TWMS J. App. and Eng. Math. V.15, N.9, 2025, pp. 2229-2238

AN INTEGRAL TYPE OPERATOR ON S^p SPACES AND ITS ESSENTIAL NORM

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Abstract. Given two analytic functions $\lambda : \mathbf{D} \to \mathbf{D}$ and $\nu : \mathbf{D} \to \mathbb{C}$ and $n \in \mathbb{N}_0$, we study boundedness of the integral-type operator $C^n_{\lambda,\nu}$ acting from derivative Hardy space into Zygmund space. We also get an approximation for the essential norm of this operator. A characterization for compactness of the operator can be obtained from the essential norm.

Keywords: Integral-type operator, Essential norm, Zygmund type space, Hardy space.

AMS Subject Classification: 47B38, 30H30

1. Introduction

Let $\mathcal{H}ol(\mathbf{D})$ denotes the space of analytic functions in the unit disk $\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$. There has been some well-known subspaces of $\mathcal{H}ol(\mathbf{D})$ which arise in the operator theory, such as Hardy space, Bergman space and so on. The space of holomorphic functions with mean square value on the circle of radius r which is bounded below as $r \to 1$, is denoted by H^2. More generally, the Hardy space H^p, $0 < p < \infty$ is the class of holomorphic functions f on the open unit disk satisfying

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}} < \infty.$$

This space is Banach space for $1 \le p < \infty$ with the above norm. See [8] for complete information on the Hardy spaces.

Let S^p be the set of analytic functions whose first derivative is in H^p:

$$S^p = \{f \in \mathcal{H}ol(\mathbf{D}) : f' \in H^p\}.$$

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§ Manuscript received: August 27, 2024; accepted: January 18, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.9;

S^p is just one of the examples of the spaces related to derivatives in another spaces. Another one is Besov-type space B^p which is the space of analytic functions with derivative in Bergman space A^p. See [18] for properties of B^p and weighted composition operators on them.

We define S^p-norm of f by

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}.$$

$\|.\|_{S^p}$ defines a norm and in the case $1 \leq p \leq \infty$, it is a complete norm in which S^p is a Banach space. Hardy inequality shows that for $p \geq 1$, $S^p \subset \ell^1$. One can now verify that S^p is an algebra and that multiplication is continuous with respect to the norm $\|.\|_{S^p}$.

The properties of composition operators on S^p first studied by Roan in [17]. Boundedness and compactness of these operators characterized in [14]. They proved that, among other results, $C_\lambda : S^p \to S^p$ is compact if and only if $\|\lambda\|_\infty < 1$, $1 \leq p < \infty$. Contreras and Hernandez-Diaz investigated boundedness, compactness, weak compactness, and complete continuity of weighted composition operators $S^p \to S^q$, $1 \leq p, q \leq \infty$, [6].

For an analytic function $\lambda : \mathbf{D} \to \mathbf{D}$, the composition operator C_λ is defined by $C_\lambda f = f \circ \lambda$. Let $\nu \in \mathcal{H}ol(\mathbf{D})$. For $n \in \mathbb{N}_0$, the integral-type operator $C^n_{\lambda,\nu}$ is

$$C^n_{\lambda,\nu} f(z) = \int_0^z f^{(n)}(\lambda(\xi))\nu(\xi)d\xi$$

which is an extension of $C^1_{\lambda,\nu} = C^\nu_\lambda$, first defined in [12] and somehow called Volterra-composition operator. The above operator also includes Volterra-type operators I_ν and J_ν, $I_\nu f(z) = \int_0^z f(\xi)\nu'(\xi)d\xi$ and $J_\nu f(z) = \int_0^z f'(\xi)\nu(\xi)d\xi$. More specially if $\nu(z) = z$ then J_ν is the integral operator and if $\nu(z) = \ln(1/(1-z))$, then J_ν is Cesàro operator. The composition operator C_λ is obtained for $n = 1$ and $\nu = \lambda'$.

