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# RELATION-THEORETIC COMMON FIXED POINTS FOR ALMOST $\mathcal{F}_{\tilde{\mathcal{R}}_{\Im}}$-CONTRACTION TYPE MAPS IN $\mathbb{B}_2$-METRIC SPACES AND APPLICATION TO NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. This paper introduces a novel class of contraction mappings called "almost $\mathcal{F}_{\tilde{\mathcal{R}}_{\Im}}$-contraction type maps" in the framework of $\mathbb{B}_2$-metric spaces. These contractions are utilized to establish results regarding coincidence points and common fixed points furnished with a binary relation. Furthermore, the paper aims to broaden the scope of these findings by offering illustrative examples. The paper concludes with an application of these concepts to prove the existence of solutions of a nonlinear fractional differential equation. Our results broaden the scope of those reported in [16] and expand on comparable findings previously documented in the literature.

Keywords: Binary relations, Almost $\mathcal{F}_{\tilde{\mathcal{R}}_{\Im}}$-contraction type maps, $\mathbb{B}_2$-metric spaces, Common coincidence points, Common fixed points, Fourth-order boundary value problems.

AMS Subject Classification: 47H10, 54H25.

## 1. Introduction

An important concept in nonlinear analysis, the Banach contraction principle is well-known for its numerous applications in various discussions. Researchers have expanded this idea over time by modifying conditions related to abstract spaces and contractions. Jaggi [14], Dass and Gupta [8], and Fisher [12] are notable for their investigation of the application of rational-type expressions in the contraction condition, leading to the widespread use of rational inequalities in fixed points, coincidence points, and proximity point problems [1,8,12,14,18,21,28,38,39].

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FPT is used to define a variety of generalized structures, including $\mathbb{B}$ metric spaces, partial metric spaces, and 2-metric spaces. Gahler [13] introduced 2-metric spaces, using the area of a triangle in $\mathbb{R}^2$ as an example. However, 2-metric spaces differ topologically from metric spaces, limiting the direct applicability of their findings. Czerwik [7] defined $\mathbb{B}$ metric spaces as a combination of 2-metric and metric spaces, with several publications investigating fixed point theory in these spaces, to which we refer [2,6,7,13,35,36,39,40].

In 2014, Mustafa et al. [19] proposed a new metric structure termed $\mathbb{B}_2$, which is an extension of both 2-metric and b-metric. Some fixed point theorems in the $\mathbb{B}_2$ metric spaces are proved. It is vital to note that a 2-metric space is a subset of $b_2$-metric spaces with coefficient $s = 1$. Several authors, including those cited in [10,13,16,19,25,27,41], have explored and established common fixed point theorems in these new $\mathbb{B}_2$ metric space, primarily employing explicit or semi-explicit contraction conditions.

Recently, many fixed point results have not met the contraction conditions between random pairs of points in the space, however, methods have been developed to restrict the domains of these contractions. One approach is in this field of metric spaces with binary relations, initiated by Turinici [32], represents a novel pathway of study. Subsequently, in the order-theoretic metric setting, Ran and Reurings [24] have extended the BCP. Recently, Alam and Imdad [4,5,6] proved the fixed point theorem for the classical BCP in a completed metric space equipped with binary relations. In this result, it discovered that the contraction condition holds only for those elements linked with the binary relation not for every pair of elements. For additional literature on relation theoretic study, we refer to [3, 4, 5, 11, 15, 17, 22, 24, 26, 27, 31, 32, 33, 34]. Wardowski [36] first proposed $F$-contraction in 2012. Multiple researchers generalized Wardowski's theorems by expanding the concept of $F$-contraction. Zada et al.[38] utilized the concept of $F_{\mathcal{R}}$ contractions developed in [9,20,23,27,29,31,37] to construct common fixed point results for rational contractions. Additionally, substantial research has been devoted to weakly contractive mappings.

Motivated by the existing literature on $\mathbb{B}_2$ metric spaces, rational expressions, $F_R$ contractions and relational theoretic study, in this paper, we define almost $\mathcal{F}_{\tilde{\mathcal{R}}_{\mathfrak{I}}}$-contraction type maps with rational expressions in $\mathbb{B}_2$-metric spaces and employ these contractions to obtain common coincidence points and common fixed points equipped with binary relations. To enhance the comprehensiveness of our findings, we provide illustrative examples. The paper concludes with an application to a fourth-order boundary value problem modeling the deformation of a fully elastic beam. Our results broaden the scope of those reported in [16] and expand on comparable findings previously documented in the literature.

## 2. Mathematical background

**Definition 2.1**.[19] $\delta : \mathbb{A}^3 \to \mathbb{R}$ be a mapping, $\mathbb{A} \neq \phi$ and $\mathfrak{s} \geq 1$. If $\delta$ satisfies the following conditions (a) to (d), then $\delta$ is a $\mathbb{B}_2$-metric on $\mathbb{A}$:

(a) for all $\varrho, \upsilon \in \mathbb{A}$, with $\varrho \neq \upsilon$ there exists $\nu \in \mathbb{A}$ such that $\delta(\varrho, \upsilon, \nu) \neq 0$.
(b) if at least two of the three points $\varrho, \upsilon, \nu$ are equal, then $\delta(\varrho, \upsilon, \nu) = 0$.
(c) $\delta(\varrho, \upsilon, \nu) = \delta(\varrho, \nu, \upsilon) = \delta(\upsilon, \varrho, \nu) = \delta(\upsilon, \nu, \varrho) = \delta(\nu, \varrho, \upsilon) = \delta(\nu, \upsilon, \varrho)$, for all $\varrho, \upsilon, \nu \in \mathbb{A}$.
(d) $\delta(\varrho, \upsilon, \nu) \leq \mathfrak{s}[\delta(\varrho, \upsilon, t) + \delta(\upsilon, \nu, t) + \delta(\nu, \varrho, t)]$, for all $\varrho, \upsilon, \nu, t \in \mathbb{A}$.

Then $(\mathbb{A}, \delta)$ is a $\mathbb{B}_2$-metric space.

Clearly, when $\mathfrak{s} = 1$, $\mathbb{B}_2-$ metric reduces to 2-metric.

**Example 2.2**.[19] Consider a 2-metric space on $\mathbb{A}$ as $\vartheta(\varrho, \upsilon, \nu) = (\delta(\varrho, \upsilon, \nu))^{\iota}$, where $\iota \geq 1$

with $\mathfrak{s} = 3^{\iota-1}$. Evidently, from convexity of the function $f(\varrho) = \varrho^p$ for $\varrho \geq 0$, then by the Jensen inequality,

$$(\varrho + \upsilon + \nu)^p \leq 3^{\iota-1}(\varrho^p + \upsilon^p + \nu^p).$$

Therefore $\vartheta$ is a $\mathbb{B}_2$-metric on $\mathbb{A}$.

**Definition 2.3**.[19] Consider a $\mathbb{B}_2$-metric space $(\mathbb{A}, \delta)$.

(1) A sequence $\{\zeta_n\}$ in $(\mathbb{A}, \delta)$ is a $\mathbb{B}_2$-*convergent to* $\zeta^*$ if $\lim_{n\to+\infty} \delta(\zeta_n, \zeta^*, \tilde{t}) = 0$. In this case, $lim_{n\to+\infty}\zeta_n = \zeta^*$.

(2) A sequence $\{\zeta_n\}$ in $(\mathbb{A}, \delta)$ is a $\mathbb{B}_2 - Cauchy$ if $\lim_{m,n\to+\infty} \delta(\zeta_n, \zeta_m, \tilde{t}) = 0$.

(3) A $\mathbb{B}_2$-metric space $(\mathbb{A}, \delta)$ is consider to be *complete* if for any Cauchy sequence in $\mathbb{A}$ is converges to a point within $\mathbb{A}$.

**Definition 2.4**.[19] Consider $(\mathbb{A}, \delta)$ and $(\overline{\mathbb{A}}, \overline{\delta})$ as two $\mathbb{B}_2$-metric spaces and $\mathcal{T} : \mathbb{A} \to \overline{\mathbb{A}}$. Then $\mathcal{T}$ is $\mathbb{B}_2$- continuous at $\Upsilon \in \mathbb{A}$ if for a given $\epsilon > 0$, there exists $\hbar > 0$ such that $\Upsilon \in \mathbb{A}$ and $\delta(\Upsilon, \Upsilon, \tilde{t}) < \hbar$ for all $\tilde{t} \in \mathbb{A}$ implies $\overline{\delta}(\mathcal{T}\Upsilon, \mathcal{T}\Upsilon, \tilde{t}) \leq \epsilon$. Further, $T$ is $\mathbb{B}_2$- continuous on $\mathbb{A}$ if it is $\mathbb{B}_2$- continuous for any $\Upsilon \in \mathbb{A}$.

**Lemma 2.5**.[19] Consider a $\mathbb{B}_2$-metric space $(\mathbb{A}, \delta)$. Assume $\{\zeta_n\}$ and $\{\ell_n\}$ are $\mathbb{B}_2$-converges to $\zeta$ and $\ell$, respectively. Then

$\frac{1}{\mathfrak{s}^2}\delta(\zeta, \ell, \tilde{t}) \leq lim\ inf_{n\to+\infty}\delta(\zeta_n, \ell_n, \tilde{t}) \leq lim\ sup_{n\to+\infty}\delta(\zeta_n, \ell_n, \tilde{t}) \leq \mathfrak{s}^2\delta(\zeta, \ell, \tilde{t})$, for all $\tilde{t} \in \mathbb{A}$. In particular if $\ell_n = \ell$, is constant, then

$\frac{1}{\mathfrak{s}}\delta(\zeta, \ell, \tilde{t}) \leq lim\ inf_{n\to+\infty}\delta(\zeta_n, \ell, \tilde{t}) \leq lim\ sup_{n\to+\infty}\delta(\zeta_n, \ell, \tilde{t}) \leq \mathfrak{s}\delta(\zeta, \ell, \tilde{t})$, for all $\tilde{t} \in \mathbb{A}$.

For relevant properties and examples on $\mathbb{B}_2$-metric spaces, we refer[10,19].

In 2012, Wardowski[36] initiated the concept of $F$-contraction.

**Definition 2.6**.[36] Let $(\mathbb{A}, \delta)$ be a metric space and $\mathcal{H}$ be self map on $\mathbb{A}$. Then $\mathcal{H}$ is $\mathcal{F}$ contraction if: there exists $\tau > 0$ such that for all $\Upsilon, \mho \in \mathbb{A}$

$$\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho) > 0\ implies\ \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho)) \leq \mathcal{F}(\delta(\Upsilon, \mho)),$$

where $\mathcal{F}$ is a mapping from $\mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(i) $\mathcal{F}$ is strictly increasing, i.e. for all $\nu, \lambda \in \mathbb{R}^+$ such that $\nu < \lambda$, $\mathcal{F}(\nu) < \mathcal{F}(\lambda)$.

(ii) For each sequence $\{\lambda_n\}$ of positive integers, $lim_{n\to+\infty}\lambda_n = 0$ if and only if $lim_{n\to+\infty}\mathcal{F}(\lambda_n) = -\infty$.

(iii) There exists $\hbar \in (0, 1)$ such that $lim_{\lambda\to 0^+}\lambda^{\hbar}\mathcal{F}(\lambda) = 0$.

**Example 2.7**.[34] Let $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\mathcal{F}(\Upsilon) = \Upsilon + log\Upsilon,\ \ for\ \Upsilon \in \mathbb{R}^+.$$

Then $\mathcal{F}$ is satisfying conditions $(i)$-$(ii)$.

Throughout we refer $\mathfrak{F}$ be the family of all functions $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfying the conditions $(i)$, $(ii)$ and $(iii)$, $\mathbb{R}$ by set of all real numbers and $\mathbb{N}$ the set of nonnegative integers.

**Definition 2.8**.[8] Let $(\mathbb{A}, \delta)$ be a metric space and $\mathcal{H}$ be self map on $\mathbb{A}$. Then $\mathcal{H}$ is a *rational contraction* if for all $\Upsilon, \mho \in \mathbb{A}$:

$$\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho) \leq \hat{\mathfrak{a}}\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho) + \hat{\mathfrak{b}}\frac{\delta(\mho, \mathcal{H}\mho)[1 + \delta(\Upsilon, \mathcal{H}\Upsilon)]}{1 + \delta(\Upsilon, \mho)},$$

where $\hat{\mathfrak{a}}$, $\hat{\mathfrak{b}} \in [0, 1)$ with $\hat{\mathfrak{a}} + \hat{\mathfrak{b}} < 1$.

