frac{1}{2q^+} - \frac{\mu C_\beta}{s^-}|\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}} y^{\tilde{s}-q^+}$. Since $q^+ < s^-$, the function h is positive in the neighborhood of the origin, hence there exist δ small enough such that $h(\delta) > 0$. In other part, define $\lambda^* = r^-\delta^{q^+-\tilde{r}}/4C_\alpha q^+|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}$.

Therefore, the above estimate implies that for all $\lambda \in (0, \lambda^*)$, $E(u) \geq \delta^{q^+}/4q^+ = \rho$, if $\|u\| = \delta$.

Let $\lambda > 0$ be given and let $\mu > 0$. Fix $\psi \in C_0^\infty(\Omega)$. Then for $\rho > 1$, One has

$$\begin{aligned} E(\rho\psi) \leq{}& \frac{\rho^{q^+}}{p^-}\int_\Omega \left(|\Delta\psi|^{p(x)} + |\nabla\psi|^{p(x)} + |\Delta\psi|^{q(x)} + |\nabla\psi|^{q(x)}\right)dx + \frac{\rho^{r^+}}{r^-}\int_\Omega |k(x)||\psi|^{r(x)} \\ &- \frac{\rho^{s^-}}{s^+}\int_\Omega \omega(x)|\psi|^{s(x)}. \end{aligned}$$

Hence $E(\rho\psi) \to -\infty$ as $\rho \to \infty$ due to $r^+ < q^+ < s^-$ and $\omega(x) > 0$ a.e $x \in \Omega$. Therefore there exists some $e \in X$ such that $\|e\| > \delta$ and $E(e) < 0$. □

The next lemma concerns the compactness condition for the functional E.

Lemma 3.1. *Assume that $\omega(x) > 0$ a.e $x \in \Omega$. For each given $\mu > 0$ and $\lambda \in \mathbb{R}$, E satisfies $(PS)_c$ condition for all $c \in \mathbb{R}$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a $(PS)_c$ sequence for E, i.e., $E(u_n) \to c$ and $E'(u_n) \to 0$ in X'. For n large, we have

$$\begin{aligned} 1 + c + \|u_n\| &\geq E(u_n) - \frac{1}{s^-} < E'(u_n), u_n > \\ &\geq \left(\frac{1}{q^+} - \frac{1}{s^-}\right)(\|u_n\|^{q^-} - 1) - 4|\lambda|\left(\frac{1}{r^-} + \frac{1}{s^-}\right)|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}(C_\alpha^{r^+}\|u_n\|^{r^+} + 1). \end{aligned}$$

This implies the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in X reflexive, then up to a subsequence $u_n \rightharpoonup u \in X$. From this and taking into account that $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence for E, we have

$$\begin{aligned}
0 &= \lim_{n\to+\infty} < E'(u_n) - E'(u), u_n - u > \\
&= \lim_{n\to+\infty} \int_\Omega (|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u)(\Delta u_n - \Delta u)\, dx \\
&\quad + \int_\Omega (|\Delta u_n|^{q(x)-2}\Delta u_n - |\Delta u|^{q(x)-2}\Delta u)(\Delta u_n - \Delta u)\, dx \\
&\quad + \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u).(\nabla u_n - \nabla u)\, dx \\
&\quad + \int_\Omega (|\nabla u_n|^{q(x)-2}\nabla u_n - |\nabla u|^{q(x)-2}\nabla u).(\nabla u_n - \nabla u)\, dx \\
&\quad - \lambda \int_\Omega k(x)(|u_n|^{r(x)-2}u_n - |u|^{r(x)-2}u)(u_n - u)\, dx \\
&\quad - \mu \int_\Omega \omega(x)(|u_n|^{s(x)-2}u_n - |u|^{s(x)-2}u)(u_n - u)\, dx. \qquad (5)
\end{aligned}$$

By means of Proposition 2.1, we have

$$|\int_\Omega k(|u_n|^{r(.)-2}u_n - |u|^{r(.)-2}u)(u_n - u)| \le 3|k|_{\frac{\alpha(.)}{\alpha(.)-r(.)}} ||u_n|^{r(.)-2}u_n - |u|^{r(.)-2}u|_{\frac{\alpha(.)}{r(.)-1}} |u_n - u|_{\alpha(.)}, \qquad (6)$$

and

$$|\int_\Omega \omega(|u_n|^{s(x)-2}u_n - |u|^{s(.)-2}u)(u_n - u)| \le 3|\omega|_{\frac{\beta(.)}{\beta(.)-s(.)}} ||u_n|^{s(.)-2}u_n - |u|^{s(.)-2}u|_{\frac{\beta(.)}{s(.)-1}} |u_n - u|_{\beta(.)}. \qquad (7)$$