The original work for I_ν and J_ν is [16] where it was shown that J_ν on the Hardy Hilbert space H^2 is bounded if and only if $\nu \in BMOA$, the space of analytic functions with bounded mean oscillation. The boundedness characterization of J_ν on the H^p and A^p has been obtained in [3, 4]. More researches on the operators I_ν, J_ν, C_λ, C^ν_λ and $C^n_{\lambda,\nu}$ mapping between spaces of analytic functions to be bounded, compact, computed the essential norm and some other properties of operators, refer to [1, 2, 5, 7, 9, 10, 11, 12, 13, 14, 15, 19, 20, 21] and references therein.

In this paper, we firstly give a necessary and sufficient condition for the boundedness of the operator $C^n_{\lambda,\nu}$ from S^p to α-Zygmund space, $n \in \mathbb{N}$. Then we will concerned with the essential norm of this operator. Finally in the case $n = 0$ we find a criteria for boundedness and compactness.

For any $\alpha > 0$, the α-Zygmund space Z^α is defined by

$$Z^\alpha = \{f \in \mathcal{H}ol(\mathbf{D}) : \|f\|_{Z^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|f''(z)| < \infty\}.$$

Here $v_\alpha(z) = (1 - |z|^2)^\alpha$. Under the norm $\|.\|_{Z^\alpha}$, Z^α is a Banach space. $Z^1 = Z$ is called classical Zygmund space. Also α-Bloch space is defined by

$$\mathcal{B}^\alpha = \{f \in \mathcal{H}ol(\mathbf{D}) : \|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|f'(z)| < \infty\}.$$

If $\mathcal{L}(X, Y)$ is space of all bounded linear operators between Banach spaces X and Y, then the essential norm of $T \in \mathcal{L}(X, Y)$ is the distance of T to the compact operators,

$$\|T\|_{e,X\to Y} = \inf\{\|T - L\|_{X\to Y} : L \in \mathcal{L}(X, Y) \textit{ is compact}\}.$$

It is a measure for finding the conditions under which the operator T is compact or not. In this paper, if there exists a constant C such that $A \leq CB$, we say that $A \preceq B$. The symbol $A \approx B$ means that $A \preceq B \preceq A$.

2. Main Results

Lemma 2.1. [11] *Let $1 < p < \infty$ and $0 \neq w \in \mathbf{D}$. For any $i \in \{1, \cdots, m\}$, there exist constants $c_2^i, \cdots, c_{m+1}^i$ such that*

$$\begin{aligned} v_{i,w} =& f_{1,w} + \sum_{j=1}^{m+1} c_j^i f_{j,w} \in S^p, \\ v_{i,w}(w) =& 0 \\ v_{i,w}^{(k)}(w) =& \begin{cases} \frac{\overline{w}^i}{(1-|w|^2)^{i+\frac{1}{p}-1}}, & k = i, \\ 0, & k \neq i. \end{cases} \end{aligned}$$

Here

$$f_{j,w}(z) = \frac{(1-|w|^2)^j}{(1-\overline{w}z)^{j+\frac{1}{p}-1}}.$$

It is easy to see that the functions $f_{j,w}$ belong to S^p and is a bounded sequence. Moreover $v_{i,w}$ converges to 0 uniformly in $\overline{\mathbf{D}}$. According to above lemma, there exist functions, denoted by h_w and k_w, such that

$$h_w^{(n)}(w) = 0, \quad h_w^{(n+1)}(w) = \frac{\overline{w}^{n+1}}{(1-|w|^2)^{n+\frac{1}{p}}} \tag{1}$$

$$k_w^{(n+1)}(w) = 0, \quad k_w^{(n)}(w) = \frac{\overline{w}^{n}}{(1-|w|^2)^{n-1+\frac{1}{p}}} \tag{2}$$

It is well-known that for $f \in H^p$,

$$|f^{(n)}(z)| \leq \frac{\|f\|_{H^p}}{(1-|z|^2)^{\frac{1}{p}+n}},$$

$n \in \mathbb{N}_0$. If $f \in S^p$ then $f' \in H^p$ and $\|f'\|_{H^p} \leq \|f\|_{S^p}$. So

$$|f^{(n+1)}(z)| \leq \frac{\|f\|_{S^p}}{(1-|z|^2)^{\frac{1}{p}+n}}. \tag{3}$$

Theorem 1 of [13] implies that $S^p \subset H^\infty$, the space of bounded analytic function with sup-norm $\|.\|_\infty$, and $\|f\|_\infty \leq \pi\|f\|_{S^p}$, $1 \leq p < \infty$. Therefore if $f \in S^p$ then

$$|f(z)| \leq \|f\|_\infty \leq \pi\|f\|_{S^p}. \tag{4}$$

The following is about conditions on the symbols induce the operator to be a bounded operator.