**Definition 2.9**.[14] Let $(\mathbb{A}, \delta)$ be a metric space and $\mathcal{H}$ be self map on $\mathbb{A}$. Then $\mathcal{H}$ is a *rational contraction* if for all $\Upsilon, \mho \in \mathbb{A}$:

$$\delta(\Im\Upsilon, \Im\mho) \leq \hat{\mathfrak{a}}\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho) + \hat{\mathfrak{b}}\frac{\delta(\mho, \mathcal{H}\mho)\delta(\Upsilon, \mathcal{H}\Upsilon)}{1 + \delta(\Upsilon, \mho)},$$

where $\hat{\mathfrak{a}}$, $\hat{\mathfrak{b}} \in [0, 1)$ with $\hat{\mathfrak{a}} + \hat{\mathfrak{b}} < 1$.

**Theorem 2.10**.[16] Consider two commuting mappings $\Im, \mathcal{H}$ on a $\mathbb{B}_2-$ metric space $(\mathbb{A}, \delta)$

into itself satisfying the inequality

$$\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, a) \leq \lambda\delta(\Im\Upsilon, \Im\mho, a), \tag{1}$$

for all $\Upsilon, \mho, a \in \mathbb{A}$, where $0 < \lambda < 1$. If the range of $\Im$ contains the range of $\mathcal{H}$ and if $\Im$ is $\mathbb{B}_2$-continuous, then $\Im$ and $\mathcal{H}$ have a unique common fixed point.

Let $\mathbb{A} \neq \emptyset$. A binary relation $\tilde{\mathcal{R}}$ is a subset of $\mathbb{A} \times \mathbb{A}$. Any two elements $\Upsilon$ and $\mho$ of $\mathbb{A}$ are $\tilde{\mathcal{R}}$- comparable if $[\Upsilon, \mho] \in \tilde{\mathcal{R}}$ *i.e.*, either $(\Upsilon, \mho) \in \tilde{\mathcal{R}}$ or $(\mho, \Upsilon) \in \tilde{\mathcal{R}}$. Also, $\tilde{\mathcal{R}}^*$ represents binary relation on $\mathbb{A}$ whenever $(\Upsilon, \mho) \in \tilde{\mathcal{R}}$ with $\Upsilon \neq \mho$ and $\tilde{\mathcal{R}}^* \subseteq \tilde{\mathcal{R}}$.

**Definition 2.11**.[4] A sequence $\Upsilon_n \subseteq \mathbb{A}$ is considered $\tilde{\mathcal{R}}$-preserving if for every $n \in \mathbb{N} \cup \{0\}$, the pair $(\Upsilon_n, \Upsilon_{n+1})$ is an element of $\tilde{\mathcal{R}}$.

**Definition 2.12**.[4] The $\tilde{\mathcal{R}}$-completeness of $\mathbb{A}$, is a property whereby every Cauchy sequence in $\mathbb{A}$ that preserves $\tilde{\mathcal{R}}$ converges in $\mathbb{A}$. Evidently, in universal relation the completeness and $\tilde{\mathcal{R}}$-completeness are same.

**Definition 2.13**.[4] For any $\tilde{\mathcal{R}}$- preserving sequence $\{\Upsilon_n\}$ with $\Upsilon_n \to \Upsilon$, there exists a subsequence $\{\Upsilon_{n(\hbar)}\}$ with $(\Upsilon_{n(\hbar)}, \Upsilon) \in \tilde{\mathcal{R}}$ for all $\hbar \in \mathbb{N}$, then $\tilde{\mathcal{R}}$ on $\mathbb{A}$ is $\delta$-self closed.

**Definition 2.14**.[4] Consider a self map $\mathcal{H}$ on $\mathbb{A}$. $\tilde{\mathcal{R}}$ is $\mathcal{H}$-closed if for any $\Upsilon, \mho \in \mathbb{A}$ with $(\Upsilon, \mho) \in \tilde{\mathcal{R}}$ implies $(\mathcal{H}\Upsilon, \mathcal{H}\mho) \in \tilde{\mathcal{R}}$.

**Definition 2.15**.[4] Consider two selfmaps $\mathcal{H}$ and $\Im$ defined on a nonempty set $\mathbb{A}$. A binary relation $\tilde{\mathcal{R}}$ on $\mathbb{A}$ is a $(\mathcal{H}, \Im)$-closed if for any $\Upsilon, \mho \in \mathbb{A}$, $(\Im\Upsilon, \Im\mho) \in \tilde{\mathcal{R}}$ implies $(\mathcal{H}\Upsilon, \mathcal{H}\mho) \in \tilde{\mathcal{R}}$.

**Definition 2.16**.[4] For any $\tilde{\mathcal{R}}$-preserving sequence $\{\Upsilon_n\} \subseteq \mathbb{A}$, with $\Upsilon_n \to \Upsilon$ if $\mathcal{H}\Upsilon_n \to \mathcal{H}\Upsilon$ then the self map $\mathcal{H}$ on $\mathbb{A}$ is termed to be an $\tilde{\mathcal{R}}$-continuous at $\Upsilon$. Moreover, if $\mathcal{H}$ exhibits this behaviour for all $x \in \mathbb{A}$, it is simply referred as $\tilde{\mathcal{R}}$- continuous.

If $\Im = I$, then Definition 2.16 leads to Definition 2.15.

**Definition 2.17**.[4] Consider two self-maps $\mathcal{H}$ and $\Im$ on $\mathbb{A}$. We say that $\mathcal{H}$ is $(\Im, \tilde{\mathcal{R}})$ continuous at a point $\Upsilon$ if there exists a sequence $\Upsilon_n \subseteq \mathbb{A}$ such that $\Im\Upsilon_n$ is a $\tilde{\mathcal{R}}$-preserving sequence and $\Im\Upsilon_n \to \Im\Upsilon$, implying $\mathcal{H}\Upsilon_n \to \mathcal{H}\Upsilon$. Additionally, $\mathcal{H}$ is said to be $(\Im, \tilde{\mathcal{R}})$-continuous if it is continuous with respect to $(\Im, \tilde{\mathcal{R}})$ at each point of $\mathbb{A}$.

**Definition 2.18**.[15] Consider a subset $Q$ of a nonempty set $\mathbb{A}$. Then, the restriction of $\tilde{\mathcal{R}}$ to $Q$ is $\tilde{\mathcal{R}}|Q$, defined by $\tilde{\mathcal{R}} \cap Q^2$.

**Definition 2.19**.[4] $\tilde{\mathcal{R}}$ is referred as transitive if for any $\Upsilon, \tilde{t}, p \in \mathbb{A}$,

$$(\Upsilon, \jmath), (\jmath, p) \in \tilde{\mathcal{R}} \; implies \; (\Upsilon, \jmath) \in \tilde{\mathcal{R}}.$$

**Lemma 2.20**.[27] Let $\mathcal{H}$ and $\Im$ be two self-maps on $\mathbb{A}$ with respect to a binary relation $\tilde{\mathcal{R}}$. Suppose $\mathcal{H}(\mathbb{A}) \subseteq \Im(\mathbb{A})$ and $\tilde{\mathcal{R}}$ is $(\mathcal{H}, \Im)$-closed, with $\tilde{\mathcal{R}} \mid \Im(\mathbb{A})$ being transitive. If there exists $\Upsilon_0 \in \mathbb{A}$ such that $(\Im\Upsilon_0, \mathcal{H}\Upsilon_0) \in \tilde{\mathcal{R}}$, and there is a sequence $\Upsilon_n$ in $\mathbb{A}$ defined by $\mathcal{H}\Upsilon_n = \Im\Upsilon_{n+1}$ for $0 \leq n$, then for all $m, n \in \mathbb{N}$ with $n > m$, we have

$$(\Im\Upsilon_m, \Im\Upsilon_n) \in \tilde{\mathcal{R}} \; and \; (\mathcal{H}\Upsilon_m, \mathcal{H}\Upsilon_n) \in \tilde{\mathcal{R}}.$$

Following on the same lines of proof of [30, Theorem 1] we have the following lemma.

**Lemma 2.21**.[27] Let $\tilde{\mathcal{R}}$ be a binary relation on $\mathbb{B}_2-$ metric space $(\mathbb{A}, \delta)$ and $\{\Im\Upsilon_n\}$ in $\mathbb{A}$ such that $lim_{n\to+\infty}\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t}) = 0$ and $\delta(\Im\Upsilon_i, \Im\Upsilon_j, \Im\Upsilon_\hbar) = 0$, for all $i, j, \hbar \in \mathbb{N}$.

If $\{\Im\Upsilon_n\}$ is not a $\mathbb{B}_2$- Cauchy sequence, we can choose a subsequences $\{\Im\Upsilon_{m_\hbar}\}$ and $\{\Im\Upsilon_{n_\hbar}\}$ of $\{\Im\Upsilon_n\}$ such that $n(\hbar) \geq m(\hbar) \geq \hbar$ for all $\hbar \in \mathbb{N}$ and

$$\delta(\Im_{m(\hbar)}, \Im_{n(\hbar)}, \tilde{t}) > \epsilon \geq \delta(\Im_{m(\hbar)}, \Im_{n(\hbar)-1}, \tilde{t}).$$

Also, we have:

(i) $\epsilon \leq lim\ sup_{\hbar\to+\infty}\delta(\Im_{m(\hbar)}, \Im_{n(\hbar)}, \tilde{t}) < \mathfrak{s}\epsilon.$

(ii) $\frac{\epsilon}{\mathfrak{s}^2} \leq lim\ sup_{\hbar\to+\infty}\delta(\Im_{m(\hbar)-1}, \Im_{n(\hbar)-1}, \tilde{t}) < \mathfrak{s}\epsilon$.
(iii) $\frac{\epsilon}{\mathfrak{s}^3} \leq lim\ sup_{\hbar\to+\infty}\delta(\Im_{m(\hbar)}, \Im_{n(\hbar)-1}, \tilde{t}) < \mathfrak{s}\epsilon$.
(iv) $\frac{\epsilon}{\mathfrak{s}} \leq lim\ sup_{\hbar\to+\infty}\delta(\Im_{m(\hbar)-1}, \Im_{n(\hbar)}, \tilde{t}) < \mathfrak{s}^2\epsilon$.

**Definition 2.22**.[1] Consider two self-maps $\mathcal{H}$ and $\Im$ on a set $\mathbb{A}$.

(i) If there exists $\Upsilon \in \mathbb{A}$ such that $\nu = \mathcal{H}(\Upsilon) = \Im(\Upsilon)$, then $\Upsilon$ is a coincidence point of $\mathcal{H}$ and $\Im$, and $\nu$ is a point of coincidence of $\mathcal{H}$ and $\Im$.
(ii) If $\mathcal{H}$ and $\Im$ share a unique coincidence point $\nu = \mathcal{H}\Upsilon = \Im\Upsilon$, then $\nu$ is the only common fixed point between $\mathcal{H}$ and $\Im$.

The following theorem was proved by Saleh et. al. [27].

