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CHARACTERIZATION FOR ADDITIVE WEIBULL DISTRIBUTION BASED ON PROGRESSIVE FIRST FAILURE CENSORING

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Abstract. In this article, we establish recurrence relations (RR) for single and product moments based on progressive first failure censoring (PFFC). Characterizations for additive Weibull distribution (AWD) using the relation between the probability density function and distribution function and using the RR of single and product moments based on PFFC are also obtained. Further, the results are specialized to the progressively type-II right censored order statistics (PTIIRCOS).

Keywords: Additive Weibull Distribution; Characterization; Progressive First Failure Censoring; Recurrence Relations.

AMS Subject Classification: 47H17; 47H05; 47H09.

1. Introduction

In life testing and reliability analysis, certain items in the study may not experience failure during the observation period (referred to as censored samples) or may be withdrawn from testing before they fail. This process of censoring arises when the precise lifetimes of only a portion of the tested items are established. One primary rationale for removing items prematurely from the experiment is to preserve them for future use, thereby reducing the expenses and time associated with ongoing testing efforts. In practical terms, the study typically involves two key variables: time and the occurrence of item failures. Censoring strategies illustrate how researchers structure experiments based on predetermined criteria. Type-I censoring involves a random selection of items at a specified time for the experiment's termination, without knowing their exact failure times. Type-II censoring, on the other hand, fixes the number of failure occurrences but allows for variability in the times of these failures. Both schemes ensure items remain in the experiment until the final stage or until they fail, facilitating the identification of defective

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items post-experimentation. A more adaptable approach, known as progressive Type-II censoring, offers greater flexibility by allowing items to be withdrawn from testing at various observed failure times.

Despite the potential for prolonged testing periods due to the inclusion of units with significant ages, many researchers favor Type II censoring. However, it should be noted that Type II censoring has a limitation in that once initiated, units cannot be removed from the experiment. Therefore, a more flexible censorship approach, allowing for the withdrawal of units during the course of the test, is termed Type II progressive censoring (PTIIC). Progressive control strategies have gained attention recently for their adaptability in permitting the removal of units at any point other than the endpoint. Various forms of progressive control and censoring systems have been introduced, including Type I, Type II, and hybrid progressive control systems. However, conducting investigations, particularly for highly reliable products, can be time-intensive with these control methods. An effective approach to address this challenge involves grouping tested units into sets of equal size, and monitoring the time until the first failure within each group using a progressive first failure censoring scheme (PFFC), see [2,4,5,6].

2. Progressive First Failure Censoring

Schematically, PFFC is as follows: Suppose that n independent groups with k items within each group are put on a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $X_{1:m:n,k}^{(R_1,R_2,\ldots,R_m)}$ has occurred and finally when the m^{th} failure $X_{m:m:n,k}^{(R_1,R_2,\ldots,R_m)}$ is observed, the remaining groups R_m are removed from the test. Then $X_{1:m:n,k}^{(R_1,R_2,\ldots,R_m)} < \cdots < X_{m:m:n,k}^{(R_1,R_2,\ldots,R_m)}$ are called progressively first-failure censored order statistics with the progressive censored scheme, where $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with cdf and pdf, the joint pdf for $X_{1:m:n,k}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n,k}^{(R_1,R_2,\ldots,R_m)}$ is defined as follows:

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_1, x_2, x_3, \ldots, x_{m-1}, x_m) = I_{(n,m-1)} k^m \prod_{i=1}^{m} f(x_i) \left[\overline{F}(x_i)\right]^{kR_i+k-1}, \quad 0 < x_1 < x_2 < x_3 < \cdots < x_{m-1} < x_m < \infty, \tag{1}$$

where,

$$I_{(n,m-1)} = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\ldots(n - R_1 - R_2 - R_3 \cdots - R_{m-1} - m + 1).$$

Recurrence relations (RR) play a pivotal role in mathematical analysis, particularly in the study of single and product moments. These relations provide a systematic way to compute moments of random variables and products thereof, offering insights into their behavior and distribution characteristics. For single moments, RR allows us to recursively calculate expectations, variances, and higher-order moments based on simpler initial conditions. This iterative approach simplifies complex calculations and facilitates the analysis of probabilistic models. Similarly, in the context of product moments, RR extends this methodology to handle the joint distribution of multiple random variables. By defining relationships between moments of products, RR enables the derivation of covariance structures and correlations, which is essential for understanding dependencies within stochastic processes and systems. The elegance of recurrence relations lies in their ability to transform intricate probabilistic problems into manageable sequences of equations, offering both theoretical depth and practical utility in the statistics field. For this reason, RR have received great attention recently, from statistical researchers, for example, Mohie El-Din et al. [9,10.11] derived RR of moments of the generalized Pareto, Gompertz and linear failure rate distributions based on general PTIIRCOS and characterization. Sadek et al. [14] derived characterization for generalized power function distribution using RR based on general PTIIRCOS. Marwa Mohie El-Din and Sharawy [8] derived RR for the generalized exponential distributions based on general PTIIRCOS. Alsadat et al. [3] derived study of RR and characterizations based on PFFC. Sharawy [15] derived RR for moment generating function of PFFC and characterizations of right truncated exponential distribution. Kotb et al. [12] derived E-Bayesian estimation for Kumaraswamy distribution using PFFC. Abu-Moussa et al. [1] derived estimation of reliability functions for the extended Rayleigh distribution under PFFC Model. Lemonte et al. and Xie and Lai [13,16] proposed the additive Weibull model based on the simple idea of combining the failure rates of two Weibull distribution, one has a decreasing failure rate and the other one has an increasing failure rate.