Theorem 2.1. *Let $\lambda : \mathbf{D} \to \mathbf{D}$ and $\nu : \mathbf{D} \to \mathbb{C}$ are analytic functions, $1 \leq p < \infty$, $\alpha > 0$ and $n \in \mathbb{N}$. Then the followings are equivalent:*

(a) *$C_{\lambda,\nu}^n : S^p \to Z^\alpha$ is continuous.*

(b)

$$\sup_{z\in\mathbf{D}} \frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n}} < \infty, \quad \sup_{z\in\mathbf{D}} \frac{v_\alpha(z)|\nu'(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n-1}} < \infty.$$

Proof. (a)⇒(b) Since $C^n_{\lambda,\nu} : S^p \to Z^\alpha$ is continuous (bounded), then we can obtain a positive constant C for which

$$\|C^n_{\lambda,\nu} f\|_{Z^\alpha} \le C\|f\|_{S^p}, \tag{5}$$

for all $f \in S^p$. Consider the function $f_1(z) = z^n \in S^p$. Then $\|C^n_{\lambda,\nu} f_1\|_{Z^\alpha} \le C$ and so by computing the norm in Z^α we have

$$\begin{aligned}\|C^n_{\lambda,\nu} f_1\|_{Z^\alpha} =& |(C^n_{\lambda,\nu} f_1)'(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|(C^n_{\lambda,\nu} f_1)''(z)| \\ =& |f_1^{(n)}(\lambda(0))||\nu(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|f_1^{(n)}(\lambda(z))\nu'(z)| \\ \le& n!|\nu(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z) n!|\nu'(z)| \le C.\end{aligned}$$

Hence

$$\sup_{z\in\mathbf{D}} v_\alpha(z)|\nu'(z)| < \infty. \tag{6}$$

Now consider the function $f_2(z) = z^{n+1} \in S^p$. Then we get

$$\begin{aligned}\|C^n_{\lambda,\nu} f_2\|_{Z^\alpha} =& |(C^n_{\lambda,\nu} f_2)'(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|(C^n_{\lambda,\nu} f_2)''(z)| \\ =& |f_2^{(n)}(\lambda(0))||\nu(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda'(z) f_2^{(n+1)}(\lambda(z))\nu(z) + f_2^{(n)}(\lambda(z))\nu'(z)| \\ =& n!|\lambda(0)||\nu(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|n!\lambda'(z)\nu(z) + n!\lambda(z)\nu'(z)|.\end{aligned}$$

Then

$$\sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda'(z)\nu(z) + \lambda(z)\nu'(z)| < \infty.$$

Therefore

$$\begin{aligned}\sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda'(z)\nu(z)| \le& \sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda'(z)\nu(z) + \lambda(z)\nu'(z)| \\ &+ \sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda(z)\nu'(z)| < \infty.\end{aligned} \tag{7}$$

Fix $w \in \mathbf{D}$ and apply (5) to the function $h_{\lambda(w)}$. Form (1) we obtain

$$\begin{aligned}C \ge \|C^n_{\lambda,\nu} h_{\lambda(w)}\|_{Z^\alpha} \ge& \sup_{z\in\mathbf{D}} v_\alpha(z)|(C^n_{\lambda,\nu} h_{\lambda(w)})''(z)| \\ \ge& v_\alpha(w)|(C^n_{\lambda,\nu} h_{\lambda(w)})''(w)| \\ =& v_\alpha(w)\frac{|\nu(w)||\lambda'(w)||\lambda(w)|^{n+1}}{(1-|\lambda(w)|^2)^{\frac{1}{p}+n}}.\end{aligned}$$