**Theorem 2.23**.[27] Consider two selfmaps $\Im, \mathcal{H}$ on a $\mathbb{B}_2-$ metric space $\mathbb{A}$ with a binary relation $\tilde{\mathcal{R}}$ such that $\mathcal{H}(\mathbb{A}) \subseteq \Im(\mathbb{A})$, $\Im(\mathbb{A})$ is $\mathbb{B}_2$- complete subspace of $\mathbb{A}$, with the following assertions:

(1) there exists $\Upsilon_0 \in \mathbb{A}$ such that $(\mathcal{H}\Upsilon_0, \Im\Upsilon_0) \in \tilde{\mathcal{R}}$.
(2) for all $\Upsilon, \mho, \tilde{t} \in \mathbb{A}$ there exists $\lambda_{\mathfrak{s}} : [0, +\infty) \to [0, \frac{1}{\mathfrak{s}})$ $lim_{n\to+\infty}\lambda_{\mathfrak{s}}(t_n) = \frac{1}{\mathfrak{s}}$ implies $lim_{n\to+\infty}t_n = 0$ and $\pounds \geq 0$ such that
$$\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t}) \leq \lambda_{\mathfrak{s}}(\mathcal{M}(\Upsilon, \mho, \tilde{t}))\mathcal{M}(\Upsilon, \mho, \tilde{t}) + \pounds\mathcal{N}(\Upsilon, \mho, \tilde{t}), \tag{2}$$
with $(\mathcal{H}\Upsilon, \mathcal{H}\mho) \in \tilde{\mathcal{R}}^*$, $(\Im\Upsilon, \Im\mho) \in \tilde{\mathcal{R}}$ where
$$\mathcal{M}(\Upsilon, \mho, \tilde{t}) = max\{\delta(\Im\Upsilon, \Im\mho, \tilde{t}), \delta(\Im\Upsilon, \mathcal{H}\Upsilon, \tilde{t}), \delta(\mathcal{H}\mho, \Im\mho, \tilde{t}), \frac{\delta(\Im\Upsilon, \mathcal{H}\mho, \tilde{t}) + \delta(\Im\mho, \mathcal{H}\Upsilon, \tilde{t})}{2s}\}$$
and
$$\mathcal{N}(\Upsilon, \mho, \tilde{t}) = min\{\delta(\Im\Upsilon, \mathcal{H}\Upsilon, \tilde{t}), \delta(\Im\mho, \mathcal{H}\mho, \tilde{t}), \delta(\Im\Upsilon, \mathcal{H}\mho, \tilde{t}), \delta(\Im\mho, \mathcal{H}\Upsilon, \tilde{t})\}.$$
(3) $\tilde{\mathcal{R}}$ is $(\mathcal{H}, \Im)$-closed and $\tilde{\mathcal{R}} \mid \mathcal{H}(\mathbb{A})$ is transitive.
(4) $\tilde{\mathcal{R}} \mid \Im(\mathbb{A})$ is $\delta-$ self closed provided (2) holds for all $\Upsilon, \mho, \tilde{t} \in \mathbb{A}$ with $(\mathcal{H}\Upsilon, \mathcal{H}\mho) \in \tilde{\mathcal{R}}^*$ and $(\Im\Upsilon, \Im\mho) \in \tilde{\mathcal{R}}$.
(5) If $\mathcal{H}$ and $\Im$ are weakly compatible mappings, there exists $w \in \mathbb{A}$ such that $(\Im u, \Im w) \in \tilde{\mathcal{R}}$ and for all coincidence points $u, v$ of $\mathcal{H}$ and $\Im$ and $(\Im v, \Im w) \in \tilde{\mathcal{R}}$, then $\mathcal{H}$ and $\Im$ have a unique common fixed point.

## 3. Common fixed point theorems for almost $\mathcal{F}_{\tilde{\mathcal{R}}_\Im}$-contractions

In the following section, we will discuss some common fixed point theorems for almost $\mathcal{F}_{\tilde{\mathcal{R}}_\Im}$ -contractions. Firstly, we define almost $\mathcal{F}_{\tilde{\mathcal{R}}_\Im}$-contraction on a $\mathbb{B}_2$-metric space.

**Definition 3.1**. Consider a $\mathbb{B}_2$-metric space $(\mathbb{A}, \delta)$ with the binary relation $\tilde{\mathcal{R}}$ and $\mathcal{H}, \Im : \mathbb{A} \to \mathbb{A}$. Suppose that for all $\Upsilon, \mho, \tilde{t} \in \mathbb{A}$, there exists $\mathcal{F} \in \mathfrak{F}$, $\pounds \geq 0$ and $\tau > 0$ such that the condition
$$\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t}) > 0 \ implies\ \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t})) \leq \mathcal{F}(\mathcal{M}_b(\Upsilon, \mho, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon, \mho, \tilde{t}) \tag{3}$$
holds, where
$$\mathcal{M}_b(\Upsilon, \mho, \tilde{t}) = max\{\delta(\Im\Upsilon, \Im\mho, \tilde{t}), \frac{\delta(\Im\Upsilon, \mathcal{H}\mho, \tilde{t})}{2\mathfrak{s}}, \delta(\Im\mho, \mathcal{H}\mho, \tilde{t}), \frac{\delta(\Im\mho, \mathcal{H}\Upsilon, \tilde{t})[1 + \delta(\Im\Upsilon, \mathcal{H}\Upsilon, \tilde{t})]}{1 + \delta(\Im\Upsilon, \Im\mho, \tilde{t})}, \frac{\delta(\Im\Upsilon, \mathcal{H}\Upsilon, \tilde{t})[1 + \delta(\Im\mho, \mathcal{H}\Upsilon, \tilde{t})]}{1 + \delta(\Im\Upsilon, \Im\mho, \tilde{t})}\}$$

and

$$\mathcal{N}_b(\Upsilon, \mho, \tilde{t}) = min\{\delta(\Im\Upsilon, \Im\mho, \tilde{t}), \delta(\Im\Upsilon, \mathcal{H}\Upsilon, \tilde{t}), \delta(\Im\mho, \mathcal{H}\mho, \tilde{t}), \frac{\delta(\Im\Upsilon, \mathcal{H}\mho, \tilde{t})\delta(\Im\mho, \mathcal{H}\Upsilon, \tilde{t})}{1 + \delta(\Im\Upsilon, \Im\mho, \tilde{t})}\}$$

with $(\Im\Upsilon, \Im\mho) \in \tilde{\mathcal{R}}$ and $(\mathcal{H}\Upsilon, \mathcal{H}\mho) \in \tilde{\mathcal{R}}^*$, then $\mathcal{H}$ called almost $\mathcal{F}_{\tilde{\mathcal{R}}_{\Im}}$-contraction.

Now we give our first new result.

**Theorem 3.2**. *Consider a $\mathbb{B}_2$-metric space $(\mathbb{A}, \delta)$ furnished with a binary relation $\tilde{\mathcal{R}}$ and mappings $\mathcal{H}, \Im : \mathbb{A} \to \mathbb{A}$. Assume that $\mathcal{H}$ is an almost $\mathcal{F}\tilde{\mathcal{R}}_{\Im}$-contraction satisfying the following conditions:*

(i) *There exists $\Upsilon_0 \in \mathbb{A}$ such that $(\Im\Upsilon_0, \mathcal{H}\Upsilon_0) \in \tilde{\mathcal{R}}$.*
(ii) *$\mathcal{H}\mathbb{A} \subseteq \Im\mathbb{A}$, where $\Im\mathbb{A}$ is a $\mathbb{B}_2$-complete subspace of $\mathbb{A}$.*
(iii) *$\tilde{\mathcal{R}}$ is $(\mathcal{H}, \Im)$-closed and $\tilde{\mathcal{R}} \mid \Im(\mathbb{A})$ is transitive.*
(iv) *$\mathcal{H}$ is $(\Im, \tilde{\mathcal{R}})$-continuous.*
(v) *$\tilde{\mathcal{R}} \mid \Im(\mathbb{A})$ is $\delta$-self-closed, provided that (3) holds for all $\Upsilon, \mho, \tilde{t} \in \mathbb{A}$ with $(\Im\Upsilon, \Im\mho) \in \tilde{\mathcal{R}}$ and $(\mathcal{H}\Upsilon, \mathcal{H}\mho) \in \tilde{\mathcal{R}}^*$.*

*Under these conditions, $\mathcal{H}$ and $\Im$ have a coincidence point.*

*Proof.* Let $\Upsilon_0 \in \mathbb{A}$ such that $(\Im\Upsilon_0, \mathcal{H}\Upsilon_0) \in \tilde{\mathcal{R}}$. If $\Im\Upsilon_0 = \mathcal{H}\Upsilon_0$, then $\Upsilon_0$ is a coincidence point of $\mathcal{H}$ and $\Im$, hence the proof. Thus, assume that $\Im\Upsilon_0 \neq \mathcal{H}\Upsilon_0$, since $\mathcal{H}\mathbb{A} \subseteq \Im\mathbb{A}$, we can choose $\Upsilon_1 \in \mathbb{A}$ such that $\mathcal{H}\Upsilon_0 = \Im\Upsilon_1$. By repeating this process, we can construct a sequence $\{\Im\Upsilon_n\}$ in $\mathbb{A}$ by $\mathcal{H}\Upsilon_n = \Im\Upsilon_{n+1}$, for all $n \in \mathbb{N}$. From Lemma 2.21, $\{\Im\Upsilon_n\}$ is $\tilde{\mathcal{R}}-$preserving that is

$$(\Im\Upsilon_n, \Im\Upsilon_{n+1}) \in \tilde{\mathcal{R}} \text{ and } (\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}) \in \tilde{\mathcal{R}} \text{ for all } n \in \mathbb{N}. \tag{4}$$

If $\mathcal{H}\Upsilon_{m_0} = \mathcal{H}\Upsilon_{m_0+1}$ for some $m_0 \in \mathbb{N}$ then $\Im\Upsilon_{m_0+1} = \mathcal{H}\Upsilon_{m_0+1}$ which implies $\Upsilon_{m_0+1}$ is a coincidence point of $\mathcal{H}$ and $\Im$.

Hence suppose that $\mathcal{H}\Upsilon_n \neq \mathcal{H}\Upsilon_{n+1}$ for all $n \in \mathbb{N}$. In view of condition (a) of Definition 2.1 and condition (4), we have

$$\begin{aligned}\tau + \mathcal{F}(\delta(\Im\Upsilon_{n+1}, \Im\Upsilon_{n+2}, \tilde{t})) &= \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t})) \\ &\leq \mathcal{F}(\mathcal{M}_b(\Upsilon_n, \Upsilon_{n+1}, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon_n, \Upsilon_{n+1}, \tilde{t}),\end{aligned} \tag{5}$$

where

$$\begin{aligned}\mathcal{M}_b(\Upsilon_n, \Upsilon_{n+1}, \tilde{t}) &= max\{\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t}), \frac{\delta(\Im\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t})}{2\mathfrak{s}}, \delta(\Im\Upsilon_{n+1}, \mathcal{H}\Upsilon_{n+1}, \tilde{t}), \\ &\quad \frac{\delta(\Im\Upsilon_{n+1}, \mathcal{H}\Upsilon_n, \tilde{t})[1 + \delta(\Im\Upsilon_n, \mathcal{H}\Upsilon_n, \tilde{t})]}{1 + \delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t})}, \\ &\quad \frac{\delta(\Im\Upsilon_n, \mathcal{H}\Upsilon_n, \tilde{t})[1 + \delta(\Im\Upsilon_{n+1}, \mathcal{H}\Upsilon_n, \tilde{t})]}{1 + \delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t})}\} \\ &= max\{\delta(\mathcal{H}\Upsilon_{n-1}, \mathcal{H}\Upsilon_n, \tilde{t}), \frac{\delta(\mathcal{H}\Upsilon_{n-1}, \mathcal{H}\Upsilon_n, \tilde{t})}{2\mathfrak{s}}, \delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t}), \\ &\quad \frac{\delta(\mathcal{H}\Upsilon_{n-1}, \mathcal{H}\Upsilon_n, \tilde{t})}{1 + \delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n-1}, \tilde{t})}\} \\ &\leq max\{\delta(\mathcal{H}\Upsilon_{n-1}, \mathcal{H}\Upsilon_n, \tilde{t}), \delta(\mathcal{H}\Upsilon_{n+1}, \mathcal{H}\Upsilon_n, \tilde{t})\}\end{aligned} \tag{6}$$

and

$$\mathcal{N}_b(\Upsilon_n, \Upsilon_{n+1}, \tilde{t}) = min\{\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t}), \delta(\Im\Upsilon_n, \mathcal{H}\Upsilon_n, \tilde{t}), \delta(\Im\Upsilon_{n+1}, \mathcal{H}\Upsilon_{n+1}, \tilde{t}), \frac{\delta(\Im\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t})\delta(\Im\Upsilon_{n+1}, \mathcal{H}\Upsilon_n, \tilde{t})}{1 + \delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t})}\}$$
$$= min\{\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t}), \delta(\Im\Upsilon_{n+1}, \Im\Upsilon_{n+2}, \tilde{t}), \delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t}), 0\} = 0. \tag{7}$$

Thus from (5), (6) and (7), we get

$$\tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t})) \leq \mathcal{F}(max\{\delta(\mathcal{H}\Upsilon_{n-1}, \mathcal{H}\Upsilon_n, \tilde{t}), \delta(\mathcal{H}\Upsilon_{n+1}, \mathcal{H}\Upsilon_n, \tilde{t})\}). \tag{8}$$

Suppose that $\delta(\mathcal{H}\Upsilon_{n-1}, \mathcal{H}\Upsilon_n, \tilde{t}) < \delta(\mathcal{H}\Upsilon_{n+1}, \mathcal{H}\Upsilon_n, \tilde{t})$ in (8), we get

$$\tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t})) \leq \mathcal{F}(\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t}))$$

which implies $\tau \leq 0$, a contradiction. Hence $\{\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t}))\}$ is a decreasing sequence of a non negative real numbers, then from (8), we get

$$\begin{aligned}\mathcal{F}(\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t})) &\leq \mathcal{F}(\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n-1}, \tilde{t})) - \tau \\ &\leq \mathcal{F}(\delta(\mathcal{H}\Upsilon_{n-1}, \mathcal{H}\Upsilon_{n-2}, \tilde{t})) - 2\tau \\ &\leq ... \leq F(\delta(\mathcal{H}\Upsilon_0, \mathcal{H}\Upsilon_1, \tilde{t})) - n\tau,\end{aligned}$$

for all $n \in \mathbb{N}$. Taking limits as $n \to +\infty$, we attain $lim_{n\to+\infty}\mathcal{F}(\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t})) = -\infty$, using $\mathcal{F}_2$, we get

$$lim_{n\to+\infty}\delta(\mathcal{H}\Upsilon_n, \mathcal{H}\Upsilon_{n+1}, \tilde{t}) = 0. \tag{9}$$

