TWMS J. App. and Eng. Math. V.15, N.9, 2025, pp. 2216-2228

A NEW NUMERICAL METHOD FOR APPROXIMATION OF HYPERSINGULAR INTEGRALS

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ABSTRACT. In this paper, we investigate the construction of a new numerical method for approximating Cauchy and Hilbert hypersingular integrals, which are important in various fields such as engineering, physics, and applied mathematics due to their significant role in the solution of singular and hypersingular integral equations. To validate the theoretical analysis, we have conducted several numerical examples implemented in the MATLAB programming language. The obtained results demonstrate the stability, accuracy, and efficiency of the suggested approach. The proposed quadrature formulas are not only straightforward to compute but also scalable, ensuring the reliability and applicability of the method to a wide range of practical problems. This makes the method particularly useful for real-world applications requiring high computational efficiency.

Keywords: hypersingular integral, Cauchy kernel, Hilbert kernel, quadrature formula, approximation

AMS Subject Classification: 41A35, 47A58, 65R20, 45E05

1. Introduction

In recent years, the approximations for hypersingular integrals have gained great attention due to their occurrence in various fields of engineering research such as fracture analysis [8], heat conduction [11], aerodynamics [28], elasticity [31,36] and acoustics [38]. Therefore, the approximations for hypersingular integrals are necessary to construct numerical algorithms for solving hypersingular integral equations, which have been utilized to study various real-world engineering problems. The approximation of singular and hypersingular integrals and the development of constructive methods for solving singular and hypersingular integral equations with Cauchy and Hilbert kernels has been extensively studied in the works of Anfinogenov, A. Yu., Lifanov, I. K., Lifanov, P. I. [7], Boikov, I. V. [9-13], Sagaria, V. [14], Eshkuvatov, Z. K. [15,16,31,32], Hu, Ch., He, X., Lu, T. [24,25], Kolm, P., Rokhlin, V. [26], Lifanov, I. K., Poltavskii, L. N., Vainikko, G. M. [27-29], Monegato, G. [30], De Bonis, M. C., Shoukralla, E. S., Ahmed, B. M. [33], Sidi, A. [34-37] and others.

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§ Manuscript received: September 04, 2024; accepted: April 19, 2025.

It should be noted that Aliev, R. A. [1-5] has developed a new constructive method for solving singular integral equations with a Cauchy kernel, where the singular integral operator is approximated by special operators that preserve the essential properties of the given singular operator. Additionally, similar approximations and their applications to hypersingular integral equations for various types of hypersingular integral operators with Cauchy and Hilbert kernels have been investigated in the author's papers. [6,17-22].

In this paper, we investigate the approximation of hypersingular integrals using a new numerical method and confirm the efficiency of the suggested approach through numerical examples. The paper is organized as follows, in Sect. 2 we give the approximation of the following Cauchy hypersingular integral operator in $L_2(\gamma_0), \gamma_0 = \{t \in \mathbb{C};\ |t| = 1\}$ (space of the square-integrable functions on γ_0):

$$(H\varphi)(t) \equiv \frac{1}{\pi i}\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^2} d\tau, \quad t \in \gamma_0\,,$$

by the sequence of operators of the form

$$(H_n\varphi)(t) = \frac{1}{\pi i}\sum_{k=0}^{n-1} \frac{\varphi\left(\tau_{2k+1}^{(t)}\right) - \varphi(t)}{\left(\tau_{2k+1}^{(t)} - t\right)^2}\Delta\tau_{2k+1}^{(t)}, \ t \in \gamma_0, \ \ n = 1, 2, ..,$$

where $\tau_k^{(t)} = e^{k\theta i}\cdot t$, $\Delta\tau_k^{(t)} = \left(\tau_{k+1}^{(t)} - \tau_{k-1}^{(t)}\right)\frac{\theta}{\sin\theta} = 2ie^{k\theta i}\cdot t\cdot\theta$, $k = \overline{0, 2n}$, $\theta = \frac{\pi}{n}$.

In Sect. 3 we give the approximation of the following Hilbert hypersingular integral operator:

$$\left(\tilde{H}\varphi\right)(t) = \frac{1}{4\pi}\int_0^{2\pi} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau, t \in T_0 = [0, 2\pi]$$

by the sequence of operators of the form

$$\left(\tilde{H}_n\varphi\right)(t) = \frac{1}{2n}\sum_{k=0}^{n-1}\csc^2\frac{\pi(2k+1)}{2n}\left(\varphi\left(t + \frac{\pi(2k+1)}{n}\right) - \varphi(t)\right), \ \ t \in T_0, n \in N$$

in the space of square-integrable functions on $T_0 = [0, 2\pi]$ $(L_2(T_0))$. In both sections, we also explain how these approximations preserve the main properties of the hypersingular integral operator and obtain an appropriate estimate of the convergence. Finally, in Sect. 4, we present numerical examples that confirm the effectiveness of the proposed method and all examples were executed using the MATLAB system.