Hence

$$\sup_{|\lambda(w)|>\delta} \frac{v_\alpha(w)|\nu(w)||\lambda'(w)|}{(1-|\lambda(w)|^2)^{\frac{1}{p}+n}} < \infty,$$

where $0 < \delta < 1$ is fixed. Also for $|\lambda(w)| \le r$,

$$\frac{v_\alpha(w)|\lambda'(w)||\nu(w)|}{(1-|\lambda(w)|^2)^{\frac{1}{p}+n}} \le \frac{v_\alpha(w)|\lambda'(w)||\nu(w)|}{(1-r^2)^{\frac{1}{p}+n}}$$

and the equation (7) implies that supremum of the above is finite. From these equations and noting that $w \in \mathbf{D}$ is arbitrary, we get

$$\sup_{z\in\mathbf{D}} \frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n}} < \infty.$$

Now employing (5) to the function $k_{\lambda(w)}$ and using (2) we obtain

$$\begin{aligned} C \geq \|C^n_{\lambda,\nu} k_{\lambda(w)}\|_{Z^\alpha} \geq & \sup_{z\in\mathbf{D}} v_\alpha(z)|(C^n_{\lambda,\nu} k_{\lambda(w)})''(z)| \\ \geq & v_\alpha(w)|(C^n_{\lambda,\nu} k_{\lambda(w)})''(w)| \\ = & v_\alpha(w)\frac{|\nu'(w)||\lambda(w)|^n}{(1-|\lambda(w)|^2)^{\frac{1}{p}+n-1}}. \end{aligned}$$

In a similar way the following condition is proved

$$\sup_{z\in\mathbf{D}} \frac{v_\alpha(z)|\nu'(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n-1}} < \infty.$$

(b)⇒(a) Suppose that $f \in S^p$. Then

$$\begin{aligned} \|C^n_{\lambda,\nu} f\|_{Z^\alpha} = & |(C^n_{\lambda,\nu} f)'(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|(C^n_{\lambda,\nu} f)''(z)| \\ = & |f^{(n)}(\lambda(0))||\nu(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda'(z)f^{(n+1)}(\lambda(z))\nu(z) + f^{(n)}(\lambda(z))\nu'(z)| \\ \leq & |f^{(n)}(\lambda(0))||\nu(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda'(z)f^{(n+1)}(\lambda(z))\nu(z)| \\ & + \sup_{z\in\mathbf{D}} v_\alpha(z)|f^{(n)}(\lambda(z))\nu'(z)| \\ \leq & \frac{|\nu(0)|}{(1-|\lambda(0)|^2)^{\frac{1}{p}+n-1}}\|f\|_{S^p} + \sup_{z\in\mathbf{D}} \frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n}}\|f\|_{S^p} \\ & + \sup_{z\in\mathbf{D}} \frac{v_\alpha(z)|\nu'(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n-1}}\|f\|_{S^p}. \end{aligned}$$

Here we use the equation (3). Definition of the operator norm implies that the operator $C^n_{\lambda,\nu} : S^p \to Z^\alpha$ is bounded (continuous). □

The following theorem is about finding an estimation for the essential norm of the operator $C^n_{\lambda,\nu} : S^p \to Z^\alpha$. As a consequence of it we can find a criteria for the compactness of the operator.

Theorem 2.2. *Let $\lambda : \mathbf{D} \to \mathbf{D}$ and $\nu : \mathbf{D} \to \mathbb{C}$ are analytic functions, $1 \leq p < \infty$, $\alpha > 0$ and $n \in \mathbb{N}$. Let the operator $C^n_{\lambda,\nu} : S^p \to Z^\alpha$ is bounded. Then*

$$\|C^n_{\lambda,\nu}\|_e \approx \left\{ \limsup_{|\lambda(z)|\to 1} \frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n}}, \limsup_{|\lambda(z)|\to 1} \frac{v_\alpha(z)|\nu'(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n-1}} \right\}.$$