We now claim that $\delta(\Im\Upsilon_i, \Im\Upsilon_j, \Im\Upsilon_\hbar) = 0$ for all $i, j, \hbar \in \mathbb{N}$.
Since, $\{\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \tilde{t})\}$ is strictly decreasing and $\delta(\Im\Upsilon_0, \Im\Upsilon_1, \Im\Upsilon_0) = 0$. We conclude that $\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \Im\Upsilon_0) = 0$ for all $m \in \mathbb{N}$.
Again, $\delta(\Im\Upsilon_{m-1}, \Im\Upsilon_m, \Im\Upsilon_m) = 0$ for all $m \in \mathbb{N}$ and $\{\delta(\Im\Upsilon_{n-1}, \Im\Upsilon_n, \tilde{t})\}$ is strictly decreasing, we obtain that

$$\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \Im\Upsilon_m) = 0, for\ all\ n \geq m - 1. \tag{10}$$

Also, for $0 \leq n \leq m - 1$, it follows that $m - 1 \geq n + 1$. Henceforth, from (10), we have

$$\delta(\Im\Upsilon_{m-1}, \Im\Upsilon_m, \Im\Upsilon_{n+1}) = \delta(\Im\Upsilon_{m-1}, \Im\Upsilon_m, \Im\Upsilon_n) = 0. \tag{11}$$

Hence by rectangular inequality, and using (11), we obtain

$$\begin{aligned}\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \Im\Upsilon_m) &\leq \mathfrak{s}[\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \Im\Upsilon_{m-1}) + \delta(\Im\Upsilon_{n+1}, \Im\Upsilon_m, \Im\Upsilon_{m-1}) \\ &\quad + \delta(\Im\Upsilon_m, \Im\Upsilon_n, \Im\Upsilon_{m-1})] \\ &= \mathfrak{s}\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \Im\Upsilon_{m-1}) \leq \mathfrak{s}\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \Im\Upsilon_{n+1}) = 0.\end{aligned}$$

Therefore, we get $\delta(\Im\Upsilon_n, \Im\Upsilon_{n+1}, \Im\Upsilon_m) = 0$, for $0 \leq n < m - 1$.
For all $i, j, \hbar \in \mathbb{N}$, $j < i$ and $\delta(\Im\Upsilon_i, \Im\Upsilon_j, \Im\Upsilon_{j-1} = \delta(\Im\Upsilon_\hbar, \Im\Upsilon_j, \Im\Upsilon_{j-1}) = 0$,
in view of rectangular inequality, we have

$$\begin{aligned}\delta(\Im\Upsilon_i, \Im\Upsilon_j, \Im\Upsilon_\hbar) &\leq \mathfrak{s}[\delta(\Im\Upsilon_i, \Im\Upsilon_j, \Im\Upsilon_{j-1}) + \delta(\Im\Upsilon_j, \Im\Upsilon_\hbar, \Im\Upsilon_{j-1}) + \delta(\Im\Upsilon_\hbar, \Im\Upsilon_i, \Im\Upsilon_{j-1})] \\ &= \mathfrak{s}[\delta(\Im\Upsilon_\hbar, \Im\Upsilon_j, \Im\Upsilon_{j-1})] \\ &\leq \mathfrak{s}^2[\delta(\Im\Upsilon_\hbar, \Im\Upsilon_j, \Im\Upsilon_{j-2})] \\ &\leq ...... \leq \mathfrak{s}^{j-1}\delta(\Im\Upsilon_\hbar, \Im\Upsilon_i, \Im\Upsilon_i) = 0.\end{aligned}$$

Therefore for all $i, j, \hbar \in \mathbb{N}$, we attain $\delta(\Im\Upsilon_i, \Im\Upsilon_j, \Im\Upsilon_\hbar) = 0$.
We now show that $\{\Im\Upsilon_n\}$ is a $\mathbb{B}_2$-Cauchy sequence.

If $\{\Im\Upsilon_n\}$ is not a $\mathbb{B}_2$- Cauchy sequence by Lemma 2.20, there exists $\epsilon > 0$ and two subsequences $\{r(\hbar)\}$ and $\{n(\hbar)\}$ with $n(\hbar) > r(\hbar) > \hbar$ such that $\delta(\Im\Upsilon_{r(\hbar)}, \Im\Upsilon_{n(\hbar)}, \tilde{t}) \geq \epsilon$ for all $\hbar$ and $\delta(\mathcal{G}\Upsilon_{r(\hbar)}, \Im\Upsilon_{n(\hbar)-1}, \tilde{t}) \leq \epsilon$ satisfying $(i)-(iv)$ of Lemma 2.20.

In view of Lemma 2.20, we have

$$(\Im\Upsilon_{r(\hbar)}, \Im\Upsilon_{n(\hbar)}) \in \tilde{\mathcal{R}} \; and \; (\mathcal{H}\Upsilon_{r(\hbar)}, \mathcal{H}\Upsilon_{n(\hbar)}) \in \tilde{\mathcal{R}},$$

for all $r(\hbar), n(\hbar) \in \mathbb{N}$ with $r(\hbar) < n(\hbar)$. On using (3), we have

$$\begin{aligned}\tau + \delta(\Im_{r(\hbar)}, \Im_{n(\hbar)}, \tilde{t}) &= \tau + \delta(\mathcal{H}_{r(\hbar)-1}, \mathcal{H}_{n(\hbar)-1}, \tilde{t}) \\ &\leq \mathcal{F}(\mathcal{M}_b(\Upsilon_{r(\hbar)-1}, \Upsilon_{n(\hbar)-1}, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon_{r(\hbar)-1}, \Upsilon_{n(\hbar)-1}, \tilde{t}), \qquad (12)\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_b(\Upsilon_{r(\hbar)-1}, \Upsilon_{n(\hbar)-1}, \tilde{t}) = max\{&\delta(\Im_{r(\hbar)-1}, \Im_{n(\hbar)-1}, \tilde{t}), \frac{\delta(\Im_{r(\hbar)-1}, \Im_{n(\hbar)}, \tilde{t})}{2\mathfrak{s}}, \\ &\delta(\Im_{n(\hbar)-1}, \Im_{n(\hbar)}, \tilde{t}), \frac{\delta(\Im_{n(\hbar)-1}, \Im_{r(\hbar)}, \tilde{t})[1 + \delta(\Im_{r(\hbar)-1}, \mathcal{H}_{r(\hbar)}, \tilde{t})]}{1 + \delta(\Im_{r(\hbar)-1}, \Im_{n(\hbar)-1}, \tilde{t})}, \\ &\frac{\delta(\Im_{r(\hbar)-1}, \Im_{r(\hbar)}, \tilde{t})[1 + \delta(\Im_{n(\hbar)-1}, \Im_{r(\hbar)}, \tilde{t})]}{1 + \delta(\Im_{r(\hbar)-1}, \Im_{n(\hbar)-1}, \tilde{t})}\},\end{aligned}$$

thus from Lemma 2.20 and (9), we have
$max\{\frac{\epsilon}{\mathfrak{s}^2}, \frac{\epsilon}{2\mathfrak{s}^2}, \frac{\epsilon}{\mathfrak{s}(1+\epsilon)}\} \leq lim\ sup_{\hbar\to+\infty}\mathcal{M}_b(\Upsilon_{r(\hbar)-1}, \Upsilon_{n(\hbar)-1}, \tilde{t}) \leq max\{\mathfrak{s}\epsilon, \epsilon, 0, \frac{\mathfrak{s}^3\epsilon}{\epsilon+\mathfrak{s}^2}\}$
this implies

$$\frac{\epsilon}{2\mathfrak{s}^2} \leq lim\ sup_{\hbar\to+\infty}\mathcal{M}_b(\Upsilon_{r(\hbar)-1}, \Upsilon_{n(\hbar)-1}, \tilde{t}) \leq \mathfrak{s}\epsilon. \qquad (13)$$

And

$$\begin{aligned}\mathcal{N}_b(\Upsilon_{r(\hbar)-1}, \Upsilon_{n(\hbar)-1}, \tilde{t}) = min\{&\delta(\Im_{r(\hbar)-1}, \Im_{n(\hbar)-1}, \tilde{t}), \delta(\Im_{r(\hbar)-1}, \mathcal{H}_{r(\hbar)-1}, \tilde{t}), \\ &\delta(\Im_{n(\hbar)-1}, \mathcal{H}_{n(\hbar)-1}, \tilde{t}), \frac{\delta(\Im_{r(\hbar)-1}, \mathcal{H}_{n(\hbar)-1}, \tilde{t})\delta(\Im_{n(\hbar)-1}, \Im_{r(\hbar)-1}, \tilde{t})}{1 + \delta(\Im_{r(\hbar)-1}, \Im_{n(\hbar)-1}, \tilde{t})}\}.\end{aligned}$$

Therefore

$$lim\ sup_{\hbar\to+\infty}\mathcal{N}_b(\Upsilon_{r(\hbar)-1}, \Upsilon_{n(\hbar)-1}, \tilde{t}) = 0. \qquad (14)$$

Hence from (12), (13) and (14), we have

$$\tau + \mathcal{F}(\mathfrak{s}\epsilon) \leq \mathcal{F}(\mathfrak{s}\epsilon),$$

this leads to a contradiction. Therefore, the sequence $\Im\Upsilon_n$ is a $\mathbb{B}_2$-Cauchy sequence in $\mathbb{A}$. In light of $\Im(\mathbb{A})$ is a complete subspace of $\mathbb{A}$, there exists an element $\mathfrak{n} \in \Im(\mathbb{A})$ such that

$$lim_{n\to+\infty}\Im\Upsilon_n = lim_{n\to+\infty}\mathcal{H}\Upsilon_n = \Im\mathfrak{n}.$$

We now demonstrate that $\mathfrak{n}$ is a coincidence point of $\mathcal{H}$ and $\Im$. To show this, we consider the following cases.
$(i)$ Suppose that $\mathcal{H}$ is $(\Im, \tilde{\mathcal{R}})$ continuous, which implies

$$lim_{n\to+\infty}\Im\Upsilon_{n+1} = lim_{n\to+\infty}\mathcal{H}\Upsilon_n = \mathcal{H}\mathfrak{n}.$$

In view of uniqueness, we get

$$\Im\mathfrak{n} = \mathcal{H}\mathfrak{n}.$$

Therefore $\mathfrak{n}$ is a coincidence point of $\Im$ and $\mathcal{H}$.
$(ii)$ Suppose that $\tilde{\mathcal{R}}|\Im(\mathbb{A})$ is $\delta$-self closed and condition (3) holds for all $\Upsilon$, $\mho$, $a \in \mathbb{A}$ with $(\Im\Upsilon, \Im\mho) \in \tilde{\mathcal{R}}$ and $(\mathcal{H}\Upsilon, \mathcal{H}\mho) \in \tilde{\mathcal{R}}^*$. Since $\{\Im\Upsilon_n\} \subseteq \Im(\mathbb{A})$, $\{\Im\Upsilon_n\}$ is $\tilde{\mathcal{R}}|\Im(\mathbb{A})$-preserving

and $\Im\Upsilon_n \to \Im\mathfrak{n}$ so that there exists $\{\Im\Upsilon_{r(\hbar)}\} \subseteq \{\Im\Upsilon_n\}$ such that $(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}) \in \tilde{\mathcal{R}}|\Im(\mathbb{A})$ for all $\hbar \in \mathbb{N}_o$ and since $\tilde{\mathcal{R}}$ is $(\Im, \mathcal{H})$ closed then $(\mathcal{H}\Upsilon_{n(\hbar)}, \mathcal{H}\mathfrak{n}) \in \tilde{\mathcal{R}}|\Im(\mathbb{A})$ for all $\hbar \in \mathbb{N}_o$.

If $\mathcal{H}\Upsilon_{n(\hbar)} = \mathcal{H}\mathfrak{n}$, for all $\hbar > \hbar_o$ and $\hbar, \hbar_o \in \mathbb{N}_o$ and then $lim_{\hbar\to+\infty}\mathcal{H}\Upsilon_{n(\hbar)} = \mathcal{H}\mathfrak{n}$, and since $lim_{n\to+\infty}\mathcal{H}\Upsilon_n = \Im\mathfrak{n}$, we have $\mathfrak{n}$ is a coincidence point of $\mathcal{H}$ and $\Im$.