Moreover, utilizing Theorem 2.1 we have $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ and $L^{\beta(x)}(\Omega)$. This yields,

$$|u_n|^{r(x)-2}u_n \to |u|^{r(x)-2}u \quad \text{in } L^{\frac{\alpha(x)}{r(x)-1}}(\Omega),$$

$$|u_n|^{s(x)-2}u_n \to |u|^{s(x)-2}u \quad \text{in } L^{\frac{\beta(x)}{s(x)-1}}(\Omega).$$

Combining this with (6) and (7) we arrive at

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega k(|u_n|^{r(x)-2}u_n - |u|^{r(x)-2}u)(u_n - u) \\
&= \lim_{n\to\infty} \int_\Omega \omega(|u_n|^{s(x)-2}u_n - |u|^{s(x)-2}u)(u_n - u) \\
&= 0.
\end{aligned}$$

From the (S_+)-property of the $((p(x), q(x))$-Laplacian and $((p(x), q(x))$-Biharmonic operator, we derive from (5) that $u_n \to u$ in X. □

Hence, E satisfies the $(PS)_c$ condition in view of Lemma 3.1. Invoking Proposition 3.1, there exist $0 < \delta < 1$ and $\rho > 0$ such that $E(u) \ge \rho$ if $\|u\| = \delta$. Thus, invoking Mountain Pass Theorem ([1]), the value c characterized by

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} E(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\},$$

is a critical value of E. Let u be a critical point of E with $E(u) = c$. Consequently, u is a nontrivial solution of the problem (1).

4. Proof of Theorem 1.2

Assume that (A1)-(A2) hold and $r^+ < q^- < q^+ < s^-$. We start by proving assertion (i) of Theorem 1.2.

Proposition 4.1. *For each given $\mu, \lambda \in \mathbb{R}$ and $k \in \mathbb{N}$, there exists $d_k = d_k(\mu) > 0$ such that*

$$\inf_{u \in Z_k,\, \|u\| = d_k} E(u) \to \infty \text{ as } k \to \infty.$$

Proof. Let $u \in Z_k$ with $\|u\| \geq 1$. Using again Proposition 2.1, Remark 2.1 and (4) we have

$$\begin{aligned} E(u) &\geq \frac{1}{q^+}\mathcal{K}(u) - \frac{|\lambda|}{r^-}\int_\Omega |k(x)||u|^{r(x)}dx - \frac{|\mu|}{s^-}\int_\Omega |\omega(x)||u|^{s(x)}dx \\ &\geq \frac{1}{q^+}\|u\|^{q^-} - \frac{4}{r^-}|\lambda||k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}(C_\alpha^{r^+}\|u\|^{r^+} + 1) - \frac{4}{s^-}|\mu||\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}(|u|_{\beta(x)}^{s^+} + 1). \end{aligned}$$

Since $q^- > r^+$, we can find d_0 large enough such that

$$\frac{4}{r^-}|\lambda||k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}C_\alpha^{r^+}\|u\|^{r^+} \leq \frac{1}{2q^+}\|u\|^{q^-} \text{ as } \|u\| \geq d_0.$$

Let $\gamma_k = \sup\{|u|_{\beta(x)} :\ u \in Z_k \text{ and } \|u\| = 1\}$, then one has $|u|_{\beta(x)} \leq \gamma_k\|u\|$ for all u in Z_k. Hence

$$E(u) \geq \frac{1}{2q^+}\|u\|^{q^-} - \frac{4}{s^-}|\mu||\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}(\gamma_k^{s^+}\|u\|^{s^+} + 1) - \frac{4}{r^-}|\lambda||k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}, \quad u \in Z_k\ \|u\| \geq d_0.$$

Take $d_k = \left(\frac{s^-}{16q^+|\mu||\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}\gamma_k^{s^+}}\right)^{\frac{1}{s^+-q^-}}$, the last inequality yields

$$E(u) \geq \frac{d_k^{q^-}}{4q^+} - \frac{4}{r^-}|\lambda||k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}} - \frac{4}{s^-}|\mu||\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}, \quad u \in Z_k\ \|u\| = d_k.$$

Utilizing Proposition 2.2 we have $\gamma_k \to 0$ as $k \to \infty$. Since $q^- < s^+$, it follows that $d_k \to \infty$ and thus we arrive at $\inf_{u \in Z_k\ \|u\| = d_k} E(u) \to \infty$. □