Proof. The proof will be done in two parts, upper and lower bounded. For $0 \leq r < 1$, set $f_r(z) = f(rz)$ and $K_r : \mathcal{H}ol(\mathbf{D}) \to \mathcal{H}ol(\mathbf{D})$ by

$$K_r f(z) = f_r(z).$$

Then $f_r \to f$ uniformly on compact subsets of $\mathbf{D}$ as $r \to 1$. Also $K_r : S^p \to S^p$ is a compact operator and noting that $C^n_{\lambda,\nu} : S^p \to Z^\alpha$ is bounded, $C^n_{\lambda,\nu} K_r : S^p \to Z^\alpha$ is compact. Let $\{r_j\}$, $0<r_j<1$, be a sequence such that for $j\to\infty$, $r_j\to 1$. So

$$\begin{aligned}\|C^n_{\lambda,\nu}\|_e &\le \limsup_{j\to\infty}\|C^n_{\lambda,\nu}-C^n_{\lambda,\nu}K_{r_j}\|\\ &=\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\|(C^n_{\lambda,\nu}-C^n_{\lambda,\nu}K_{r_j})f\|_{Z^\alpha}\\ &=\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}|\nu(0)||(f-f_{r_j})^{(n)}(\lambda(0))|+\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}}\Theta^f_j,\end{aligned}$$

where

$$\Theta^f_j=v_\alpha(z)|\lambda'(z)\nu(z)(f-f_{r_j})^{(n+1)}(\lambda(z))+\nu'(z)(f-f_{r_j})^{(n)}(\lambda(z))|.$$

As $f_{r_j}\to f$ uniformly on compact subsets of $\mathbf{D}$ and the subset $\{\lambda(0)\}$ is compact then

$$\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}|\nu(0)||(f-f_{r_j})^{(n)}(\lambda(0))|=0.$$

Then

$$\begin{aligned}\|C^n_{\lambda,\nu}\|_e &\le \limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}}\Theta^f_j\\ &\le \limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}_1}\Theta^f_j+\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}_2}\Theta^f_j,\end{aligned}$$

where

$$\mathbf{D}_1=\{z\in\mathbf{D}:|\lambda(z)|\le r\},\quad \mathbf{D}_2=\{z\in\mathbf{D}:|\lambda(z)|>r\}$$

and $r\in(0,1)$ is fixed. Again using uniformly convergence of $f_{r_j}\to f$ on compact subsets of $\mathbf{D}$ and also employing equations (6) and (7) we have

$$\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}_1}\Theta^f_j=0.$$

Therefore according to (3) we get

$$\begin{aligned}\|C^n_{\lambda,\nu}\|_e &\le \limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}_2}\Theta^f_j\\ &=\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}_2}v_\alpha(z)|\lambda'(z)\nu(z)(f-f_{r_j})^{(n+1)}(\lambda(z))\\ &\quad+\nu'(z)(f-f_{r_j})^{(n)}(\lambda(z))|\\ &\preceq 2\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}_2}\frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n}}\|f\|_{S^p}\\ &\quad+2\limsup_{j\to\infty}\sup_{\|f\|_{S^p}\le 1}\sup_{z\in\mathbf{D}_2}\frac{v_\alpha(z)|\nu'(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n-1}}\|f\|_{S^p}\\ &\le 2\limsup_{j\to\infty}\sup_{z\in\mathbf{D}_2}\frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n}}\\ &\quad+2\limsup_{j\to\infty}\sup_{z\in\mathbf{D}_2}\frac{v_\alpha(z)|\nu'(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n-1}}\end{aligned}$$

and this completes the proof of upper bound.

Let $\{z_j\}$ be a sequence in $\mathbf{D}$ such that $|\lambda(z_j)|\to 1$, $j\to+\infty$. Then the sequence $\{h_{\lambda(z_j)}\}$ is bounded and converges uniformly on compact subsets of $\mathbf{D}$ to 0. If $K:S^p\to Z^\alpha$ is any compact operator, then $\lim_{j\to\infty}\|Kh_{\lambda(z_j)}\|_{Z^\alpha}=0$. So