If $\mathcal{H}\Upsilon_{n(\hbar)} \neq \mathcal{H}\mathfrak{n}$ for all $\hbar > \hbar_o$ and $\hbar, \hbar_o \in \mathbb{N}_o$, then $(\mathcal{H}\Upsilon_{n(\hbar)}, \mathcal{H}\mathfrak{n}) \in \tilde{\mathcal{R}}|\Im(\mathbb{A})$ and $(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}) \in \tilde{\mathcal{R}}|\Im(\mathbb{A})$. From contraction condition (3), we have

$$\begin{aligned}\tau + \mathcal{F}(\delta(\Im\Upsilon_{n(\hbar)+1}, \mathcal{H}\mathfrak{n}, \tilde{t})) &= \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon_{n(\hbar)}, \mathcal{H}\mathfrak{n}, \tilde{t})) \\ &\leq \mathcal{F}(\mathcal{M}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t})), \end{aligned} \tag{15}$$

where

$$\begin{aligned}\mathcal{M}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t}) &= max\{\delta(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t}), \frac{\delta(\Im\Upsilon_{n(\hbar)}, \mathcal{H}\mathfrak{n},\ \tilde{t})}{2\mathfrak{s}}, \delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n}, \tilde{t}), \\ &\quad \frac{\delta(\Im\mathfrak{n}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t})[1 + \delta(\Im\Upsilon_{n(\hbar)}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t})]}{1 + \delta(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t})}, \\ &\quad \frac{\delta(\Im\Upsilon_{n(\hbar)}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t})[1 + \delta(\Im\mathfrak{n}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t})]}{1 + \delta(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t})}\} \\ &= max\{\delta(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t}), \frac{\delta(\Im\Upsilon_{n(\hbar)}, \mathcal{H}\mathfrak{n}, \tilde{t})}{2\mathfrak{s}}, \delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n}, \tilde{t}), \\ &\quad \frac{\delta(\Im\mathfrak{n}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t})[1 + \delta(\Im\mathfrak{n}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t})]}{1 + \delta(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t})}, \\ &\quad \frac{\delta(\Im\Upsilon_{n(\hbar)}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t})[1 + \delta(\mathcal{H}\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t})]}{1 + \delta(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t})]}\}\end{aligned}$$

$$lim\ sup_{n\to+\infty}\mathcal{M}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t}) = max\{\delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n}, \tilde{t}), lim\ sup_{\hbar\to+\infty}\frac{\delta(\Im\Upsilon_{n(\hbar)}, \mathcal{H}\mathfrak{n}, \tilde{t})}{2\mathfrak{s}}\}. \tag{16}$$

Hence by Lemma 2.5, we get

$$\begin{aligned}max\{\delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n},\ \tilde{t}), \frac{\delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n},\ \tilde{t})}{\mathfrak{s}}\} &\leq lim\ sup_{\hbar\to+\infty}\mathcal{M}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t}) \\ &\leq max\{\delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n}, \tilde{t}), \frac{\delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n}, \tilde{t})}{2}\}.\end{aligned} \tag{17}$$

Also

$$\begin{aligned}\mathcal{N}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t}) &= min\{\delta(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t}), \delta(\Im\Upsilon_{n(\hbar)}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t}), \delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n}, \tilde{t}), \\ &\quad \frac{\delta(\Im\Upsilon_{n(\hbar)}, \mathcal{H}\mathfrak{n}, \tilde{t})\delta(\Im\mathfrak{n}, \mathcal{H}\Upsilon_{n(\hbar)}, \tilde{t})}{1 + \delta(\Im\Upsilon_{n(\hbar)}, \Im\mathfrak{n}, \tilde{t})}\},\end{aligned}$$

thus,

$$lim\ sup_{\hbar\to+\infty}\mathcal{N}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t}) = 0. \tag{18}$$

Now, on using (16), (17) and (18) in (15), we get
$\tau + lim\ sup_{\hbar\to+\infty}\mathcal{F}(\delta(\mathcal{H}\Upsilon_{n(\hbar)}, \mathcal{H}\mathfrak{n}, \tilde{t}))$
$\leq lim\ sup_{\hbar\to+\infty}\mathcal{F}(\mathcal{M}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t})) + \pounds lim\ sup_{\hbar\to+\infty}\mathcal{N}_b(\Upsilon_{n(\hbar)}, \mathfrak{n}, \tilde{t})$,
which implies $\mathcal{F}(\delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n}, \tilde{t})) \leq \mathcal{F}(\delta(\Im\mathfrak{n}, \mathcal{H}\mathfrak{n}, \tilde{t})) - \tau$
this implies $\delta(\mathcal{H}\mathfrak{n}, \Im\mathfrak{n}, \tilde{t}) = 0$, for all $\tilde{t} \in \mathbb{A}$.

Hence $\mathcal{H}\mathfrak{n} = \Im\mathfrak{n}$.

Thus, $\Im$ and $\mathcal{H}$ have a coincidence point. □

**Theorem 3.3**. *In addition to the conditions of Theorem 3.2, assume that the pair $(\mathcal{H}, \Im)$ is weakly compatible. Furthermore, for all coincidence points $\phi$ and $\xi$ of $\mathcal{H}$ and $\Im$, there exists $\zeta \in \mathbb{A}$ such that $(\Im\phi, \Im\zeta) \in \tilde{\mathcal{R}}$ and $(\Im\xi, \Im\zeta) \in \tilde{\mathcal{R}}$. Under these assumptions, it follows that $\mathcal{H}$ and $\Im$ share a unique common fixed point.*

*Proof.* In view of Theorem 3.2, the set of coincidence points of $\mathcal{H}$ and $\Im$ is nonempty. Further, assume that $\phi$ and $\xi$ are two coincidence points of $\Im$ and $\mathcal{H}$ i.e., $\Im\phi = \mathcal{H}\phi$ and $\Im\xi = \mathcal{H}\xi$. We now claim that $\Im\phi = \Im\xi$. In light of our assumption, there exists $\zeta \in \mathbb{A}$ and

$$(\Im\phi, \Im\zeta) \in \tilde{\mathcal{R}} \text{ and } (\Im\xi, \Im\zeta) \in \tilde{\mathcal{R}}.$$

Following similarly from the proof of Theorem 3.2, we can define a sequence $\{\zeta_n\}$ in $\mathbb{A}$ such that $\mathcal{H}\zeta_n = \Im\zeta_{n+1}$ for all $n \in \mathbb{N}$, and $\zeta_o = \zeta$, with $lim_{n\to+\infty}\delta(\Im\zeta_n, \Im\zeta_{n+1}, \tilde{t}) = 0$. Since $(\Im\phi, \Im\zeta_0) \in \tilde{\mathcal{R}}$, $(\Im\xi, \Im\zeta_0) \in \tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}$ is $(\Im, \mathcal{H})$ closed, it follows that $(\mathcal{H}\phi, \mathcal{H}\zeta_0) \in \tilde{\mathcal{R}}$ and $(\mathcal{H}\xi, \mathcal{H}\zeta_0) \in \tilde{\mathcal{R}}$. Hence $(\Im\phi, \Im\zeta_1) \in \tilde{\mathcal{R}}$ and $(\Im\xi, \Im\zeta_1) \in \tilde{\mathcal{R}}$ . Thus by induction, we have

$$(\Im\phi, \Im\zeta_n) \in \tilde{\mathcal{R}} \text{ and } (\Im\xi, \Im\zeta_n) \in \tilde{\mathcal{R}}, \tag{19}$$

for all $n \in \mathbb{N}$.
From (3) and (19), we get

$$\begin{aligned}\tau + \mathcal{F}(\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t})) &= \tau + \mathcal{F}(\delta(\mathcal{H}\phi, \mathcal{H}\zeta_n, \tilde{t})) \\ &\leq \mathcal{F}(\mathcal{M}_b(\phi, \zeta_n, \tilde{t})) + \pounds\mathcal{N}_b(\phi, \zeta_n, \tilde{t}) \end{aligned}\tag{20}$$

where

$$\begin{aligned}\mathcal{M}_b(\phi, \zeta_n, \tilde{t}) &= max\{\delta(\Im\phi, \Im\zeta_n, \tilde{t}), \delta(\Im\zeta_n, \mathcal{H}\zeta_n, \tilde{t}), \frac{\delta(\Im\phi, \mathcal{H}\zeta_n, \tilde{t})}{2\mathfrak{s}}, \\ &\quad \frac{\delta(\Im\zeta_n, \mathcal{H}\phi, \tilde{t})[1 + \delta(\mathcal{H}\phi, \Im\phi, \tilde{t})]}{1 + \delta(\Im\phi, \Im\zeta_n, \tilde{t})}, \frac{\delta(\mathcal{H}\phi, \Im\phi, \tilde{t})[1 + \delta(\Im\zeta_n, \mathcal{H}\phi, \tilde{t})]}{1 + \delta(\Im\phi, \Im\zeta_n, \tilde{t})}\} \\ &= max\{\delta(\Im\phi, \Im\zeta_n, \tilde{t}), \delta(\Im\zeta_n, \Im\zeta_{n+1}, \tilde{t}), \frac{\delta(\Im\phi, \mathcal{H}\zeta_n, \tilde{t})}{2\mathfrak{s}}, \\ &\quad \frac{\delta(\Im\zeta_n, \mathcal{H}\phi, \tilde{t})[1 + \delta(\mathcal{H}\phi, \Im\phi, \tilde{t})]}{1 + \delta(\Im\phi, \Im\zeta_n, \tilde{t})}, \frac{\delta(\mathcal{H}\phi, \Im\phi, \tilde{t})[1 + \delta(\Im\zeta_n, \mathcal{H}\phi, \tilde{t})]}{1 + \delta(\Im\phi, \Im\zeta_n, \tilde{t})}\} \\ &= max\{\delta(\Im\phi, \Im\zeta_n, \tilde{t}), \frac{\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t})}{2\mathfrak{s}}, \frac{\delta(\Im\zeta_n, \mathcal{H}\phi, \tilde{t})}{1 + \delta(\Im\phi, \Im\zeta_n, \tilde{t})}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_b(\phi, \zeta_n, \tilde{t}) &= min\{\delta(\Im\zeta_n, \mathcal{H}\zeta_n, \tilde{t}), \delta(\Im\phi, \mathcal{H}\phi, \tilde{t}), \delta(\Im\zeta_n, \mathcal{H}\phi, \tilde{t}), \\ &\quad \frac{\delta(\Im\phi, \mathcal{H}\zeta_n, \tilde{t})\delta(\Im\zeta_n, \mathcal{H}\phi, \tilde{t})}{1 + \delta(\Im\phi, \Im\zeta_n, \tilde{t})}\} = 0.\end{aligned}$$

If $\mathcal{M}_b(\phi, \zeta_n, \tilde{t}) = \delta(\Im\phi, \Im\zeta_n, \tilde{t})$ then from (20), we have

$$\begin{aligned}\mathcal{F}(\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t})) &\leq \mathcal{F}(\delta(\Im\phi, \Im\zeta_n, \tilde{t})) - \tau \\ &\leq \mathcal{F}(\delta(\Im\phi, \Im\zeta_{n-1}, \tilde{t})) - 2\tau \\ &\leq .... \\ &\leq .... \\ &\leq .... \\ &\leq \mathcal{F}(\delta(\Im\phi, \Im\zeta_o, \tilde{t})) - n\tau.\end{aligned}$$

Taking limits $n \to +\infty$ in the above inequality, we get

$$lim_{n\to+\infty}\mathcal{F}(\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t})) = -\infty.$$

Using condition (2) of $\mathcal{F}$, we get $lim_{n\to+\infty}\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t}) = 0$.
If $\mathcal{M}_b(\phi, \zeta_n, \tilde{t}) = \delta(\Im\zeta_n, \Im\zeta_{n+1}, \tilde{t})$, then
$\tau + \mathcal{F}(\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t})) \leq \mathcal{F}(\delta(\Im\zeta_n, \Im\zeta_{n+1}, \tilde{t}))$, which implies

$$\begin{aligned}\mathcal{F}(\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t})) &\leq \mathcal{F}(\delta(\Im\zeta_n, \Im\zeta_{n+1}, \tilde{t})) - \tau \\ &\leq \mathcal{F}(\delta(\Im\zeta_{n-1}, \Im\zeta_n, \tilde{t})) - 2\tau \\ &\leq .... \\ &\leq .... \\ &\leq ..... \\ &\leq \mathcal{F}(\delta(\Im\zeta_0, \Im\zeta_1, \tilde{t})) - n\tau.\end{aligned}$$

Taking limits as $n \to +\infty$, we get
$lim_{n\to+\infty}\mathcal{F}(\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t})) = -\infty$. Again, by using condition (2) of $\mathcal{F}$, we get $\delta(\Im\phi, \Im\zeta_{n+1}, \tilde{t}) = 0$.
Thus from the above arguments, we can conclude that

$$lim_{n\to+\infty}\delta(\Im\phi, \Im\zeta_n, \tilde{t}) = 0. \tag{21}$$

Similarly, we can show that

$$lim_{n\to+\infty}\delta(\Im\xi, \Im\zeta_n, \tilde{t})) = 0. \tag{22}$$

Hence from (21) and (22), we get $\Im\phi = \Im\xi$.