2. APPROXIMATION OF CAUCHY HYPERSINGULAR INTEGRALS

Hypersingular integrals were introduced by J. Hadamard to solve the Cauchy problem for linear partial differential equations of a hyperbolic type [23]. Thus, we will begin by defining Cauchy hypersingular integrals (hypersingular integrals with Cauchy kernel) and then discuss some key theorems related to the approximation of the given hypersingular integral.

Consider the integral

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^2} d\tau\ , \quad t \in \gamma_0 \tag{1}$$

where the function $\varphi(t)$ is Lebesgue integrable on $\gamma_0 = \{t \in \mathbb{C};\ |t| = 1\}$.

If we define this integral in a similar manner to the Cauchy integral, even $\varphi \equiv 1$ we get the divergent integral, therefore, using the idea of Hadamard finite part integral [23], we will define the integral (1) as follows.

Definition 2.1. [4] *If a finite limit*

$$\lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau-\frac{2\varphi(t)}{i\varepsilon\cdot t}\right),$$

exists, then the value of this limit is called the hypersingular integral of the function $\frac{\varphi(\tau)}{(\tau-t)^2}$*, and is denoted by* $\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau$ *, where* $\gamma_\varepsilon=\{\tau\in\gamma_0:\,|\tau-t|>\varepsilon\}$.

Let $L_2(\gamma_0)$ be the space of square-integrable functions on γ_0 with the norm

$$\|\varphi\|_{L_2(\gamma_0)}=\left(\frac{1}{2\pi}\int_{\gamma_0}|\varphi(\tau)|^2\,|d\tau|\right)^{\frac{1}{2}},$$

and let $W_2^1(\gamma_0)$ be the space of absolutely continuous on γ_0 functions, which their derivatives belong to the space $L_2(\gamma_0)$, with the norm $\|\varphi\|_{W_2^1(\gamma_0)}=\|\varphi\|_{L_2(\gamma_0)}+\|\varphi'\|_{L_2(\gamma_0)}$. Consider the *Cauchy hypersingular integral operator*

$$(H\varphi)(t)\equiv\frac{1}{\pi i}\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau,\ t\in\gamma_0, \tag{2}$$

where the function $\varphi(t)$ is Lebesgue integrable on $\gamma_0=\{t\in\mathbb{C}:\,|t|=1\}$.

Since, the following singular integral operator with Cauchy kernel

$$(S\varphi)(t)=\frac{1}{\pi i}\int_{\gamma_0}\frac{\varphi(\tau)}{\tau-t}d\tau\ \ ,\ \ t\in\gamma_0$$

is bounded from the space $L_2(\gamma_0)$ into the space $L_2(\gamma_0)$ [29], then (2) is bounded from the space $W_2^1(\gamma_0)$ into the space $L_2(\gamma_0)$ (See [4], Theorem 2.3, p.1058) and

$$\|H\|_{W_2^1(\gamma_0)\to L_2(\gamma_0)}\le 1.$$

Let's approximate (2) by the following sequence of operators

$$(H_n\varphi)(t)=\frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left(\tau_{2k+1}^{(t)}-t\right)^2}\Delta\tau_{2k+1}^{(t)},\ t\in\gamma_0,n=1,2,..,\tag{3}$$

where $\tau_k^{(t)}=e^{k\theta i}\cdot t$, $\Delta\tau_k^{(t)}=\left(\tau_{k+1}^{(t)}-\tau_{k-1}^{(t)}\right)\frac{\theta}{\sin\theta}=2ie^{k\theta i}\cdot t\cdot\theta$, $k=\overline{0,2n}$, $\theta=\frac{\pi}{n}$.

State the following properties of the operators $H_n, n=1,2,3,....$

Theorem 2.1. *The operators* $H_n, n=1,2,...$ *are bounded from the space* $W_2^1(\gamma_0)$ *into the space* $L_2(\gamma_0)$*, and*

$$\|H_n\|_{W_2^1(\gamma_0)\to L_2(\gamma_0)}\le 1,\tag{4}$$

and for any polynomial $q(t)=\sum_{k=-n+1}^{n-1}q_kt^k$ *of a degree not higher than n the following relation holds*

$$(H_nq)(t)=(Hq)(t).\tag{5}$$

Proof. Before starting the proof, first calculate $H_n(t^m)$ for any $m\in\mathbb{Z}$ ($\mathbb{Z}$- set of integer numbers):

$$H_n(t^m)=\frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\left(\tau_{2k+1}^{(t)}\right)^m-t^m}{\left(\tau_{2k+1}^{(t)}-t\right)^2}\Delta\tau_{2k+1}^{(t)}=$$