Proposition 4.2. *For every $\mu, \lambda \in \mathbb{R}$ such that $\mu\omega(x) > 0$ a.e $x \in \Omega$ and $k \in \mathbb{N}$, there exists $\rho_k > d_k > 0$ such that*

$$\sup_{u \in Y_k,\, \|u\| = \rho_k} E(u) \leq 0.$$

Proof. Let u in Y_k with $\|u\| = 1$. As $dim Y_k = k$, norms are equivalents on Y_k. Since $\mu\omega(x) > 0$ a.e $x \in \Omega$, $r^+ < q^+ < s^-$ and $\rho_k \to \infty$ as $k \to \infty$, then for ρ_k large enough ($\rho_k > d_k$), this completes the proof. □

We note that Lemma 3.1 holds true for $\mu \in \mathbb{R}$ given such that $\mu\omega(x) > 0$ a.e $x \in \Omega$, thus according to Fontaine theorem ([12]), Proposition 4.1, Proposition 4.2 and Lemma 3.1, we achieve the proof of assertion (i) of Theorem 1.2. Next we will prove the assertion (ii) of Theorem 1.2.

Proposition 4.3. *For any given $\lambda, \mu \in \mathbb{R}$, there exists an integer k_0, such that for each $k > k_0$, there exists $\tau_k(\lambda) > 0$ satisfying* $\inf_{u \in Z_k,\, \|u\| = \tau_k} E(u) \geq 0$.

Proof. Let u in Z_k with $\|u\|\leq 1$, we have

$$E(u)\geq \frac{1}{q^+}\mathcal{K}(u)-\frac{2|\lambda|}{r^-}|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}|u|^{\tilde{r}}_{\alpha(x)}-\frac{2|\mu|}{s^-}|\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}|u|^{\tilde{s}}_{\beta(x)}$$
$$\geq \frac{1}{q^+}\|u\|^{q^+}-\frac{2|\lambda|}{r^-}|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}|u|^{\tilde{r}}_{\alpha(x)}-\frac{|\mu|C_\beta}{s^-}|\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}\|u\|^{\tilde{s}}.$$

Since $q^+<s^-$, we can find $\tau_0>0$ small enough such that

$$\frac{|\mu|C_\beta}{s^-}|\omega|_{\frac{\beta(x)}{\beta(x)-s(x)}}\|u\|^{\tilde{s}}\leq \frac{1}{2q^+}\|u\|^{q^+} \text{ as } 0<\|u\|\leq\tau_0.$$

Let $\theta_k=\sup\{|u|_{\alpha(x)}:\ u\in Z_k \ and\ \|u\|=1\}$, then one has $|u|_{\alpha(x)}\leq\theta_k\|u\|$ for all $u\in Z_k$. Thus

$$E(u)\geq \frac{1}{2q^+}\|u\|^{q^+}-\frac{2|\lambda|}{r^-}|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}\theta_k^{\tilde{r}}\|u\|^{\tilde{r}},\quad u\in Z_k\ \|u\|\leq\tau_0. \tag{8}$$

Take $\tau_k=\left(\frac{4q^+|\lambda|\theta_k^{\tilde{r}}}{r^-}|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}\right)^{\frac{1}{q^+-\tilde{r}}}$. From (8) we get $E(u)\geq\frac{1}{2q^+}\tau_k^{q^+}-\frac{1}{2q^+}\tau_k^{q^+}=0,\ u\in Z_k$ with $\|u\|=\tau_k$. The proof is completed. □

Proposition 4.4. *For each given $\lambda,\mu\in\mathbb{R}$ such that $\lambda k(x)>0$ a.e $x\in\Omega$ and all $k>k_0$, there exists $0<l_k<\tau_k$ fulfilling $\max_{u\in Y_k,\ \|u\|=l_k}E(u)<0$.*

Proof. Let u in Y_k with $\|u\|=1$ and $0<l<\tau_k<1$. Since $\lambda k(x)>0$ a.e $x\in\Omega$, $r^+<q^-<s^-$ and norms are equivalents on Y_k, one can find $l_k<\tau_k$ small enough such that $E(u)<0$ if $\|u\|=l_k$. This proves the result. □

Proposition 4.5. *For any given $\mu,\lambda\in\mathbb{R}$ and $k>k_0$, there exists $\tau_k>0$ given by Proposition 4.3 such that*

$$\inf_{u\in Z_k,\ \|u\|\leq\tau_k}E(u)\to 0,\ \text{as } k\to\infty.$$

Proof. By definition of Y_k and Z_k we have $Y_k\cap Z_k\neq\emptyset$. Since $l_k<\tau_k$, where l_k is given in Proposition 4.4, it follows that

$$b_k=\inf_{u\in Z_k,\ \|u\|\leq\tau_k}E(u)\leq\max_{u\in Y_k,\ \|u\|=l_k}E(u)\leq 0.$$