The Additive Weibull Distribution (AWD) originated as an extension of the traditional Weibull distribution. AWD is widely used in reliability engineering to model the lifetimes of components and materials, where the failure rate changes over time. The AWD enhances this by allowing for the combination of multiple independent AWD variables, making it suitable for scenarios involving complex systems or compounded failures, which benefit from a quite flexible distribution for fitting lifetime data with a bathtub-shaped failure rate function. Historically, the AWD has evolved through theoretical advancements in reliability theory and statistical modeling, expanding its applicability beyond traditional Weibull settings. Its features include the ability to model scenarios where failures result from the simultaneous occurrence of multiple independent events, making it valuable in systems engineering and quality control. The probability density function (pdf) of the AWD is

$$f\left(x,\alpha,\beta,\sigma,\theta\right)=(\alpha\theta x^{\theta-1}+\beta\sigma x^{\beta-1})e^{-\alpha x^{\theta}-\sigma x^{\beta}},\quad \alpha,\sigma>0,\quad \theta>\beta>0\ ,\qquad x>0, \tag{2}$$

and the corresponding cumulative distribution function (cdf) is given by

$$F\left(x,\alpha,\beta,\sigma,\theta\right)=1-e^{-\alpha x^{\theta}-\sigma x^{\beta}},\qquad \alpha,\sigma>0,\quad \theta>\beta>0\ ,\qquad x>0. \tag{3}$$

It may be noticed that from (2) and (3) the relation between pdf and cdf is given by,

$$f\left(x\right)=\left(\alpha\theta x^{\theta-1}+\beta\sigma x^{\beta-1}\right)\left[1-F\left(x\right)\right]. \tag{4}$$

For any continuous distribution, we shall denote the i^{th} single moment of the PFFC in view of Eq. (1) as

$$\begin{aligned}\mu_{q:m:n,k}^{(kR_1+k-1,kR_2+k-1,\ldots,kR_m+k-1)^{(i)}} &= E\Big[X_{q:m:n,k}^{(kR_1+k-1,kR_2+k-1,\ldots,kR_m+k-1)}\Big]^{i}\\ = I_{(n,m-1)}\iint\cdots\int_{0<x_1<\cdots<x_m<\infty} & x_q^{i}k^{m}f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1}\times\\ f\left(x_2\right)\left[\overline{F}\left(x_2\right)\right]^{kR_2+k-1} & \ldots f\left(x_m\right)\left[\overline{F}\left(x_m\right)\right]^{kR_m+k-1}dx_1\ldots dx_m,\end{aligned} \tag{5}$$

where $\left[\overline{F}\left(x\right)\right]=\left[1-F\left(x\right)\right]$, and the i^{th} and j^{th} product moments as

$$\begin{aligned}\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,r)}} &= E\left[X_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{i}}\ X_{s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{r}}\right]\\ = I_{(n,m-1)}\iint\cdots\int_{0<x_1<\cdots<x_m<\infty} & x_q^{i}x_s^{j}k^{m}f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1}\times\\ f\left(x_2\right)\left[\overline{F}\left(x_2\right)\right]^{kR_2+k-1} & \ldots f\left(x_m\right)\left[\overline{F}\left(x_m\right)\right]^{kR_m+k-1}dx_1\ldots dx_m.\end{aligned} \tag{6}$$

3. Recurrence Relations of Single and Product Moments

RR are invaluable tools in statistical analysis, particularly when applied to moments in the context of PFFC schemes. These schemes involve systematically removing units from testing upon their first failure, allowing for flexibility in experiment duration. For single moments, RR in this framework enables iterative computation of expected values, variances, and higher-order moments, starting from initial conditions and updating as failures occur. This method not only simplifies complex calculations but also enhances understanding of how moments evolve over time. In the case of product moments, RR extends their utility by facilitating the calculation of joint distributions and covariance structures amidst ongoing censoring events. Such analyses are crucial in fields like reliability engineering and clinical trials, where understanding the dynamics of failure and its impact on statistical outcomes is paramount. Through RR, researchers can effectively model and predict the behavior of systems subject to progressive first failure censoring, thereby advancing both theoretical insights and practical applications in probabilistic analysis. In the next theorem we introduce the RR for single moments based on PFFC.

Theorem 3.1. *If $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n following AWD, for $2 = \mathrm{q} \leq m-1, \quad m \leq n$ and $i \geq 0$, then*

$$\begin{aligned}
&\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i)}} = \\
&(kR_q+k)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+\beta)}} + \frac{\alpha\theta}{\theta+i}\,\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+\theta)}}\right) \\
&-\frac{\beta\sigma\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\beta+i}\mu_{q-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)^{(i+\beta)}} \\
&+\frac{\beta\sigma\left(n-R_1-\cdots-R_q-q\right)}{\beta+i}\,\mu_{q:m-1:n,k}^{(kR_1+k-1,\dots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\dots,kR_m+k-1)^{(i+\beta)}} \\
&-\frac{\alpha\theta\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\theta+i}\mu_{q-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)^{(i+\theta)}} \\
&+\frac{\alpha\theta\left(n-R_1-\cdots-R_q-q\right)}{\theta+i}\,\mu_{q:m-1:n,k}^{(kR_1+k-1,\dots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\dots,kR_m+k-1)^{(i+\theta)}}.
\end{aligned} \tag{7}$$