$$\begin{aligned}\|C^n_{\lambda,\nu}-K\| &\succeq \limsup_{j\to\infty}\|(C^n_{\lambda,\nu}-K)h_{\lambda(z_j)}\|_{Z^\alpha}\\ &=\limsup_{j\to\infty}\|C^n_{\lambda,\nu}h_{\lambda(z_j)}\|_{Z^\alpha}\\ &=\limsup_{j\to\infty}(|(C^n_{\lambda,\nu}h_{\lambda(z_j)})'(0)|+\sup_{z\in\mathbf{D}}v_\alpha(z)|(C^n_{\lambda,\nu}h_{\lambda(z_j)})''(z)|)\\ &\geq \limsup_{j\to\infty}v_\alpha(z_j)|\lambda'(z_j)||\nu(z_j)||h^{(n+1)}_{\lambda(z_j)}(\lambda(z_j))|\\ &=\limsup_{j\to\infty}\frac{v_\alpha(z_j)|\lambda'(z_j)||\lambda(z_j)|^{n+1}|\nu(z_j)|}{(1-|\lambda(z_j)|^2)^{\frac{1}{p}+n}}\\ &=\limsup_{j\to\infty}\frac{v_\alpha(z_j)|\lambda'(z_j)||\nu(z_j)|}{(1-|\lambda(z_j)|^2)^{\frac{1}{p}+n}}.\end{aligned}$$

Hence

$$\|C^n_{\lambda,\nu}\|_e=\inf_K\|C^n_{\lambda,\nu}-K\|\succeq\limsup_{|\lambda(z)|\to1}\frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n}}. \tag{8}$$

By a similar discussion applying to the sequence $\{k_{\lambda(z_j)}\}$ the following can be obtained

$$\|C^n_{\lambda,\nu}\|_e\succeq\limsup_{|\lambda(z)|\to1}\frac{v_\alpha(z)|\nu'(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}+n-1}}. \tag{9}$$

Now the lower bound comes from (8) and (9). □

In the two following theorems we consider $C^n_{\lambda,\nu}:S^p\to Z^\alpha$ in the case $n=0$.

Theorem 2.3. *Let $\lambda:\mathbf{D}\to\mathbf{D}$ and $\nu:\mathbf{D}\to\mathbb{C}$ are analytic functions. Then*

$$C^0_{\lambda,\nu}:S^p\to Z^\alpha \text{ is bounded} \Longleftrightarrow \nu\in\mathcal{B}^\alpha \text{ and } M=\sup_{z\in\mathbf{D}}\frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}}}<\infty.$$

Proof. If $\nu\in\mathcal{B}^\alpha$ and $M<\infty$ then as in the proof of Theorem 2.1, using (3) and (4) we can prove that $C^0_{\lambda,\nu}:S^p\to Z^\alpha$ is bounded. Converse part is proved in Theorem 2.1. □

Set

$$N_1=\limsup_{|\lambda(z)|\to1}\frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}}},\quad N_2=\sup_{z\in\mathbf{D}}v_\alpha(z)|\lambda'(z)||\nu(z)|.$$

Theorem 2.4. *Let $\lambda:\mathbf{D}\to\mathbf{D}$ and $\nu:\mathbf{D}\to\mathbb{C}$ are analytic functions. Then*

$$C^0_{\lambda,\nu}:S^p\to Z^\alpha \text{ is compact} \Longleftrightarrow \nu\in\mathcal{B}^\alpha,\ N_2<\infty \text{ and } N_1=0.$$

Proof. If $C^0_{\lambda,\nu}:S^p\to Z^\alpha$ is a compact operator then it is bounded and so $\nu\in\mathcal{B}^\alpha$ and $N_2<\infty$, see (6) and (7). Now let $\{z_j\}$ be a sequence in $\mathbf{D}$ such that $|\lambda(z_j)|\to1$, $j\to\infty$. Then the sequence $\{h_{\lambda(z_j)}\}$ is bounded and converges uniformly on compact subsets of $\mathbf{D}$ to 0. Since $C^0_{\lambda,\nu}:S^p\to Z^\alpha$ is compact then we have $\lim_{j\to\infty}\|C^0_{\lambda,\nu}h_{\lambda(z_j)}\|_{Z^\alpha}=0$. By computing the norm in Z^α we get