Using (8), for every u in Z_k with $\|u\|\leq\tau_k$ one has

$$E(u)\geq-\frac{2|\lambda|}{r^-}|k|_{\frac{\alpha(x)}{\alpha(x)-r(x)}}(\theta_k\tau_k)^{\tilde{r}}.$$

By Proposition 2.2 we have $\theta_k\tau_k\to 0$ as $k\to\infty$. Consequently $b_k\to 0$. The proof is completed. □

Lemma 4.1. *For every given $\mu\in\mathbb{R}$ such that $\mu\omega(x)$ keeps a constant sign, The functional E satisfies $(PS)^*_c$ condition for all $c\in[b_{k_0},0)$.*

Proof. Let $\{v_{n_j}\}$ be a sequence in X such that $n_j\to\infty$, $v_{n_j}\in Y_{n_j}$, $E(v_{n_j})\to c$ and $(E|_{Y_{n_j}})'(v_{n_j})\to 0$.

For n_j large enough, we have

$$E(v_{n_j})-\frac{<E'(v_{n_j}),v_{n_j}>}{s^-}\geq(\frac{1}{q^+}-\frac{1}{s^-})(\|v_{n_j}\|^{q^-}-1)$$
$$-4|\lambda|(\frac{1}{r^-}+\frac{1}{s^-})|k|_{\frac{\alpha(.)}{\alpha(.)-r(.)}}(C_\alpha^{\tilde{r}}\|v_{n_j}\|^{\tilde{r}}+1). \tag{9}$$

As $r^+ < q^-$, $\{v_{n_j}\}$ is a bounded sequence in X reflexive, up to a subsequence, $v_{n_j} \rightharpoonup u$ in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, there exists $\psi_{n_j} \in Y_{n_j}$ such that $\psi_{n_j} \to v$, hence

$$\lim_{n_j \to +\infty} < E^{'}(v_{n_j}), v_{n_j} - v > = \lim_{n_j \to \infty} < E^{'}(v_{n_j}), v_{n_j} - \psi_{n_j} > + \lim_{n_j \to \infty} < E^{'}(v_{n_j}), \psi_{n_j} - v >$$
$$= \lim_{n_j \to \infty} < (E|_{Y_{n_j}})^{'}(v_{n_j}), v_{n_j} - \psi_{n_j} >= 0.$$

In a similar manner used in the proof of $(PS)_c$ condition in Lemma 3.1, we arrive at $v_{n_j} \to v$. This yields $E^{'}(v_{n_j}) \to E^{'}(v)$. In other hand, for every $\phi_k \in Y_k$ one has

$$< E^{'}(v), \phi_k > = \lim_{n_j \to \infty} < E^{'}(v_{n_j}), \phi_k >= \lim_{n_j \to \infty} < (E|_{Y_{n_j}})^{'}(v_{n_j}), \phi_k >= 0.$$

So $E^{'}(v) = 0$. Finally E satisfies $(PS)_c^*$ condition for all $c \in \mathbb{R}$.

Utilizing dual Fountain theorem ([12]), Proposition 4.3, Proposition 4.4, Proposition 4.5, and Lemma 4.1, we prove (ii) of Theorem 1.2. □

5. Conclusions

Invoking variational method and critical point theory, we established the existence of at least one solution when $\omega > 0$ a.e. in Ω to the double phase (1), then the existence of infinitely many solutions to (1) when $\mu\omega(x)$ is positive and when it keeps a constant sign.

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Abdessamad El katit is a research scholar in the department of mathematics at Mohamed First University. He completed his master degree from Mohamed first University. His research area of interests are Partial Differential Equations, Difference Equations, Nonlinear Analysis and Critical point theory.

Abdelrachid El Amrouss is a full professor in department of mathematics at the faculty of sciences, Mohamed First University. He completed his PhD in the field of Nonlinear Analysis in 1998. He has supervised numerous researchers in the fields of Partial Differential Equations and Nonlinear Analysis and has made significant contributions in critical point theory to various prestigious academic journals.

Fouad Kissi has been a full professor at FSJES, Mohamed First University, since 2014. He completed his PhD in the field of Applied Analysis in 2012. He is member of the research team in Nonlinear Analysis applied to partial differential equations, focusing on approximation and numerical simulation at the LAMAO Laboratory since 2018.