Proof. From (4) and (5), we get

$$\begin{aligned}
&\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i)}} = I_{(n,m-1)}\iint\cdots\int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} W_1\left(x_{q-1},x_{q+1}\right)k^m\times \\
&f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1}\dots f\left(x_{q-1}\right)\left[\overline{F}\left(x_{q-1}\right)\right]^{kR_{q-1}+k-1}\times \\
&f\left(x_{q+1}\right)\left[\overline{F}\left(x_{q+1}\right)\right]^{kR_{q+1}+k-1}\dots f\left(x_m\right)\left[\overline{F}\left(x_m\right)\right]^{kR_m+k-1}dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m,
\end{aligned} \tag{8}$$

where

$$W_1\left(x_{q-1},x_{q+1}\right) = \int_{x_{q-1}}^{x_{q+1}}\left(\alpha\theta x_q^{i+\theta-1}+\beta\sigma x_q^{i+\beta-1}\right)\left[\overline{F}\left(x_q\right)\right]^{kR_q+k}\,dx_q. \tag{9}$$

Now, integrating by parts gives

$$\begin{aligned} W_1\left(x_{q-1},x_{q+1}\right)=&\frac{x_{q+1}^{i+\beta}\left[\overline{F}\left(x_{q+1}\right)\right]^{kR_q+k}-x_{q-1}^{i+\beta}\left[\overline{F}\left(x_{q-1}\right)\right]^{kR_q+k}}{i+\beta}\\ &+\left(\frac{kR_q+k}{i+\beta}\right)\int_{x_{q-1}}^{x_{q+1}}x_q^{i+\beta}f\left(x_q\right)\left[\overline{F}\left(x_q\right)\right]^{kR_q+k-1}\,dx_q\\ &+\frac{x_{q+1}^{i+\vartheta}\left[\overline{F}\left(x_{q+1}\right)\right]^{kR_q+k}-x_{q-1}^{i+\theta}\left[\overline{F}\left(x_{q-1}\right)\right]^{kR_q+k}}{i+\theta}\\ &+\left(\frac{kR_q+k}{i+\theta}\right)\int_{x_{q-1}}^{x_{q+1}}x_q^{i+\theta}f\left(x_q\right)\left[\overline{F}\left(x_q\right)\right]^{kR_q+k-1}\,dx_q. \end{aligned} \tag{10}$$

Substituting by (10) in (8) and simplifying, yields (7).
This completes the proof. □

Special Cases.

1- Theorem 3.1 will be valid for the PTIIRCOS as a special case from the PFFC when $k=1$,

$$\begin{aligned} \mu_{q:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}}=&\left(R_q+1\right)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{q:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+\beta)}}+\frac{\alpha\theta}{\theta+i}\,\mu_{q:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+\theta)}}\right)\\ &-\frac{\beta\sigma}{\beta+i}\left(n-R_1-\cdots-R_{q-1}-q+1\right)\mu_{q-1:m-1:n}^{\left(R_1,R_2,\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m\right)^{(i+\beta)}}\\ &+\frac{\beta\sigma}{\beta+i}\,\left(n-R_1-\cdots-R_q-q\right)\mu_{q:m-1:n}^{\left(R_1,R_2,\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m\right)^{(i+\beta)}}\\ &-\frac{\alpha\theta}{\theta+i}\left(n-R_1-\cdots-R_{q-1}-q+1\right)\mu_{q-1:m-1:n}^{\left(R_1,R_2,\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m\right)^{(i+\theta)}}\\ &+\frac{\alpha\theta}{\theta+i}\,\left(n-R_1-\cdots-R_q-q\right)\mu_{q:m-1:n}^{\left(R_1,R_2,\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m\right)^{(i+\theta)}}. \end{aligned}$$

2- For $k=1$ *and* $q=m$

$$\begin{aligned} \mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}}=&\left(R_m+1\right)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+\beta)}}+\frac{\alpha\theta}{\theta+i}\,\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+\theta)}}\right)\\ &-\frac{\beta\sigma}{\beta+i}\left(n-R_1-\cdots-R_{m-1}-m+1\right)\mu_{m-1:m-1:n}^{\left(R_1,R_2,\ldots,(R_{m-1}+R_m+1)\right)^{(i+\beta)}}\\ &-\frac{\alpha\theta}{\theta+i}\left(n-R_1-\cdots-R_{m-1}-m+1\right)\mu_{m-1:m-1:n}^{\left(R_1,R_2,\ldots,(R_{m-1}+R_m+1)\right)^{(i+\theta)}}. \end{aligned}$$

3- For $k=1$ *and* $2\le m\le n$

$$\begin{aligned} \mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}}=&\left(R_1+1\right)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+\beta)}}+\frac{\alpha\theta}{\theta+i}\,\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+\theta)}}\right)\\ &+\frac{\beta\sigma}{\beta+i}\,\left(n-R_1-\cdots-R_m-m\right)\mu_{1:m-1:n}^{\left((R_1+R_2+1),R_3,\ldots,R_m\right)^{(i+\beta)}}, \end{aligned}$$

and for $k=0$, $m=1$ *and* $n=1,2,\ldots$

$$\mu_{1:1:n}^{(n-1)^{(i)}}=\frac{\beta\sigma}{\beta+i}\,\mu_{1:1:n}^{(n-1)^{(i+\beta)}}+\frac{\alpha\theta}{\theta+i}\,\mu_{1:1:n}^{(n-1)^{(i+\theta)}}.$$

In the next two theorems, we shall introduce RR for product moments based on PFFC.