$$= \frac{1}{\pi i} \sum_{k=0}^{n-1} \frac{e^{(2k+1)m\theta i} \cdot t^m - t^m}{\left(e^{(2k+1)\theta i} \cdot t - t\right)^2} \cdot 2ie^{(2k+1)\theta i} \cdot t \cdot \theta =$$

$$= \frac{2t^{m-1}}{n} \sum_{k=0}^{n-1} \frac{e^{(2k+1)m\theta i} - 1}{\left(e^{(2k+1)\theta i} - 1\right)^2} \cdot e^{(2k+1)\theta i} = \mu_m^{(n)} \cdot t^{m-1} \tag{6}$$

where $\mu_m^{(n)} = \frac{2}{n} \sum\limits_{k=0}^{n-1} \frac{e^{(2k+1)m\theta i}-1}{\left(e^{(2k+1)\theta i}-1\right)^2} \cdot e^{(2k+1)\theta i}$.

Since

$$\mu_0^{(n)} = 0, \tag{7}$$

$$\mu_{m+1}^{(n)} - \mu_m^{(n)} = \frac{2}{n} \sum_{k=0}^{n-1} \frac{e^{(2k+1)(m+1)\theta i} - e^{(2k+1)m\theta i}}{\left(e^{(2k+1)\theta i} - 1\right)^2} \cdot e^{(2k+1)\theta i} =$$

$$= \frac{2}{n} \sum_{k=0}^{n-1} \frac{e^{(2k+1)(m+1)\theta i}}{e^{(2k+1)\theta i} - 1} = \frac{1}{in} \sum_{k=0}^{n-1} \frac{e^{(2k+1)\left(m+\frac{1}{2}\right)\theta i}}{\sin\left(2k+1\right)\frac{\theta}{2}} = \lambda_m^{(n)}, \quad m \in \mathbb{Z}. \tag{8}$$

Let's compute $\lambda_m^{(n)}$, $m \in \mathbb{Z}$. Since $\lambda_{m\pm 2n}^{(n)} = \lambda_m^{(n)}$, $m \in \mathbb{Z}$, it is enough to compute $\lambda_m^{(n)}$ for $m = \overline{0, 2n-1}$. $\lambda_m^{(n)}$ can be written as follows:

$$\lambda_m^{(n)} = \frac{1}{in} \sum_{k=0}^{n-1} \frac{e^{(2k+1)\left(m+\frac{1}{2}\right)\theta i}}{\sin\left(2k+1\right)\frac{\theta}{2}} = \frac{1}{in} \sum_{k=0}^{n-1} \frac{e^{\left(m+\frac{1}{2}\right)(2n-(2k+1))\theta i}}{\sin\left(2n-(2k+1)\right)\frac{\theta}{2}} =$$

$$= \frac{1}{in} \sum_{k=0}^{n-1} \frac{e^{(2\pi m+\pi)i} e^{-\left(m+\frac{1}{2}\right)(2k+1)\theta i}}{\sin\left(\pi-(2k+1)\right)\frac{\theta}{2}} = -\frac{1}{in} \sum_{k=0}^{n-1} \frac{e^{-\left(m+\frac{1}{2}\right)(2k+1)\theta i}}{\sin\left(2k+1\right)\frac{\theta}{2}}. \tag{9}$$

From (8) - (9), we get

$$\lambda_m^{(n)} = \frac{1}{2ni} \sum_{k=0}^{n-1} \frac{e^{(2k+1)\left(m+\frac{1}{2}\right)\theta i} - e^{-\left(m+\frac{1}{2}\right)(2k+1)\theta i}}{\sin\left(2k+1\right)\frac{\theta}{2}} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sin\left(2m+1\right)\left(2k+1\right)\frac{\theta}{2}}{\sin\left(2k+1\right)\frac{\theta}{2}} .$$

It follows that,

$$\lambda_0^{(n)} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sin\left(2k+1\right)\frac{\theta}{2}}{\sin\left(2k+1\right)\frac{\theta}{2}} = 1,$$

$$\lambda_m^{(n)} = \lambda_{m-1}^{(n)} + \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sin\left(2m+1\right)\left(2k+1\right)\frac{\theta}{2} - \sin\left(2m-1\right)\left(2k+1\right)\frac{\theta}{2}}{\sin\left(2k+1\right)\frac{\theta}{2}} =$$

$$= \lambda_{m-1}^{(n)} + \frac{2}{n} \sum_{k=0}^{n-1} \cos\left(2k+1\right)m\theta$$

for $m = \overline{1, 2n}$.