Thus, $\Im$ and $\mathcal{H}$ have a unique coincidence point. Now by utilizing Definition 2.21, it follows that $\mathcal{H}$ and $\Im$ have a unique common fixed point. □

**Theorem 3.4**. *Along with the axioms of Theorem 3.2, suppose that the all the coincidence points of $\mathcal{H}$ and $\Im$ are $\tilde{\mathcal{R}}-$ comparable and one of $\mathcal{H}$ or $\Im$ is one-one then coincidence points of $\mathcal{H}$ and $\Im$ is unique. Moreover, the pair ($\mathcal{H}$, $\Im$) is weakly compatible then $\mathcal{H}$ and $\Im$ have a unique common fixed point.*

*Proof.* On the account of proof of Theorem 3.2, coincidence points of $\mathcal{H}$ and $\Im$ is nonempty. If is coincidence points of $\mathcal{H}$ and $\Im$ is singleton set, then the proof is completed. Otherwise, choose $\nu$ and $\lambda$ be two coincidence points with $\nu \neq \lambda$ so that

$$\mathcal{H}\nu = \Im\nu \ and \ \mathcal{H}\lambda = \Im\lambda. \tag{23}$$

By our assumption, we have $(\Im\nu, \Im\lambda) \in \tilde{\mathcal{R}}$. Since $\tilde{\mathcal{R}}$ is $(\mathcal{H}, \Im)$ closed, we have $(\mathcal{H}\nu, \mathcal{H}\lambda) \in \tilde{\mathcal{R}}$.

Using condition (3) and (23), we have

$$\tau + \mathcal{F}(\delta(\mathcal{H}\nu, \mathcal{H}\lambda, \tilde{t})) \leq \mathcal{F}(\mathcal{M}_b(\nu, \mho, \tilde{t})) + \pounds\mathcal{N}_b(\nu, \lambda, \tilde{t}) \tag{24}$$

where

$$\begin{aligned}\mathcal{M}_b(\nu,\lambda,\tilde{t}) &= max\{\delta(\Im\nu,\Im\lambda,\tilde{t}), \frac{\delta(\Im\nu,\mathcal{H}\lambda,\tilde{t})}{2\mathfrak{s}}, \delta(\Im\lambda,\mathcal{H}\lambda,\tilde{t}),\\ &\qquad \frac{\delta(\Im\nu,\mathcal{H}\nu,\tilde{t})[1+\delta(\Im\lambda,\mathcal{H}\nu,\tilde{t})]}{1+\delta(\Im\nu,\Im\lambda,\tilde{t})}, \frac{\delta(\Im\lambda,\mathcal{H}\nu,\tilde{t})[1+(\Im\nu,\mathcal{H}\nu,\tilde{t})]}{1+d(\Im\nu,\Im\lambda,\tilde{t})}\}\\ &= max\{\delta(\Im\lambda,\Im\nu,\tilde{t}), \frac{\delta(\Im\nu,\Im\lambda,\tilde{t})}{2\mathfrak{s}}, \frac{\delta(\Im\nu,\Im\lambda,\tilde{t})}{1+\delta(\Im\nu,\Im\lambda,\tilde{t})}\}\\ &= \delta(\Im\nu,\Im\lambda,\tilde{t}) \qquad (25)\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_b(\nu,\lambda,\tilde{t}) &= min\{\delta(\Im\nu,\Im\lambda,\tilde{t}), \delta(\Im\nu,\mathcal{H}\nu,\tilde{t}), \delta(\Im\lambda,\mathcal{H}\lambda,\tilde{t}),\\ &\qquad \frac{\delta(\Im\nu,\mathcal{H}\lambda,\tilde{t}),\delta(\Im\lambda,\mathcal{H}\nu,\tilde{t})}{1+\delta(\Im\nu,\Im\lambda,\tilde{t})}\} = 0. \qquad (26)\end{aligned}$$

On utilizing (23), (25) and (26) in (24), we get
$\tau + \mathcal{F}(\delta(\mathcal{H}\nu,\mathcal{H}\lambda,a)) \leq \mathcal{F}(\delta(\mathcal{H}\nu,\mathcal{H}\lambda,a))$ , which yields $\mathcal{H}\lambda = \mathcal{H}\nu$.
Thus, $\Im$ and $\mathcal{H}$ have a unique coincidence point. Now by utilizing Definition 2.21, it follows that $\mathcal{H}$ and $\Im$ have a unique common fixed point. □

**Example 3.5**. Let $\mathbb{A} = \{(a,0)|a \in [0,8]\} \bigcup \{(0,2)\} \subseteq \mathbb{R}^2$ and let $\delta(\Upsilon,\mho,\vartheta)$ denote the square of the area of triangle with vertices $\Upsilon,\mho$ and $\vartheta \in \mathbb{A}$,
*e.g.*, $\delta((\Upsilon,0),(\mho,0),(0,2)) = (\mho-\Upsilon)^2$. Clearly, $(\mathbb{A},\delta)$ is a $\mathbb{B}_2$-metric space with $\mathfrak{s}=2$. We define relation $\tilde{\mathcal{R}}$ on $\mathbb{A}$ by

$$\begin{aligned}\tilde{\mathcal{R}} = \{&((0,0),(3,0)),((1,0),(\frac{5}{2},0)),((1,0),(2,0)),((1,0),(\frac{3}{2},0)),((1,0),(3,0)),\\ &((1,0),(1,0)),((\frac{3}{2},0)),(\frac{5}{2},0),((0,0),(1,0)),((0,0),(2,0)),((\frac{3}{2},0),(2,0)),\\ &((0,0),(\frac{3}{2},0)),((0,0),(\frac{5}{2},0))\}.\end{aligned}$$

We define $\Im,\mathcal{H} : \mathbb{A} \to \mathbb{A}$ by

$$\mathcal{H}(\Upsilon,0) = \begin{cases} (1,0) & if\ \Upsilon \in [0,1] \\ (\frac{\Upsilon}{2},0) & if\ \Upsilon \in (1,8] \end{cases} \quad and$$

$$\Im(\Upsilon,0) = \begin{cases} (1-\Upsilon^2,0) & if\ \Upsilon \in [0,1] \\ (\frac{\Upsilon+1}{2},0) & if\ \Upsilon \in (1,8] \end{cases}$$

and $\Im(0,2) = \mathcal{H}(0,2) = (1,0)$.
We now verify the postulates of the Theorem 3.2. There exists an $\Upsilon_0 = (0,0)$ such that $(\Im\Upsilon_0,\mathcal{H}\Upsilon_0) = ((1,0),(1,0)) \in \tilde{\mathcal{R}}$. Clearly, $\mathcal{H}\mathbb{A} \subseteq \Im\mathbb{A}$, $\Im\mathbb{A}$ is a complete subspace of $\mathbb{A}$. $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}|\Im\mathbb{A}$ is transitive and $\tilde{\mathcal{R}}|\Im\mathbb{A}$ is $\delta-$ self closed. It is easy to verify that $\tilde{\mathcal{R}}$ is $(\mathcal{H},\Im)$ closed.
We define $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$ by $\mathcal{F}(\nu) = log\nu + \frac{\nu}{2}$ for all $\nu \in \mathbb{R}^+$. Clearly, $\mathcal{F} \in \mathfrak{F}$. We verify contraction condition (3) when $(\Im\Upsilon,\Im\mho) \in \tilde{\mathcal{R}}$, $(\mathcal{H}\Upsilon,\mathcal{H}\mho) \in \tilde{\mathcal{R}}$, $q = (0,2), L = 3$, $\tau = ln\sqrt{3}$ and $\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,\tilde{t}) > 0$.

Case(i): When $(\Upsilon, 0) = (1, 0)$,$(\mho, 0) = (5, 0)$ or $(\Upsilon, 0) = (2, 0)$ ,$(\mho, 0) = (3, 0)$ then

$$ln\sqrt{3} + ln(\frac{9}{4}) + \frac{9}{8} = \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t}))$$
$$\leq ln9 + \frac{9}{2} + \frac{3}{4} = \mathcal{F}(\mathcal{M}_b(\Upsilon, \mho, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon, \mho, \tilde{t})$$

Case(ii): When $(\Upsilon, 0) = (0, 0)$, $(\mho, 0) = (4, 0)$ then

$$ln\sqrt{3} + \frac{1}{2} = \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t}))$$
$$\leq ln(\frac{9}{4}) + \frac{9}{8} = \mathcal{F}(\mathcal{M}_b(\Upsilon, \mho, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon, \mho, \tilde{t})$$

Case(iii): When $(\Upsilon, 0) = (0, 0)$, $(\mho, 0) = (2, 0)$ then

$$ln\sqrt{3} + ln(\frac{1}{4}) + \frac{1}{8} = \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t}))$$
$$\leq \frac{1}{2} = \mathcal{F}(\mathcal{M}_b(\Upsilon, \mho, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon, \mho, \tilde{t})$$

Case(iv): When $(\Upsilon, 0) = (2, 0)$, $(\mho, 0) = ((3, 0)$ then

$$ln\sqrt{3} + ln(\frac{1}{4}) + \frac{1}{8} = \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t}))$$
$$\leq \frac{1}{2} = \mathcal{F}(\mathcal{M}_b(\Upsilon, \mho, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon, \mho, \tilde{t})$$

Case(v): When $(\Upsilon, 0) = (2, 0)$, $(\mho, 0) = (4, 0)$ then

$$ln\sqrt{3} + \frac{1}{2} = \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t}))$$
$$\leq ln(\frac{45}{32}) + \frac{45}{64} + \frac{3}{4} = \mathcal{F}(\mathcal{M}_b(\Upsilon, \mho, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon, \mho, \tilde{t})$$

Case(vi): When $(\Upsilon, 0) = (1, 0)$, $(\mho, 0) = (3, 0)$ then

$$ln\sqrt{3} + ln(\frac{1}{8}) + \frac{1}{4} = \tau + \mathcal{F}(\delta(\mathcal{H}\Upsilon, \mathcal{H}\mho, \tilde{t}))$$
$$\leq ln(4) + \frac{4}{2} + \frac{3}{4} = \mathcal{F}(\mathcal{M}_b(\Upsilon, \mho, \tilde{t})) + \pounds\mathcal{N}_b(\Upsilon, \mho, \tilde{t}).$$

Thus, all the conditions of Theorem 3.2 are fulfilled, where $\mathcal{H}$ and $\Im$ have two coincidence points, $(0, 0)$ and $(0, 2)$. However, the uniqueness of the coincidence point fails since $(0, 0)$ and $(0, 2)$ are not comparable under $\tilde{\mathcal{R}}$.