Theorem 3.2. *If $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n following AWD, for $1 \le q < s \le m-1$, $m \le n$ and $i, j \ge 0$, then*

$$\begin{aligned}
&\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,j)}} = \\
&(kR_q+k)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}} + \frac{\alpha\theta}{\theta+i}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}}\right)\\
&-\frac{\beta\sigma\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\beta+i}\mu_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}}\\
&+\frac{\beta\sigma\left(n-R_1-\cdots-R_q-q\right)}{\beta+i}\,\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}}\\
&-\frac{\alpha\theta\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\theta+i}\mu_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}}\\
&+\frac{\alpha\theta\left(n-R_1-\cdots-R_q-q\right)}{\theta+i}\,\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}}.
\end{aligned} \tag{11}$$

Proof. From (4) and (6), we get

$$\begin{aligned}
&(kR_q+k)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}} + \frac{\alpha\theta}{\theta+i}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}}\right)\\
&-\frac{\beta\sigma\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\beta+i}\mu_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}}\\
&+\frac{\beta\sigma\left(n-R_1-\cdots-R_q-q\right)}{\beta+i}\,\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}}\\
&-\frac{\alpha\theta\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\theta+i}\mu_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}}\\
&+\frac{\alpha\theta\left(n-R_1-\cdots-R_q-q\right)}{\theta+i}\,\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}}.
\end{aligned} \tag{12}$$

Substituting the resultant expression of $W_1\left(x_{q-1}, x_{q+1}\right)$ from Eq. (10) in Eq. (12) and simplifying, yields Eq. (11).

This completes the proof. □

Special Case. For $k=1$, we obtain the recurrence relation of PTIICOS.

$$\begin{aligned}
&(R_q+1)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{q,s:m:n,k}^{(R_1,R_2,\ldots,R_m)^{(i+\beta,j)}} + \frac{\alpha\theta}{\theta+i}\,\mu_{q,s:m:n,k}^{(R_1,R_2,\ldots,R_m)^{(i+\theta,j)}}\right)\\
&-\frac{\beta\sigma}{\beta+i}\left(n-R_1-\cdots-R_{q-1}-q+1\right)\mu_{q-1,s-1:m-1:n,k}^{(R_1,R_2,\ldots,R_{q-1}+R_q+1,R_{q+1},\ldots,R_m)^{(i+\beta,j)}}\\
&+\frac{\beta\sigma}{\beta+i}\,\left(n-R_1-\cdots-R_q-q\right)\mu_{q,s-1:m-1:n,k}^{(R_1,R_2,\ldots,R_q+R_{q+1}+1,R_{q+2},\ldots,R_m)^{(i+\beta,j)}}\\
&-\frac{\alpha\theta}{\theta+i}\left(n-R_1-\cdots-R_{q-1}-q+1\right)\mu_{q-1,s-1:m-1:n,k}^{(R_1,R_2,\ldots,R_{q-1}+R_q+1,R_{q+1},\ldots,R_m)^{(i+\theta,j)}}\\
&+\frac{\alpha\theta}{\theta+i}\,\left(n-R_1-\cdots-R_q-q\right)\mu_{q,s-1:m-1:n,k}^{(R_1,R_2,\ldots,R_q+R_{q+1}+1,R_{q+2},\ldots,R_m)^{(i+\theta,j)}}.
\end{aligned}$$

Theorem 3.3. *If $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n following AWD, for $1 \le q < s \le m-1$, $m \le n$ and $i, j \ge 0$, then*

$$
\begin{aligned}
&\mu_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i,j)}} = \\
&(kR_s+k)\left(\frac{\beta\sigma}{\beta+j}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i,j+\beta)}} + \frac{\alpha\theta}{\theta+j}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i,j+\theta)}}\right) \\
&-\frac{\beta\sigma\left(n-R_1-\cdots-R_{s-1}-s+1\right)}{\beta+j}\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\dots,kR_m+k-1)^{(i,j+\beta)}} \\
&+\frac{\beta\sigma\left(n-R_1-\cdots-R_s-s\right)}{\beta+j}\,\mu_{q,s:m-1:n,k}^{(kR_1+k-1,\dots,kR_s+kR_{s+1}+2k-1,kR_{s+2}+k-1,\dots,kR_m+k-1)^{(i,j+\beta)}} \\
&-\frac{\alpha\theta\left(n-R_1-\cdots-R_{s-1}-s+1\right)}{\theta+j}\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\dots,kR_m+k-1)^{(i,j+\theta)}} \\
&+\frac{\alpha\theta\left(n-R_1-\cdots-R_s-s\right)}{\theta+i}\,\mu_{q,s:m-1:n,k}^{(kR_1+k-1,\dots,kR_s+kR_{s+1}+2k-1,kR_{s+2}+k-1,\dots,kR_m+k-1)^{(i,j+\theta)}}.
\end{aligned} \tag{13}
$$