Therefore,

$$\lambda_m^{(n)} = \lambda_{m-1}^{(n)} + \frac{2}{n} \cdot \frac{\sin 2n \cdot m\theta}{2\sin m\theta} = \lambda_{m-1}^{(n)}, m = \overline{1, n-1}$$

and we have

$$\lambda_m^{(n)} = 1, \text{ for } m = 0,\ 1,\ 2, ..., \ n-1.$$

Since

$$\lambda_m^{(n)} = \lambda_{m-1}^{(n)} + \frac{2}{n}\sum_{k=0}^{n-1}\cos\left(2k+1\right)n\theta \; = \; \lambda_{n-1}^{(n)} + \frac{2}{n}\sum_{k=0}^{n-1}(-1) \; = \; 1-2=-1,$$

when $m = n, n+1, ..., 2n-1$.

To summarize the results obtained above, we get the following values for $\lambda_m^{(n)}$:

$$\begin{cases} \lambda_m^{(n)} = 1, for\, m = 0,\, 1,\, 2, ...,\, n-1, \\ \lambda_m^{(n)} = -1, for\, m = n, n+1, ...,\, 2n-1, \\ \lambda_{m\pm 2n}^{(n)} = \lambda_m^{(n)}, m \in \mathbb{Z}. \end{cases} \tag{10}$$

Then, from equations (7), (8), and (10), we can conclude that:

$$\begin{cases} \mu_m^{(n)} = m, for\, m = \overline{0,\, n}, \\ \mu_m^{(n)} = 2n-m, for\, m = \overline{n+1, 2n} \quad , \\ \mu_{m\pm 2n}^{(n)} = \mu_m^{(n)}, m \in \mathbb{Z}. \end{cases} \tag{11}$$

Since from (11) it implies that, the coefficients $\mu_m^{(n)}$ satisfy the following inequality:

$$\left|\mu_m^{(n)}\right| \leq |m|\,, m \in \mathbb{Z}. \tag{12}$$

The proof of (12) can be easily derived from (11). However, for clarity, we will briefly show the validity of this inequality according to (11) as follows:

if $m = \overline{0, n}$

$$\left|\mu_m^{(n)}\right| = |m| \leq |m|\,,$$

if $m = \overline{n+1, 2n}$,

$$\left|\mu_m^{(n)}\right| = |2n-m| \leq |2m-m| = |m| \leq |m|\,,$$

if $m = -n, -n+1, ..., 0$,

$$\left|\mu_m^{(n)}\right| = |-m| = |m|\,,$$

if $m = -2n, ..., -n-1$,

$$\left|\mu_m^{(n)}\right| = |m+2n| \leq |n| \leq |m|\,,$$

if $|m| \geq 2n$,

$$\left|\mu_m^{(n)}\right| = |2n| \leq |m|\,,$$

and this leads us to the desired result.

Now prove the theorem. Suppose that $\varphi\left(t\right) = \sum\limits_{k=-\infty}^{+\infty} c_k t^k \in W_2^1\left(\gamma_0\right)$. Then, by taking (6) into account, we obtain

$$\left(H_n\varphi\right)(t) = \left(H_n\left(\sum_{k=-\infty}^{+\infty} c_k t^k\right)\right) = \sum_{k=-\infty}^{+\infty} c_k\left(H_n\left(t^k\right)\right) = \sum_{k=-\infty}^{+\infty} c_k \mu_k^{(n)} t^{k-1}. \tag{13}$$

Since [39]

$$\left\|H_n\varphi\left(t\right)\right\|_{L_2(\gamma_0)} = \left[\sum_{k=-\infty}^{+\infty} |c_k|^2 \left|\mu_k^{(n)}\right|^2 \left|t^{(k-1)}\right|^2\right]^{\frac{1}{2}},$$

and

$$\|H_n\varphi(t)\|_{L_2(\gamma_0)} \overset{(12)}{\leq} \left[\sum_{k=-\infty}^{+\infty} k^2 |c_k|^2 \left|t^{(k-1)}\right|^2\right]^{\frac{1}{2}} = \left\|\varphi'(t)\right\|_{L_2(\gamma_0)} \leq$$

$$\leq \left\|\varphi'(t)\right\|_{L_2(\gamma_0)} + \|\varphi(t)\|_{L_2(\gamma_0)} = \|\varphi(t)\|_{W_2^1(\gamma_0)},$$

then, we get

$$\|H_n\varphi(t)\|_{L_2(\gamma_0)} \leq \|\varphi(t)\|_{W_2^1(\gamma_0)}. \tag{14}$$

(14) implies the boundedness of the operators H_n, $n = 1, 2, ...$ from the space $W_2^1(\gamma_0)$ into the space $L_2(\gamma_0)$ and (4).