**Example 3.6**. Let $\mathbb{A} = \{1, 2, 3, 4\}$ and define $\delta : \mathbb{A} \times \mathbb{A} \times \mathbb{A} \to \mathbb{R}$ by

$\delta(2, 4, 1) = \delta(2, 4, 2) = \delta(3, 1, 1) = \delta(3, 2, 3) = 2$
$\delta(4, 4, 1) = \delta(2, 1, 1) = \delta(2, 1, 2) = \delta(2, 1, 3) = \delta(3, 1, 3) = 1$
$\delta(1, 4, 3) = \delta(2, 4, 3) = \delta(2, 4, 4) = 3$
$\delta(4, 4, 3) = \delta(3, 2, 2) = 4$

with symmetry in all the variables and $\delta(\Upsilon, \mho, a) = 0$, otherwise. Then $(\mathbb{A}, \delta)$ is a $\mathbb{B}_2$-metric space with $\mathfrak{s} = \frac{4}{3}$. We define relation $\tilde{\mathcal{R}}$ on $\mathbb{A}$ by

$$\tilde{\mathcal{R}} = \{(1, 1), (3, 3), (1, 4), (1, 2), (4, 4), (3, 2), (2, 2)\}.$$

We define $\Im, \mathcal{H} : \mathbb{A} \to \mathbb{A}$ by

$\mathcal{H}1 = \mathcal{H}2 = 1, \mathcal{H}3 = 2, \mathcal{H}4 = 4.$
$\Im 1 = 2, \Im 2 = 1, \Im 3 = 3, \Im 4 = 4.$

We now verify the postulates of the Theorem 3.2. There exists an $\Upsilon_0 = 1$ such that $(\Im\Upsilon_0, \mathcal{H}\Upsilon_0) \in \tilde{\mathcal{R}}$. Clearly, $\mathcal{H}\mathbb{A} \subseteq \Im\mathbb{A}$, $\Im\mathbb{A}$ is a complete subspace of $\mathbb{A}$. $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}|\Im\mathbb{A}$ is

transitive and $\tilde{\mathcal{R}}|\Im\mathbb{A}$ is $\delta-$ self closed. It is easy to verify that $\tilde{\mathcal{R}}$ is $(\mathcal{H},\Im)$ closed.
We define $\mathcal{F}:\mathbb{R}^+\to\mathbb{R}^+$ by $\mathcal{F}(\nu)=log\nu+\frac{\nu}{2}$ for all $\nu\in\mathbb{R}^+$. Clearly, $\mathcal{F}\in\mathfrak{F}$. We verify contraction condition (3) when $(\Im\Upsilon,\Im\mho)\in\tilde{\mathcal{R}}$, $(\mathcal{H}\Upsilon,\mathcal{H}\mho)\in\tilde{\mathcal{R}}$. The nontrivial case is when $q=(3,2), L=2$, $\tau=ln\sqrt{3}$ and $\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,a)>0$.

$$ln\sqrt{3}+\delta(2,1,a)=\tau+\mathcal{F}(\delta(\mathcal{H}3,\mathcal{H}2,a))\leq\mathcal{F}(\mathcal{M}_b(3,2,a))+\pounds\mathcal{N}_b(3,2,a)$$

Case(i): When $a=1$ and $1=\delta(\mathcal{H}3,\mathcal{H}2,1)>0$ then

$$\begin{aligned}ln\sqrt{3}+\frac{1}{2}&=\tau+\mathcal{F}(\delta(\mathcal{H}3,\mathcal{H}2,a))\\&\leq ln2+1+\pounds(0)=\mathcal{F}(\mathcal{M}_b(3,2,a))+\pounds\mathcal{N}_b(3,2,a)\end{aligned}$$

Case(ii): When $a=2$ and $1=\delta(\mathcal{H}3,\mathcal{H}2,1)>0$ then

$$\begin{aligned}ln\sqrt{3}+\frac{1}{2}&=\tau+\mathcal{F}(\delta(\mathcal{H}3,\mathcal{H}2,a))\\&\leq\frac{1}{2}+\pounds(\frac{1}{2})=\mathcal{F}(\mathcal{M}_b(3,2,a))+\pounds\mathcal{N}_b(3,2,a)\end{aligned}$$

Case(iii): When $a=3$ and $1=\delta(\mathcal{H}3,\mathcal{H}2,1)>0$ then

$$\begin{aligned}ln\sqrt{3}+\frac{1}{2}&=\tau+\mathcal{F}(\delta(\mathcal{H}3,\mathcal{H}2,a))\\&\leq ln2+1+\pounds(\frac{1}{2})=\mathcal{F}(\mathcal{M}_b(3,2,a))+\pounds\mathcal{N}_b(3,2,a)\end{aligned}$$

Case(iv): When $a=4$ and $2=\delta(\mathcal{H}3,\mathcal{H}2,1)>0$ then

$$\begin{aligned}ln\sqrt{3}+ln2+1&=\tau+\mathcal{F}(\delta(\mathcal{H}3,\mathcal{H}2,a))\\&\leq ln3+\frac{3}{2}+\pounds(\frac{3}{2})=\mathcal{F}(\mathcal{M}_b(3,2,a))+\pounds\mathcal{N}_b(3,2,a)\end{aligned}$$

Thus, all the conditions of Theorem 3.2 are fulfilled, where $\mathcal{H}$ and $\Im$ have two coincidence points, 2 and 4. Hence, the uniqueness of the coincidence point holds.

Additionally, for the values of $\Upsilon=3,\mho=3$, and $a=2$, there is no $\lambda\in(0,1)$ with $\lambda<1$ for which the contraction condition (1) holds. Since

$$1=\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,3)\nleq\lambda(1)=\lambda\delta(\Im\Upsilon,\Im\mho,3).$$

Hence, our results generalizes the results stated in [16].

## 4. Some Consequences

If $\Im=I$, we have the following Corollary.

**Corollary 4.1**. Let $\tilde{\mathcal{R}}$ transitive binary relation on a $\mathbb{B}_2$-metric space $(\mathbb{A},\delta)$ and $\mathcal{H}:\mathbb{A}\to\mathbb{A}$ such that:

(i) for all $\Upsilon,\mho,\tilde{t}\in\mathbb{A}$, if there exists $\mathcal{F}\in\mathfrak{F}$, $\pounds\geq 0$ and $\tau>0$ such that

$$\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,\tilde{t})>0 \textit{ implies } \tau+\mathcal{F}(\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,\tilde{t}))\leq\mathcal{F}(\mathcal{M}_b(\Upsilon,\mho,\tilde{t}))+\pounds\mathcal{N}_b(\Upsilon,\mho,\tilde{t}) \quad (27)$$

where

$$\begin{aligned}\mathcal{M}_b(\Upsilon,\mho,\tilde{t})=max\{&\delta(\Upsilon,\mho,\tilde{t}),\frac{\delta(\Upsilon,\mathcal{H}\mho,\tilde{t})}{2\mathfrak{s}},\delta(\mho,\mathcal{H}\mho,\tilde{t}),\\&\frac{\delta(\mho,\mathcal{H}\Upsilon,\tilde{t})[1+\delta(\Upsilon,\mathcal{H}\Upsilon,\tilde{t})]}{1+\delta(\Upsilon,\mho,\tilde{t})},\frac{\delta(\Upsilon,\mathcal{H}\Upsilon,\tilde{t})[1+\delta(\mho,\mathcal{H}\Upsilon,\tilde{t})]}{1+\delta(\Upsilon,\mho,\tilde{t})}\}\end{aligned}$$

and

$$\mathcal{N}_b(\Upsilon,\mho,\tilde{t}) = min\{\delta(\Upsilon,\mho,\tilde{t}),\delta(\Upsilon,\mathcal{H}\Upsilon,\tilde{t}),\delta(\mho,\mathcal{H}\mho,\tilde{t}),\frac{\delta(\Upsilon,\mathcal{H}\mho,\tilde{t})\delta(\mho,\mathcal{H}\Upsilon,\tilde{t})}{1+\delta(\Upsilon,\mathcal{H}\mho,\tilde{t})}\}$$

with $(\Upsilon,\mho)\in\tilde{\mathcal{R}}$ and $(\mathcal{H}\Upsilon,\mathcal{H}\mho)\in\tilde{\mathcal{R}}^*$.

(ii) there exists $\Upsilon_0\in\mathbb{A}$ such that $(\Upsilon_0,\mathcal{H}\Upsilon_0)\in\tilde{\mathcal{R}}$.
(iii) $\tilde{\mathcal{R}}$ is $\mathcal{H}$-closed.
(iv) $\tilde{\mathcal{R}}$ is $\delta-$ self closed provided (27) holds for all $\Upsilon,\mho,\tilde{t}\in\mathbb{A}$ with $(\mathcal{H}\Upsilon,\mathcal{H}\mho)\in\tilde{\mathcal{R}}^*$.

Then $\mathcal{H}$ has a fixed point. Moreover, if for all coincidence points of $\phi,\xi$ there exists $\zeta$ in $\mathbb{A}$ such that $(\phi,\zeta)\in\tilde{\mathcal{R}}$ and $(\xi,\zeta)\in\tilde{\mathcal{R}}$, then $\mathcal{H}$ has a unique fixed point in $\mathbb{A}$.
**Definition 4.2.**[22] Let $(\mathbb{A},\preceq)$ be a partially ordered set and $\mathcal{H},\Im$ be two self maps on $\mathbb{A}$. If for any $\Upsilon,\ \mho\in\mathbb{A}$, if $\Im\Upsilon\preceq\Im\mho$ implies $\mathcal{H}\Upsilon\preceq\mathcal{H}\mho$, then $\mathcal{H}$ is $\Im$ non-decreasing.
$\tilde{\mathcal{R}}=\preceq$, we have the following Corollaries.
**Corollary 4.3**. Consider two self maps $\mathcal{H}$ and $\Im$ on an ordered complete $\mathbb{B}_2$-metric space $(\mathbb{A},\delta,\preceq)$. Assume that:

(i) there exists $\Upsilon_o\in\mathbb{A}$ such that $\Im\Upsilon_0\preceq\mathcal{H}\Upsilon_0$.
(ii) $\mathcal{H}$ is $\Im$ nondecreasing.
(iii) $\mathcal{H}\mathbb{A}\subseteq\Im(\mathbb{A})$, $\Im(\mathbb{A})$ is a $\mathbb{B}_2$- complete subspace of $\mathbb{A}$.
(iv) for all $\Upsilon,\mho,a\in\mathbb{A}$, if there exists $\mathcal{F}\in\mathfrak{F}$, $\pounds\geq 0$ and $\tau>0$ such that

$$\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,a)>0\ implies\ \tau+\mathcal{F}(\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,a))\leq\mathcal{F}(\mathcal{M}_b(\Upsilon,\mho,\tilde{t}))+\pounds\mathcal{N}_b(\Upsilon,\mho,\tilde{t})$$

where

$$\mathcal{M}_b(\Upsilon,\mho,\tilde{t}) = max\{\delta(\Im\Upsilon,\Im\mho,\tilde{t}),\frac{\delta(\Im\Upsilon,\mathcal{H}\mho,\tilde{t})}{2\mathfrak{s}},\delta(\Im\mho,\mathcal{H}\mho,\tilde{t}),\\ \frac{\delta(\Im\mho,\mathcal{H}\Upsilon,\tilde{t})[1+\delta(\Im\Upsilon,\mathcal{H}\Upsilon,\tilde{t})]}{1+\delta(\Im\Upsilon,\Im\mho,\tilde{t})},\frac{\delta(\Im\Upsilon,\mathcal{H}\Upsilon,\tilde{t})[1+(\Im\mho,\mathcal{H}\Upsilon,\tilde{t})]}{1+d(\Im\Upsilon,\Im\mho,\tilde{t})}\}$$

and

$$\mathcal{N}_b(\Upsilon,\mho,\tilde{t}) = min\{\delta(\Im\Upsilon,\Im\mho,\tilde{t}),\delta(\Im\Upsilon,\mathcal{H}\Upsilon,\tilde{t}),\delta(\Im\mho,\mathcal{H}\mho,\tilde{t}),\\ \frac{\delta(\Im\Upsilon,\mathcal{H}\mho,\tilde{t})\delta(\Im\mho,\mathcal{H}\Upsilon,\tilde{t})}{1+\delta(\Im\Upsilon,\mathcal{H}\mho,\tilde{t})}\}$$

with $\Im\Upsilon\preceq\Im\mho$.

(v) If $\{\Im\Upsilon_n\}$ is nondecreasing sequence in $\mathbb{A}$ with $\Im\Upsilon_n\to\Im\wp$ as $n\to+\infty$, then $\Im\Upsilon_n\preceq\Im\wp$ for all $n\in\mathbb{N}$. Then $\Im$ and $\mathcal{H}$ have a coincidence point.
(vi) If $\Im$ and $\mathcal{H}$ are weakly compatible mappings, and for all coincidence points $\phi,\xi$, there exists $w\in\mathbb{A}$ such that either $\Im\phi\preceq\Im w$ or $\Im\xi\preceq\Im w$, then $\Im$ and $\mathcal{H}$ have a unique common fixed point in $\mathbb{A}$.