Proof. From (4) and (6), we get

$$
\begin{aligned}
&\mu_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i,j)}} = I_{(n,m-1)} \iint\cdots\int_{0<x_1<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty} k^m \times \\
&W_2\left(x_{s-1},x_{s+1}\right) f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1}\dots f\left(x_{s-1}\right)\left[\overline{F}\left(x_{s-1}\right)\right]^{kR_{s-1}+k-1}\times \\
&f\left(x_{s+1}\right)\left[\overline{F}\left(x_{s+1}\right)\right]^{kR_{s+1}+k-1}\dots f\left(x_m\right)\left[\overline{F}\left(x_m\right)\right]^{kR_m+k-1} dx_1\dots dx_{s-1}dx_{s+1}\dots dx_m,
\end{aligned} \tag{14}
$$

where

$$
W_2\left(x_{s-1},x_{s+1}\right) = \int_{x_{s-1}}^{x_{s+1}} \left(\alpha\theta x_s^{i+\theta-1}+\beta\sigma x_s^{i+\beta-1}\right)\left[\overline{F}\left(x_s\right)\right]^{kR_s+k}\,dx_s. \tag{15}
$$

Now, integrating by parts gives

$$
\begin{aligned}
W_2\left(x_{s-1},x_{s+1}\right) =& \frac{x_{s+1}^{i+\beta}\left[\overline{F}\left(x_{s+1}\right)\right]^{kR_s+k} - x_{s-1}^{i+\beta}\left[\overline{F}\left(x_{s-1}\right)\right]^{kR_s+k}}{i+\beta} \\
&+\left(\frac{kR_s+k}{i+\beta}\right)\int_{x_{s-1}}^{x_{s+1}} x_s^{i+\beta} f\left(x_s\right)\left[\overline{F}\left(x_s\right)\right]^{kR_s+k-1}\,dx_s \\
&+\frac{x_{s+1}^{i+\vartheta}\left[\overline{F}\left(x_{s+1}\right)\right]^{kR_s+k} - x_{s-1}^{i+\theta}\left[\overline{F}\left(x_{s-1}\right)\right]^{kR_s+k}}{i+\theta} \\
&+\left(\frac{kR_s+k}{i+\theta}\right)\int_{x_{s-1}}^{x_{s+1}} x_s^{i+\theta} f\left(x_s\right)\left[\overline{F}\left(x_s\right)\right]^{kR_s+k-1}\,dx_s.
\end{aligned} \tag{16}
$$

Substituting by (16) in (14) and simplifying, yields (13). □

Special Case. For $k=1$, we obtain the recurrence relation of PTIICOS.

$$
\begin{aligned}
&(R_s+1)\left(\frac{\beta\sigma}{\beta+j}\ \mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+\beta)}}+\frac{\alpha\theta}{\theta+j}\ \mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+\theta)}}\right)\\
&-\frac{\beta\sigma}{\beta+j}\left(n-R_1-\cdots-R_{s-1}-s+1\right)\mu_{q,s-1:m-1:n,k}^{(R_1,R_2,\ldots,R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(i,j+\beta)}}\\
&+\frac{\beta\sigma}{\beta+j}\ \left(n-R_1-\cdots-R_s-s\right)\mu_{q,s:m-1:n}^{(R_1,R_2,\ldots,R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(i,j+\beta)}}\\
&-\frac{\alpha\theta}{\theta+j}\left(n-R_1-\cdots-R_{s-1}-s+1\right)\mu_{q,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(i,j+\theta)}}\\
&+\frac{\alpha\theta}{\theta+i}\ \left(n-R_1-\cdots-R_s-s\right)\mu_{q,s:m-1:n}^{(R_1,R_2,\ldots,R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(i,j+\theta)}},
\end{aligned}
$$

and for $s=m$

$$
\begin{aligned}
&(R_m+1)\left(\frac{\beta\sigma}{\beta+j}\ \mu_{q,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+\beta)}}+\frac{\alpha\theta}{\theta+j}\ \mu_{q,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+\theta)}}\right)\\
&-\frac{\beta\sigma}{\beta+j}\left(n-R_1-\cdots-R_{s-1}-s+1\right)\mu_{q,m-1:m-1:n,k}^{(R_1,R_2,\ldots,R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(i,j+\beta)}}\\
&-\frac{\alpha\theta}{\theta+j}\left(n-R_1-\cdots-R_{s-1}-s+1\right)\mu_{q,m-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(i,j+\theta)}}.
\end{aligned}
$$

4. The Characterizations

Characterizations of distributions hold significant importance for researchers in applied fields, particularly when assessing the suitability of their models to specific distributions. Investigators are keen to ascertain if their models meet the criteria of a given distribution, relying heavily on these characterizations. These definitions outline conditions that confirm whether the underlying distribution aligns with the desired theoretical model. Numerous characterizations of distributions have been formulated across various avenues of research. In this study, several characterizations of the AWD are detailed, including characterizations via differential equation for the general distribution, via RR for a single moment, and via RR for product moments.

4.1. **Characterization via differential equation for the general distribution.** In the next theorem, we introduce the characterization of the AWD using relation between pdf and cdf.

Theorem 4.1. *Let X be a continuous random variable with pdf $f(\bullet)$, cdf $F(\bullet)$ and survival function $\overline{F}$. Then X has AWD iff*

$$
f(x)=\left(\alpha\theta x^{\theta-1}+\beta\sigma x^{\beta-1}\right)\left[\overline{F}(x)\right]. \tag{17}
$$

Proof. **Necessity:** From (2) and (3) we can easily obtain (17).

Sufficiency: Suppose that (17) is true. Then we have:

$$
\frac{-d\left[\overline{F}(x)\right]}{\overline{F}(x)}=(\alpha\theta x^{\theta-1}+\beta\sigma x^{\beta-1})dx.
$$

By integrating, we get

$$
-\ln\left|\overline{F}(x)\right| \ =(\alpha x^{\theta}+\sigma x^{\beta})+C, \tag{18}
$$

where C is an arbitrary constant.