Thus, it remains to prove (5). As we know for any polynomial of the form $q(t) = \sum\limits_{k=-n+1}^{n-1} q_k t^k$ the following equation is true (See [4], Theorem 2.3, p.1058):

$$(Hq)(t) = \left(Sq'\right)(t). \tag{15}$$

Since [2]

$$\begin{cases} S(t^m) = t^m, m \geq 0, \\ S(t^m) = -t^m, m < 0, \end{cases}, m \in \mathbb{Z},$$

then from (15) it follows that,

$$(Hq)(t) = \left(Sq'\right)(t) = S\left(\sum_{k=-n+1}^{n-1} kq_k t^{k-1}\right) =$$

$$= \sum_{k=-n+1}^{n-1} kq_k S\left(t^{k-1}\right) = \sum_{k=1}^{n-1} kq_k t^{k-1} - \sum_{k=-n+1}^{-1} kq_k t^{k-1}, \tag{16}$$

and we have

$$(H_n q)(t) = H_n\left(\sum_{k=-n+1}^{n-1} q_k t^k\right) =$$

$$= \sum_{k=-n+1}^{n-1} q_k H_n\left(t^k\right) \overset{(6)}{=} \sum_{k=-n+1}^{n-1} q_k \mu_k^{(n)} t^{k-1} \overset{(11)}{=} \sum_{k=1}^{n-1} kq_k t^{k-1} + \sum_{k=-n+1}^{-1} q_k \cdot (-k) \cdot t^{k-1}. \tag{17}$$

By combining (16) and (17), we obtain (5) and this completes the proof of the theorem. □

Theorem 2.2. *The sequence of operators* $\{H_n\}$, $n = 1, 2, ...$ *strongly converges to the operator* H, *and for any* $\varphi \in W_2^1(\gamma_0)$ *the following estimate holds:*

$$\|H\varphi - H_n\varphi\|_{L_2(\gamma_0)} \leq 2E_n\left(\varphi; W_2^1\right),$$

where $E_n\left(\varphi; W_2^1\right) = \inf\limits_{q \in T_n} \|\varphi(\cdot) - q_n(\cdot)\|_{W_2^1(\gamma_0)}$, $n = 1, 2, ...$ *is the best approximation of the function* $\varphi \in W_2^1(\gamma_0)$ *by polynomials from* T_n, *and* T_n *is the set of polynomials of the form* $\sum\limits_{k=-n}^{n} \alpha_k t^k$, $\alpha_k \in C$.

Proof. Suppose $q_n(t) = \sum\limits_{k=-n+1}^{n-1} q_k^{(n)} t^k$ is the best polynomial approximation for the function $\varphi \in W_2^1(\gamma_0)$ from T_n.

Then, we can write

$$(H\varphi - H_n\varphi)(t) = H\varphi(t) - H_n\varphi(t) + Hq_n(t) - Hq_n(t) \overset{(5)}{=}$$

$$= H\varphi(t) - H_n\varphi(t) + H_n q_n(t) - Hq_n(t) = H(\varphi - q_n)(t) - H_n(\varphi - q_n)(t).$$

This yields

$$\|H\varphi - H_n\varphi\|_{L_2(\gamma_0)} = \|H(\varphi - q_n)(t) - H_n(\varphi - q_n)\|_{L_2(\gamma_0)} \leq$$

$$\leq \left(\|H\|_{W_2^1(\gamma_0)\to L_2(\gamma_0)} + \|H_n\|_{W_2^1(\gamma_0)\to L_2(\gamma_0)}\right) \times$$

$$\times \|\varphi - q_n\|_{W_2^1(\gamma_0)} \overset{(4)}{\leq} 2E_n\left(\varphi;\, W_2^1\right),$$

and

$$\|H\varphi - H_n\varphi\|_{L_2(\gamma_0)} \leq 2E_n\left(\varphi;\, W_2^1\right).$$

Hence, we get the required result and the proof is complete. □

3. APPROXIMATION OF HILBERT HYPERSINGULAR INTEGRALS

Now we will examine the approximation of the hypersingular integral with a special type of kernel known as the *Hilbert hypersingular integral.* Alternatively, when considering this integral from a geometric perspective, it can be referred to as the hypersingular integral on the circle. The latter name is derived from the fact that the inner Neumann problem for the Laplace equation on the circle is reduced to the hypersingular integral equation with the integral of this form (See [27], Ch.4, §4.3). Here, we will define Hilbert hypersingular integral (hypersingular integral with Hilbert kernel), conduct theoretical analysis for approximate computation, and give an error estimate analogously to Sect. 2

Consider the integral

$$\int_0^{2\pi} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau,\ t \in T_0 = [0, 2\pi], \tag{18}$$

where $\varphi(t)$ is Lebesgue integrable on T_0 and 2π-periodic function.