$$\begin{aligned}\limsup_{j\to\infty}\|C^0_{\lambda,\nu}h_{\lambda(z_j)}\|_{Z^\alpha} =& \limsup_{j\to\infty}\sup_{z\in\mathbf{D}} v_\alpha(z)|(C^0_{\lambda,\nu}h_{\lambda(z_j)})''(z)|\\ \geq& \limsup_{j\to\infty} v_\alpha(z_j)|\lambda'(z_j)||\nu(z_j)||h'_{\lambda(z_j)}(\lambda(z_j))|\\ =& \limsup_{j\to\infty}\frac{v_\alpha(z_j)|\lambda'(z_j)||\lambda(z_j)||\nu(z_j)|}{(1-|\lambda(z_j)|^2)^{\frac{1}{p}+n}}\\ =& \limsup_{j\to\infty}\frac{v_\alpha(z_j)|\lambda'(z_j)||\nu(z_j)|}{(1-|\lambda(z_j)|^2)^{\frac{1}{p}}},\end{aligned}$$

from which $N_1 = 0$. On the other hand suppose that $\nu \in \mathcal{B}^\alpha$, $N_2 < \infty$ and $N_1 = 0$. To show that $C^0_{\lambda,\nu}$ is compact we need to show that for any bounded sequence $\{f_j\}$ in S^p which converges to 0 uniformly on compact subsets of $\mathbf{D}$, $\|C^0_{\lambda,\nu}f_j\|_{Z^\alpha} \to$, $j \to \infty$. From $N_1 = 0$, for every $\epsilon > 0$ there exists $0 < \delta < 1$ such that if $\delta < |\lambda(z)| < 1$ then

$$\frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}}} < \epsilon.$$

It follows from above and equations (3) and (4) that

$$\begin{aligned}\|C^0_{\lambda,\nu}f_j\|_{Z^\alpha} =& |f_j(\lambda(0))||\nu(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda'(z)f'_j(\lambda(z))\nu(z) + f_j(\lambda(z))\nu'(z)|\\ \leq& |f_j(\lambda(0))||\nu(0)| + \sup_{z\in\mathbf{D}} v_\alpha(z)|\lambda'(z)f'_j(\lambda(z))\nu(z)|\\ &+ \sup_{z\in\mathbf{D}} v_\alpha(z)|f_j(\lambda(z))\nu'(z)|\\ \leq& |f_j(\lambda(0))||\nu(0)| + N_2\sup_{|\lambda(z)|\leq\delta}|f'_j(\lambda(z))|\\ &+ \sup_{\delta<|\lambda(z)|<1}\frac{v_\alpha(z)|\lambda'(z)||\nu(z)|}{(1-|\lambda(z)|^2)^{\frac{1}{p}}}\|f_j\|_{S^p} + \|\nu\|_{\mathcal{B}^\alpha}\sup_{z\in\mathbf{D}}|f_j(\lambda(z))|\\ \leq& |f_j(\lambda(0))||\nu(0)| + N_2\sup_{|\lambda(z)|\leq\delta}|f'_j(\lambda(z))| + \epsilon\|f_j\|_{S^p} + \|\nu\|_{\mathcal{B}^\alpha}\sup_{z\in\overline{\mathbf{D}}}|f_j(\lambda(z))|.\end{aligned}$$

Noting to the facts: 1. $f_j \to 0$ and $f'_j \to 0$ uniformly on compact subsets of $\mathbf{D}$,
2. the sets $\{\lambda(0)\}$ and $\{|\lambda(z)| \leq \delta\}$ are compact subsets of $\mathbf{D}$,
we obtain that there exists a positive constant C such that

$$\limsup_{j\to\infty}\|C^0_{\lambda,\nu}f_j\|_{Z^\alpha} < C\epsilon.$$

This completes the proof. □

Remark 2.1. *It should be mentioned that the results of the paper can be stated for the operators included in $C^n_{\lambda,\nu}$. The boundedness, compactness and essential norm of composition operator C_λ, Volterra-type operators I_ν and J_ν, Volterra-composition operator C^ν_λ can be obtained.*

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