Now, since $\overline{F}(0)=1$, then putting $x=0$ in (18), we get $C=0$.

Therefore,

$$\ln\left|\left[\overline{F}(x)\right]\right| = -\left(\alpha x^{\theta} + \sigma x^{\beta}\right),$$

or,

$$\left[\overline{F}(x)\right] = \exp\left\{-\alpha x^{\theta} - \sigma x^{\beta}\right\}.$$

Hence,

$$F(x) = 1 - \exp\left\{-\alpha x^{\theta} - \sigma x^{\beta}\right\}.$$

That is the distribution function of AWD.
This completes the proof. □

4.2. **Characterization via RR for single moment.** In the next theorem, we will introduce the characterization of the AWD using recurrence relation for single moments based on PFFC.

Theorem 4.2. *Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order stasistics of a random sample of size n. Then X has AWD iff, for $2 = q \leq m-1$, $m \leq n$ and $i \geq 0$,*

$$\begin{aligned}
&\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i)}} = \\
&(kR_q+k)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\beta)}} + \frac{\alpha\theta}{\theta+i}\,\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\theta)}}\right) \\
&-\frac{\beta\sigma\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\beta+i}\mu_{q-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\beta)}} \\
&+\frac{\beta\sigma\left(n-R_1-\cdots-R_q-q\right)}{\beta+i}\,\mu_{q:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\beta)}} \\
&-\frac{\alpha\theta\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\theta+i}\mu_{q-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\theta)}} \\
&+\frac{\alpha\theta\left(n-R_1-\cdots-R_q-q\right)}{\theta+i}\,\mu_{q:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\theta)}}.
\end{aligned} \tag{19}$$

Proof. **Necessity:** Theorem 3.1 proved the necessary part of this theorem.
Sufficiency: Assuming that Eq. (19) holds, then we have:

$$\begin{aligned}
&\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i)}} = \\
&(kR_q+k)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\beta)}} + \frac{\alpha\theta}{\theta+i}\,\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\theta)}}\right) \\
&-\frac{\beta\sigma\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\beta+i}\mu_{q-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\beta)}} \\
&+\frac{\beta\sigma\left(n-R_1-\cdots-R_q-q\right)}{\beta+i}\,\mu_{q:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\beta)}} \\
&-\frac{\alpha\theta\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\theta+i}\mu_{q-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\theta)}} \\
&+\frac{\alpha\theta\left(n-R_1-\cdots-R_q-q\right)}{\theta+i}\,\mu_{q:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\theta)}}.
\end{aligned} \tag{20}$$

where,

$$\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\beta)}} = I_{(n,m-1)} \iint \cdots \int_{0<x_1\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} W_2\left(x_{q-1},x_{q+1}\right) \times$$
$$k^m f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1} \ldots f\left(x_{q-1}\right)\left[\overline{F}\left(x_{q-1}\right)\right]^{kR_{q-1}+k-1} \times$$
$$f\left(x_{q+1}\right)\left[\overline{F}\left(x_{q+1}\right)\right]^{kR_{q+1}+k-1} \ldots f\left(x_m\right)\left[\overline{F}\left(x_m\right)\right]^{kR_m+k-1} dx_1 \ldots dx_{q-1} dx_{q+1} \ldots dx_m, \tag{21}$$

where

$$W_2\left(x_{q-1},x_{q+1}\right) = \int_{x_{q-1}}^{x_{q+1}} x_q^{i+\beta} f\left(x_q\right)\left[\overline{F}\left(x_q\right)\right]^{kR_q+k-1} dx_q. \tag{22}$$

Integrating by parts, we obtain

$$\begin{aligned} W_2\left(x_{q-1},x_{q+1}\right) =& \frac{-1}{kR_q+k} x_{q+1}^{i+\beta}\left[\overline{F}\left(x_{q+1}\right)\right]^{kR_q+k} + \frac{1}{kR_q+k} x_{q-1}^{i+\beta}\left[\overline{F}\left(x_{q-1}\right)\right]^{kR_q+k} \\ &+ \frac{i+\beta}{kR_q+k} \int_{x_{q-1}}^{x_{q+1}} x_q^{i+\beta-1}\left[\overline{F}\left(x_q\right)\right]^{kR_q+k} dx_q. \end{aligned} \tag{23}$$