If we define (18) in a similar manner to the Cauchy integral, even $\varphi \equiv 1$, we get the divergent integral. Therefore, using the idea of Hadamard finite part integral [23], we will define (18) as follows:

Definition 3.1. [18] *If a finite limit*

$$\lim_{\varepsilon\to 0+}\left(\int_{t-\pi}^{t-\varepsilon} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau + \int_{t+\varepsilon}^{t+\pi} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau - \frac{8\varphi(t)}{\varepsilon}\right)$$

exists, then the value of this limit is referred to as the Hilbert hypersingular integral of the function $\csc^2\frac{\tau-t}{2}\varphi(\tau)$ *on* T_0*, and is denoted by* $\int_0^{2\pi} \csc^2\frac{\tau-t}{2}\varphi(\tau)\,d\tau$.

Let $L_2(T_0)$ be the space of square-integrable functions on $T_0 = [0, 2\pi]$ with the norm

$$\|\varphi\|_{L_2(T_0)} = \left(\frac{1}{2\pi}\int_0^{2\pi} |\varphi(\tau)|^2\,d\tau\right)^{\frac{1}{2}},$$

and let $W_2^1(T_0)$ be the space of absolutely continuous functions on T_0, which their derivatives belong to the space $L_2(T_0)$, with the norm $\|\varphi\|_{W_2^1(T_0)} = \|\varphi\|_{L_2(T_0)} + \|\varphi'\|_{L_2(T_0)}$.

Consider the Hilbert hypersingular integral operator

$$\left(\tilde{H}\varphi\right)(t) = \frac{1}{4\pi}\int_0^{2\pi} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau, t \in T_0 = [0, 2\pi] \tag{19}$$

where $\varphi(t)$ is Lebesgue integrable on T_0 and 2π-periodic function.

Since, (19) is bounded from the space $W_2^1(T_0)$ into the space $L_2(T_0)$ (See [18], [29], [39]) and

$$\left\|\tilde{H}\right\|_{W_2^1(T_0)\to L_2(T_0)} \le 1.$$

Now approximate (19) by the following sequence of operators:

$$\left(\tilde{H}_n\varphi\right)(t) = \frac{1}{2n}\sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n}\left(\varphi\left(t+\frac{\pi(2k+1)}{n}\right)-\varphi(t)\right), \quad t\in T_0, n\in N. \quad (20)$$

State the following properties of the operators $\tilde{H}_n, n = 1, 2, 3, \ldots$.

Theorem 3.1. [18] *The operators $\tilde{H}_n$, $n = 1, 2, \ldots$ are bounded from the space $W_2^1(T_0)$ into the space $L_2(T_0)$, and*

$$\left\|\tilde{H}_n\right\|_{W_2^1(T_0)\to L_2(T_0)} \le 1,$$

and for any trigonometric polynomial $q(t) = \sum_{k=-n}^{n} q_k e^{ikt}$ the following relation holds:

$$\left(\tilde{H}_n q\right)(t) = \left(\tilde{H} q\right)(t).$$

Theorem 3.2. [18] *The sequence of operators $\left\{\tilde{H}_n\right\}$, $n = 1, 2, \ldots$ strongly converges to the operator $\tilde{H}$ and, for any $\varphi \in W_2^1$, the following estimate holds:*

$$\left\|\tilde{H}\varphi - \tilde{H}_n\varphi\right\|_{L_2(T_0)} \le 2E_n\left(\varphi; W_2^1\right),$$

where $E_n\left(\varphi; W_2^1\right) = \inf_{q\in T_n}\|\varphi(\cdot) - q_n(\cdot)\|_{W_2^1}$, $n = 1, 2, \ldots$ is the best approximation of the function $\varphi \in W_2^1$ by polynomials from T_n, and T_n is the set of trigonometric polynomials of the form $\sum_{k=-n}^{n} \alpha_k e^{ikt}$, $\alpha_k \in C$.

For the proofs of Theorem 3.1 and Theorem 3.2, see [18].

4. NUMERICAL EXAMPLES

To demonstrate the performance of the numerical algorithm and justify the theoretical analysis presented in Sections 2 and 3, we will consider two examples, implemented using the MATLAB programming language.

Firstly, provide an example to illustrate the features of the numerical method and verify the theoretical conclusions stated in Section 2.

Example 4.1. Consider the hypersingular integral of the form

$$H(t^2) = \frac{1}{\pi i}\int_{\gamma_0} \frac{t^2}{(t-t_0)^2}dt, \; t_0 \in \gamma_0, \quad (21)$$

where $\varphi(t) = t^2$, $\gamma_0 = \{t \in \mathbb{C} : |t| = 1\}$. The exact values of (21) at the different values of $t_0 \in \gamma_0$, calculated by (16), are shown in Table 1. The results of the numerical evaluation of the given hypersingular integral at different values of $t_0 \in \gamma_0$, computed using the approximate method mentioned above (Sect. 2) and the quadrature formula (3), are given in Table 2.

TABLE 1. *Exact results of* $H(t^2) = \frac{1}{\pi i}\int\limits_{\gamma_0} \frac{t^2}{(t-t_0)^2}dt$ *at the points* $t_0 = e^{i0}$, $t_0 = e^{i\frac{\pi}{6}}$, $t_0 = e^{i\frac{\pi}{4}}$, $t_0 = e^{i\frac{\pi}{3}}$, $t_0 = e^{i\frac{\pi}{2}}$.

t_0	e^{i0}	$e^{i\frac{\pi}{6}}$	$e^{i\frac{\pi}{4}}$	$e^{i\frac{\pi}{3}}$	$e^{i\frac{\pi}{2}}$
$H(t^2)$	2.0000 -0.0000i	1.7321+1.0000i	1.4142+1.4142i	1.0000 + 1.7321i	0.0000 + 2.0000i

TABLE 2. *Numerical results of* $H(t^2) = \frac{1}{\pi i}\int\limits_{\gamma_0} \frac{t^2}{(t-t_0)^2}dt$ *at the points* $t_0 = e^{i0}$, $t_0 = e^{i\frac{\pi}{6}}$, $t_0 = e^{i\frac{\pi}{4}}$, $t_0 = e^{i\frac{\pi}{3}}$, $t_0 = e^{i\frac{\pi}{2}}$, *for* $n = 10, 40, 70, 100$.

$H_n(t^2)$, $n = 10, 40, 70, 100$				
t_0	$H_{10}(t^2)$	$H_{40}(t^2)$	$H_{70}(t^2)$	$H_{100}(t^2)$
e^{i0}	2.0000 -0.0000i	2.0000 -0.0000i	2.0000 -0.0000i	2.0000 -0.0000i
$e^{i\frac{\pi}{6}}$	1.7321+1.0000i	1.7321+1.0000i	1.7321+1.0000i	1.7321+1.0000i
$e^{i\frac{\pi}{4}}$	1.4142+1.4142i	1.4142+1.4142i	1.4142+1.4142i	1.4142+1.4142i
$e^{i\frac{\pi}{3}}$	1.0000 + 1.7321i	1.0000 + 1.7321i	1.0000 + 1.7321i	1.0000 + 1.7321i
$e^{i\frac{\pi}{2}}$	0.0000 + 2.0000i	0.0000 + 2.0000i	0.0000 + 2.0000i	0.0000 + 2.0000i

Additionally, the following graph (Figure 1) presents the comparison of numerical and exact results of $H(t^2)$ which are stated above (See Table 1 and Table 2).

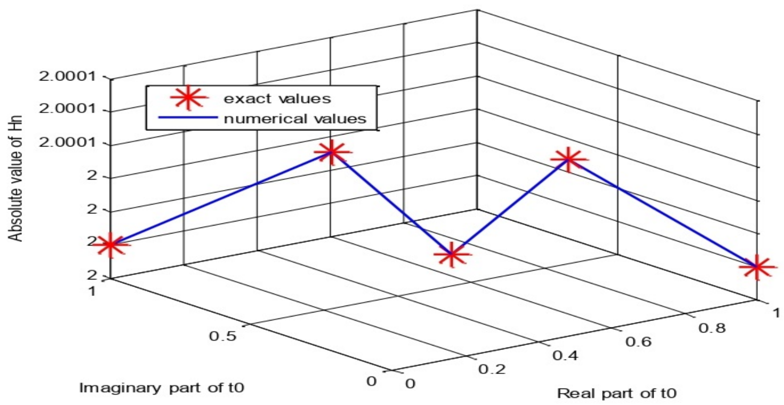

FIGURE 1. *Comparison of numerical values and exact values of* $H(t^2)$.

The above results (see Table 1 and Table 2) show that the computational errors coincide with that of theoretical analysis. Since Figure 1 illustrate that the algorithm outlined by formula (3), relation (5) in Theorem 2.1, and the estimation in Theorem 2.2 yield results that are both highly accurate and consistent with theoretical results. Consequently, the numerical performance is consistent with the theoretical conclusions.

Now consider an example that demonstrates the justification of the theoretical analyses cited in Section 3.