Substituting in Eq. (21), we get

$$\mu_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\beta)}} = \frac{i+\beta}{kR_q+k} I_{(n,m-1)} \iint \cdots \int_{0<x_1\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} k^m \times$$
$$f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1} \ldots f\left(x_{q-1}\right)\left[\overline{F}\left(x_{q-1}\right)\right]^{kR_{q-1}+k-1} \int_{x_{q-1}}^{x_{q+1}} x_q^{i+\beta-1}\left[\overline{F}\left(x_q\right)\right]^{kR_q+k} dx_q$$
$$f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{kR_{q+1}+k-1} \ldots f\left(x_m\right)\left[1-F\left(x_m\right)\right]^{kR_m+k-1} dx_1 \ldots dx_{q-1} dx_{q+1} \ldots dx_m$$
$$+ \frac{I_{(n,m-1)}}{kR_q+k} \iint \cdots \int_{0<x_1\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} x_{q-1}^{i+\beta} k^m f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1} \cdots \times$$
$$\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1} \ldots f\left(x_{q-1}\right)\left[\overline{F}\left(x_{q-1}\right)\right]^{kR_{q-1}+kR_q+2k-1} f\left(x_{q+1}\right)\left[\overline{F}\left(x_{q+1}\right)\right]^{kR_{q+1}+k-1} \cdots \times$$
$$f\left(x_m\right)\left[\overline{F}\left(x_m\right)\right]^{kR_m+k-1} dx_1 \ldots dx_{q-1} dx_{q+1} \ldots dx_m$$
$$- \frac{I_{(n,m-1)}}{kR_q+k} \iint \cdots \int_{0<x_1\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} x_{q+1}^{i+\beta} k^m \times$$
$$f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1} \ldots f\left(x_{q+1}\right)\left[\overline{F}\left(x_{q+1}\right)\right]^{kR_q+kR_{q+1}+2k-1} \cdots \times$$
$$f\left(x_m\right)\left[\overline{F}\left(x_m\right)\right]^{R_m} dx_1 \ldots dx_{q-1} dx_{q+1} \ldots dx_m$$
$$= I_{(n,m-1)} \frac{i+\beta}{kR_q+k} \iint \cdots \int_{0<x_1\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} k^m \cdots \times$$
$$\int_{x_{q-1}}^{x_{q+1}} x_q^{i+\beta-1}\left[\overline{F}\left(x_q\right)\right]^{kR_q+k} dx_q f\left(x_1\right)\left[\overline{F}\left(x_1\right)\right]^{kR_1+k-1} \ldots f\left(x_{q-1}\right)\left[\overline{F}\left(x_{q-1}\right)\right]^{kR_{q-1}+k-1}$$
$$f\left(x_{q+1}\right)\left[\overline{F}\left(x_{q+1}\right)\right]^{kR_{q+1}+k-1} \ldots f\left(x_m\right)\left[\overline{F}\left(x_m\right)\right]^{kR_m+k-1} dx_1 \ldots dx_{q-1} dx_{q+1} \ldots dx_m$$
$$- \frac{\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{kR_q+k} \mu_{q-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\beta)}}$$
$$+ \frac{\left(n-R_1-\cdots-R_q-q\right)}{kR_q+k} \mu_{q:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\beta)}}, \tag{24}$$

and

$$\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+\theta)}} = I_{(n,m-1)} \frac{i+\theta}{kR_q+k} \iint \cdots \int_{0<x_1\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} k^m \cdots \times$$
$$\int_{x_{q-1}}^{x_{q+1}} x_q^{i+\theta-1} \left[\overline{F}(x_q)\right]^{kR_q+k} dx_q f(x_1) \left[\overline{F}(x_1)\right]^{kR_1+k-1} \dots f(x_{q-1}) \left[\overline{F}(x_{q-1})\right]^{kR_{q-1}+k-1}$$
$$f(x_{q+1}) \left[\overline{F}(x_{q+1})\right]^{kR_{q+1}+k-1} \dots f(x_m) \left[\overline{F}(x_m)\right]^{kR_m+k-1} dx_1 \dots dx_{q-1} dx_{q+1} \dots dx_m$$
$$- \frac{(n-R_1-\cdots-R_{q-1}-q+1)}{kR_q+k} \mu_{q-1:m-1:n}^{(kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)^{(i+\theta)}}$$
$$+ \frac{(n-R_1-\cdots-R_q-q)}{kR_q+k} \mu_{q:m-1:n,k}^{(kR_1+k-1,\dots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\dots,kR_m+k-1)^{(i+\theta)}}. \tag{25}$$

Substituting for $\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+\beta)}}$ and $\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i+\theta)}}$ from (24) and (25) in (20), we get

$$\mu_{q:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i)}} = I_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} x_q^i (\alpha\theta x_q^{\theta-1} + \beta\sigma x_q^{\beta-1})$$
$$k^m \left[\overline{F}(x_q)\right]^{kR_q+k} f(x_1) \left[\overline{F}(x_1)\right]^{kR_1+k-1} \dots f(x_{q-1}) \left[\overline{F}(x_{q-1})\right]^{kR_{q-1}+k-1} \times \tag{26}$$
$$f(x_{q+1}) \left[\overline{F}(x_{q+1})\right]^{kR_{q+1}+k-1} \dots f(x_m) \left[\overline{F}(x_m)\right]^{kR_m+k-1} dx_1 \dots dx_m,$$

we get

$$I_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} x_q^i k^m f(x_q) \left[\overline{F}(x_q)\right]^{kR_q+k-1} \times$$
$$f(x_1) \left[\overline{F}(x_1)\right]^{kR_1+k-1} \dots f(x_{q-1}) \left[\overline{F}(x_{q-1})\right]^{kR_{q-1}+k-1}$$
$$f(x_{q+1}) \left[\overline{F}(x_{q+1})\right]^{kR_{q+1}+k-1} \dots f(x_m) \left[\overline{F}(x_m)\right]^{kR_m+k-1} dx_1 \dots dx_m$$
$$= I_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} x_q^i (\alpha\theta x_q^{\theta-1} + \beta\sigma x_q^{\beta-1}) k^m \left[\overline{F}(x_q)\right]^{kR_q+k} \cdots \times$$
$$f(x_1) \left[\overline{F}(x_1)\right]^{kR_1+k-1} \dots f(x_{q-1}) \left[\overline{F}(x_{q-1})\right]^{kR_{q-1}+k-1} f(x_{q+1}) \left[\overline{F}(x_{q+1})\right]^{kR_{q+1}+k-1} \cdots \times$$
$$f(x_m) \left[\overline{F}(x_m)\right]^{kR_m+k-1} dx_1 \dots dx_m. \tag{27}$$