Example 4.2. Compute the hypersingular integral of the form

$$\left(\tilde{H}\varphi\right)(t) = \frac{1}{4\pi i}\int_0^{2\pi} \csc^2\frac{t-t_0}{2}\sin t dt,\ t \in T_0 = [0, 2\pi], \tag{22}$$

where $\varphi(t) = \sin t$. The exact values of (22) at the different values of $t_0 \in [0, 2\pi]$, are calculated using the following formula (See [27], Ch.4, §4.3.1)

$$\frac{1}{4\pi}\int_0^{2\pi} \csc^2\frac{t-t_0}{2}(a_n \cos nt + b_n \sin nt)dt = -n(a_n \cos nt_0 + b_n \sin nt_0),$$

and the numerical results for the given hypersingular integral at different values of $t_0 \in [0\ , 2\pi]$, computed by the application of approximate method mentioned above (Sect. 3) and using quadrature formula (20) are shown correspondingly in Table 3 and Table 4.

TABLE 3. *Exact results of* $\tilde{H}(\sin t) = \frac{1}{4\pi i}\int_0^{2\pi} \csc^2\frac{t-t_0}{2}\sin t dt$ *at the points* $t_0 = \frac{\pi}{6}$, $t_0 = \frac{\pi}{4}$, $t_0 = \frac{\pi}{3}$, $t_0 = \frac{\pi}{2}$, $t_0 = \pi$.

t_0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\tilde{H}(\sin t)$	-0.5000	-0.7071	-0.8660	-1.000000	-3.5527e-016

TABLE 4. *Numerical results of* $\tilde{H}(\sin t) = \frac{1}{4\pi i}\int_0^{2\pi} \csc^2\frac{t-t_0}{2}\sin t dt$ *at the points* $t_0 = \frac{\pi}{6}$, $t_0 = \frac{\pi}{4}$, $t_0 = \frac{\pi}{3}$, $t_0 = \frac{\pi}{2}$, $t_0 = \pi$, *for* $n = 10, 40, 70, 100$.

$\tilde{H}_n(\sin t)$, $n = 10, 40, 70, 100$				
t_0	$\tilde{H}_{10}(\sin t)$	$\tilde{H}_{40}(\sin t)$	$\tilde{H}_{70}(\sin t)$	$\tilde{H}_{100}(\sin t)$
$\frac{\pi}{6}$	-0.5000	-0.5000	-0.5000	-0.5000
$\frac{\pi}{4}$	-0.7071	-0.7071	-0.7071	-0.7071
$\frac{\pi}{3}$	-0.8660	-0.8660	-0.8660	-0.8660
$\frac{\pi}{2}$	-1.000000	-1.000000	-1.000000	-1.000000
π	-3.5527e-016	-3.5527e-016	-3.5527e-016	-3.5527e-016

In addition, the following figure (Figure 2) illustrates the comparison of numerical and exact values of $\tilde{H}(\sin t)$ which are stated above (See Table 3 and Table 4).

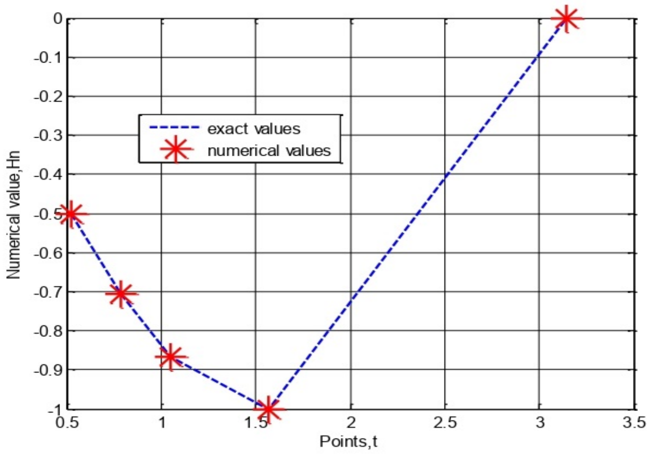

FIGURE.2. *Comparison of numerical values and exact values of* $\tilde{H}(\sin t)$.

A comparison of the numerical and exact values for the given hypersingular integral indicates that, the numerical value matches the exact value of $\tilde{H}(\varphi(t))$ for any trigonometric polynomial of the specified form, confirming the theoretical analysis in Sect. 3 and demonstrating the feasibility and effectiveness of our algorithm (see Table 3, Table 4, Figure 2).

It should be noted that in Example 4.1 and Example 4.2, the values for n, $t_0 \in \gamma_0$ and $t_0 \in [0\ ,2\pi]$ are chosen arbitrarily. Changing these values does not alter the results obtained from our algorithms.

5. CONCLUSION

In this paper, we have introduced a new numerical method for approximating hypersingular integrals of types (2) and (19) with optimal accuracy. The proposed quadrature formulas can be applied to solve real-world engineering problems and have various successful applications in numerical implementations. To confirm our theoretical analysis, we have included numerical examples. It's important to note that the numerical algorithm described above applies to a wide range of hypersingular integrals, including the popular ones discussed in our work. Finally, note that our new method can provide good numerical results with optimal accuracy in all cases.

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