We get

$$I_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} x_q^i \, k^m \left[\overline{F}(x_q)\right]^{kR_q+k-1} f(x_1) \left[\overline{F}(x_1)\right]^{Rk_1+k-1} \dots \times$$
$$f(x_{q-1}) \left[\overline{F}(x_{q-1})\right]^{kR_{q-1}+k-1} f(x_{q+1}) \left[\overline{F}(x_{q+1})\right]^{kR_{q+1}+k-1} \dots f(x_m) \left[\overline{F}(x_m)\right]^{kR_m+k-1}$$
$$[f(x_q) - (\alpha\theta x_q^{\theta-1} + \beta\sigma x_q^{\beta-1})[\overline{F}(x_q)]] dx_1 \dots dx_m = 0. \tag{28}$$

Using Muntz-Szasz theorem, [See, Hwang and Lin [7]], we get

$$f(x_q) = (\alpha\theta x_q^{\theta-1} + \beta\sigma x_q^{\beta-1})[\overline{F}(x_q)] .$$

Using Theorem 4.1, we get

$$F(x) = 1 - e^{-\alpha x^{\theta} - \sigma x^{\beta}}.$$

That is the distribution function of AWD.

This completes the proof. □

4.3. **Characterization via RR for product moments.** In the next two theorems, we will introduce the characterization of AWD using RR for product moments based on PFFC.

Theorem 4.3. *Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics of a random sample of size n. Then X has AWD iff, for $1 = q < s \leq m-1$, $m \leq n$ and $i, j \geq 0$,*

$$\begin{aligned}
&\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,j)}} = \\
&(kR_q+k)\left(\frac{\beta\sigma}{\beta+i}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}} + \frac{\alpha\theta}{\theta+i}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}}\right) \\
&-\frac{\beta\sigma\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\beta+i}\mu_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}} \\
&+\frac{\beta\sigma\left(n-R_1-\cdots-R_q-q\right)}{\beta+i}\,\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\beta,j)}} \\
&-\frac{\alpha\theta\left(n-R_1-\cdots-R_{q-1}-q+1\right)}{\theta+i}\mu_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}} \\
&+\frac{\alpha\theta\left(n-R_1-\cdots-R_q-q\right)}{\theta+i}\,\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1)^{(i+\theta,j)}}.
\end{aligned} \tag{29}$$

Proof. **Necessity:** Theorem 3.2 proved the necessary part of this theorem.

Sufficiency: Similarly as proved in theorem 4.2 we obtain the distribution function of AWD given by

$$F(x) = 1 - e^{-\alpha x^{\theta} - \sigma x^{\beta}}.$$

This completes the proof. □

Theorem 4.4. *Let $X_{1:n} \leq \ldots \leq X_{n:n}$ be the order statistics of a random sample of size n. Then X has AWD iff, for $1 = q < s \leq m-1$, $m \leq n$ and $i, j \geq 0$,*

$$\begin{aligned}
&\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,j)}} = \\
&(kR_s+k)\left(\frac{\beta\sigma}{\beta+j}\,\mu_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,j+\beta)}} + \frac{\alpha\theta}{\theta+j}\,\mu_{q,s:m:n}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(i,j+\theta)}}\right) \\
&-\frac{\beta\sigma\left(n-R_1-\cdots-R_{s-1}-s+1\right)}{\beta+j}\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\ldots,kR_m+k-1)^{(i,j+\beta)}} \\
&+\frac{\beta\sigma\left(n-R_1-\cdots-R_s-s\right)}{\beta+j}\,\mu_{q,s:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{s+1}+2k-1,kR_{s+2}+k-1,\ldots,kR_m+k-1)^{(i,j+\beta)}} \\
&-\frac{\alpha\theta\left(n-R_1-\cdots-R_{s-1}-s+1\right)}{\theta+j}\mu_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\ldots,kR_m+k-1)^{(i,j+\theta)}} \\
&+\frac{\alpha\theta\left(n-R_1-\cdots-R_s-s\right)}{\theta+i}\,\mu_{q,s:m-1:n,k}^{(kR_1+k-1,\ldots,kR_q+kR_{s+1}+2k-1,kR_{s+2}+k-1,\ldots,kR_m+k-1)^{(i,j+\theta)}}.
\end{aligned} \tag{30}$$

Proof. **Necessity:** Theorem 3.3 proved the necessary part of this theorem.

Sufficiency: Similarly as proved in theorem 4.2 we obtain the distribution function of AWD given by

$$F(x) = 1 - e^{-\alpha x^{\theta} - \sigma x^{\beta}}.$$

This completes the proof. □

5. Conclusion

This study introduces novel recurrence relations for single and product moments derived from the AWD through the framework of PFFC. The characterizations of the AWD are explored via differential equations governing the general distribution, recurrence relations for single moments (RR), and recurrence relations for product moments. The research underscores the significance of RR in facilitating the expression of higher-order moments of order statistics in terms of lower-order moments, thereby enhancing the computation of these higher-order moments. Moreover, RR play a crucial role in verifying the accuracy of moment computations for order statistics. Ultimately, these recurrence relations provide a foundation for deriving comprehensive characterizations of distributions.

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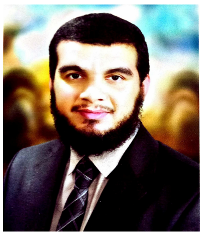

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