**Corollary 4.4**. Consider a self map $\mathcal{H}$ on a complete partially ordered $\mathbb{B}_2$-metric space $(\mathbb{A},\delta,\preceq)$. Assume that:

(i) there exists $\Upsilon_o\in\mathbb{A}$ such that $\Upsilon_0\preceq\mathcal{H}\Upsilon_0$.
(ii) $\mathcal{H}$ is nondecreasing.
(iii) for all $\Upsilon,\mho,a\in\mathbb{A}$, if there exists $\mathcal{F}\in\mathfrak{F}$, $\pounds\geq 0$ and $\tau>0$ such that

$$\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,a)>0\ implies\ \tau+\mathcal{F}(\delta(\mathcal{H}\Upsilon,\mathcal{H}\mho,a))\leq\mathcal{F}(\mathcal{M}_b(\Upsilon,\mho,\tilde{t}))+\pounds\mathcal{N}_b(\Upsilon,\mho,\tilde{t})$$

where

$$\mathcal{M}_b(\Upsilon,\mho,\tilde{t}) = max\{\delta(\Upsilon,\mho,\tilde{t}), \frac{\delta(\Upsilon,\mathcal{H}\mho,\tilde{t})}{2\mathfrak{s}}, \delta(\mho,\mathcal{H}\mho,\tilde{t}),$$
$$\frac{\delta(\mho,\mathcal{H}\Upsilon,\tilde{t})[1+\delta(\Upsilon,\mathcal{H}\Upsilon,\tilde{t})]}{1+\delta(\Upsilon,\mho,\tilde{t})}, \frac{\delta(\Upsilon,\mathcal{H}\Upsilon,\tilde{t})[1+(\mho,\mathcal{H}\Upsilon,\tilde{t})]}{1+d(\Upsilon,\mho,\tilde{t})}\}$$

and

$$\mathcal{N}_b(\Upsilon,\mho,\tilde{t}) = min\{\delta(\Upsilon,\mho,\tilde{t}), (\Upsilon,\mathcal{H}\Upsilon,\tilde{t}), d(\mho,\mathcal{H}\mho,\tilde{t}),$$
$$\frac{\delta(\Upsilon,\mathcal{H}\mho,\tilde{t})\delta(\mho,\mathcal{H}\Upsilon,\tilde{t})}{1+\delta(\Upsilon,\mathcal{H}\mho,\tilde{t})}\}$$

with $\Upsilon \preceq \mho$.

(iv) If the nondecreasing sequence $\{\Upsilon_n\}$ in $\mathbb{A}$ with $\Upsilon_n \to \wp$ as $n \to +\infty$, then $\Upsilon_n \preceq \wp$, for all $n \in \mathbb{N}$. Then $\mathcal{H}$ has a fixed point.

(v) For all $\phi, \xi \in Fix(\mathcal{H})$, if there exists $\zeta$ in $\mathbb{A}$ such that $\phi \preceq \zeta$ and $\xi \preceq \zeta$ , then $\mathcal{H}$ has an unique fixed point in $\mathbb{A}$.

*Proof.* The proof of this corollary follows by setting $\Im = I$ in Corollary 4.3. □

## 5. Application

In this section, we will provide an application of the corollary 4.1 for proving the existence of a solution of the following nonlinear fractional differential equation.

$$\begin{cases} {}^cD^{\beta}(\vartheta(\mathfrak{s})) + f(\mathfrak{s},\vartheta(\mathfrak{s})) = 0; (0 \le \mathfrak{s} \le 1; \beta < 1) \\ \vartheta(0) = 0 = \vartheta(1), \end{cases} \tag{28}$$

where $f \in C([0,1]\times[0,\infty),[0,\infty))$. The operator $\mathcal{H}$ is defined by

$$\mathcal{H}u(\mathfrak{s}) = \int_0^1 \mathfrak{G}(\mathfrak{s},\mathfrak{t}) f(\mathfrak{t},\vartheta(\mathfrak{t}))dt$$

The Green function related to (27) is

$$\mathfrak{G}(\mathfrak{s},\mathfrak{t}) = \begin{cases} \frac{(\mathfrak{s}(1-\mathfrak{t}))^{\alpha-1}-(\mathfrak{s}-\mathfrak{t})^{\alpha-1}}{\Gamma(\alpha)} & if 0 \le \mathfrak{t} \le \mathfrak{s} \le 1 \\ \frac{(\mathfrak{s}(1-\mathfrak{t}))^{\alpha-1}}{\Gamma(\alpha)} & if\ 0 \le \mathfrak{s} \le \mathfrak{t} \le 1. \end{cases}$$

where $\Gamma$ is a gamma function. Let $\mathbb{A} = C[[0,1],\mathbb{R}]$ be set of all continuous functions defined on $[0,1]$ and we define $\rho : \mathbb{A}\times\mathbb{A} \to \mathbb{R}$ by

$$\rho(\Upsilon,\mho) = max_{\mathfrak{s}\in[0,1]}|\Upsilon(\mathfrak{s}) - \mho(\mathfrak{s})|,$$

for all $\Upsilon,\mho \in \mathbb{A}$ equip $\mathbb{A}$ with the 2-metric given by $\eta : \mathbb{A}^3 \to \mathbb{R}^+$ which is defined by

$$\eta(\Upsilon,\mho,\tilde{t}) = max_{0\le s\le 1}\{min\{|\Upsilon(s)-\mho(s)|, |\mho(s)-\tilde{t}(s)|, |\tilde{t}(s)-\Upsilon(s)|\}\},$$

for all $\Upsilon,\mho,\tilde{t} \in \mathbb{A}$. As $(\mathbb{A},\rho)$ is complete metric space, $(\mathbb{A},\eta)$ is a complete 2-metric space. We define a $\mathbb{B}_2$-metric on $\mathbb{A}$ by $\delta(\Upsilon,\mho,\tilde{t}) = \eta^3(\Upsilon,\mho,\tilde{t})$, for all $\Upsilon,\mho,\tilde{t} \in \mathbb{A}$.
Therefore, $(\mathbb{A},\delta)$ is a complete $\mathbb{B}_2-$metric space with $\mathfrak{s} = 9$.
We define a relation $\tilde{\mathcal{R}}$ on $\mathbb{A}$ by

$$\tilde{\mathcal{R}} = \{(\Upsilon,\mho) \in \mathbb{A}^2 : \Upsilon(\mathfrak{s}) \le \mho(\mathfrak{s})\ for\ all\ \mathfrak{s} \in [0,+\infty)\}$$

We prove the following main result of this section.

**Theorem 5.1.** *Consider the differential equation* (28)*. Suppose that:*

$$|f(\mathfrak{s}, \Upsilon(\mathfrak{s}) - f(\mathfrak{s}, \mho(\mathfrak{s}))| \leq \frac{min\{|\Upsilon(\mathfrak{s}) - \mho(\mathfrak{s})|, |\Upsilon(\mathfrak{s}) - \tilde{t}(\mathfrak{s})|, |\mho(\mathfrak{s}) - \tilde{t}(\mathfrak{s})|\}}{(1 + \tau\mathcal{M}_b(\Upsilon, \mho, \tilde{t}))^{\frac{1}{3}}}$$

*for all* $\Upsilon, \mho, \tilde{t} \in \mathbb{A}$*,* $\mathfrak{s} \in [0, 1]$*, where*

$$\mathcal{M}_b(\Upsilon, \mho, \tilde{t}) = max\{\delta(\Upsilon, \mho, \tilde{t}), \frac{\delta(\Upsilon, \mathcal{H}\mho, \tilde{t})}{2\mathfrak{s}}, \delta(\mho, \mathcal{H}\mho, \tilde{t}),$$
$$\frac{\delta(\mho, \mathcal{H}\Upsilon, \tilde{t})[1 + \delta(\Upsilon, \mathcal{H}\Upsilon, \tilde{t})]}{1 + \delta(\Upsilon, \mho, \tilde{t})}, \frac{\delta(\Upsilon, \mathcal{H}\Upsilon, \tilde{t})[1 + (\mho, \mathcal{H}\Upsilon, \tilde{t})]}{1 + d(\Upsilon, \mho, \tilde{t})}\}$$

*Under the above postulates the equation* (28) *has a unique solution.*

*Proof.* The equation (28) can be written as

$$\vartheta(\mathfrak{s}) = \int_0^1 \mathfrak{S}(\mathfrak{s}, \mathfrak{t}) f(\mathfrak{t}, \vartheta(\mathfrak{t}))dt, \tag{29}$$

for all $\mathfrak{s} \in [0, 1]$. Now $\vartheta \in \mathbb{A}$ is a solution of (29) if and only if it is a solution of an nonlinear fractional differential equation (28). We define a map $\mathcal{H} : \mathbb{A} \to \mathbb{A}$ by

$$\mathcal{H}\vartheta(\mathfrak{t}) = \int_0^1 \mathfrak{S}(\mathfrak{t}, \mathfrak{s})(f(\mathfrak{s}, \vartheta(\mathfrak{s}))d\mathfrak{s}.$$

We choose an $\tilde{\mathcal{R}}$ preserving sequence $\{\vartheta_n\}$ such that $\vartheta_n(t) \to \varpi(\mathfrak{t})$. Then for all $\mathfrak{t} \in [0, 1]$, we get

$$\vartheta_0(\mathfrak{t}) \leq \vartheta_1(\mathfrak{t}) \leq \vartheta_2(\mathfrak{t}) \leq ....... \leq \vartheta_n(\mathfrak{t}) \leq ...........$$

and converges to $\varpi(\mathfrak{t})$ which implies $\vartheta_n(\mathfrak{t}) \leq \varpi(\mathfrak{t})$ for all $\mathfrak{t} \in [0, 1]$. We can choose subsequence $\{\vartheta_{n(\hbar)}(\mathfrak{t})\}$ of $\vartheta_n(\mathfrak{t})$ such that $[\vartheta_{n(\hbar)}(\mathfrak{t}), \varpi(\mathfrak{t})] \in \tilde{\mathcal{R}}$ for all $n \in \mathbb{N}$. Hence $\tilde{\mathcal{R}}$ is $\delta$-self closed.

For $(\vartheta, \varpi) \in \tilde{\mathcal{R}}$, we have $\vartheta(\mathfrak{t}) \leq \varpi(\mathfrak{t})$ and $\mathfrak{S}(\mathfrak{t}, \mathfrak{s}) > 0$ for all $\mathfrak{t}, \mathfrak{s} \in [0, 1]$.

$$\begin{aligned} \mathcal{H}\vartheta(\mathfrak{t}) &= sup_{t\in[0,1]} \int_0^1 \mathfrak{S}(\mathfrak{t}, \mathfrak{s})(f(\mathfrak{s}, \vartheta(\mathfrak{s})d\mathfrak{s} \\ &\leq sup_{t\in[0,1]} \int_0^1 \mathfrak{S}(\mathfrak{t}, \mathfrak{s})(f(\mathfrak{s}, \varpi(\mathfrak{s}))d\mathfrak{s} \\ &= \mathcal{H}\varpi(\mathfrak{t}) \ for\ all\ \mathfrak{t} \in [0, 1] \end{aligned}$$

which implies $(\mathcal{H}\vartheta(\mathfrak{t}), \mathcal{H}\varpi(\mathfrak{t})) \in \tilde{\mathcal{R}}$ *i.e.*, $\tilde{\mathcal{R}}$ is $S$- closed.
Now for $(\vartheta, \varpi) \in \tilde{\mathcal{R}}$, we have
$|\mathcal{H}\vartheta(\mathfrak{t}) - \mathcal{H}\varpi(\mathfrak{t})|$

$$\begin{aligned} &\leq |\int_0^1 \mathfrak{S}(\mathfrak{t}, \mathfrak{s})[f(\mathfrak{s}, \vartheta(\mathfrak{s})) \ - f(\mathfrak{s}, \varpi(\mathfrak{s}))]d\mathfrak{s}| \\ &\leq \int_0^1 \mathfrak{S}(\mathfrak{t}, \mathfrak{s})d\mathfrak{s} \frac{min\{|\vartheta(\mathfrak{s}) - \varpi(\mathfrak{s})|, |\vartheta(\mathfrak{s}) - \tilde{t}(\mathfrak{s})|, |\varpi(\mathfrak{s}) - \tilde{t}(\mathfrak{s})|\}}{(\tau\mathcal{M}_b(\vartheta, \varpi, \tilde{t}) + 1)^{\frac{1}{3}}} \\ &\leq \frac{min\{|\vartheta(\mathfrak{s}) - \varpi(\mathfrak{s})|, |\vartheta(\mathfrak{s}) - \tilde{t}(\mathfrak{s})|, |\varpi(\mathfrak{s}) - \tilde{t}(\mathfrak{s})|\}}{(\tau\mathcal{M}_b(\vartheta, \varpi, \tilde{t}) + 1)^{\frac{1}{3}}} \int_0^1 \mathfrak{S}(\mathfrak{t}, \mathfrak{s})d\mathfrak{s} \end{aligned} \tag{30}$$

Since $\int_0^1 \mathfrak{S}(\mathfrak{t},\mathfrak{s}) < 1$ and taking supremum in both sides we get
$sup_{\mathfrak{t}\in[0,1]}|\mathcal{H}\vartheta(\mathfrak{t})-\mathcal{H}\varpi(\mathfrak{t})| \leq sup_{\mathfrak{t}\in[0,1]} \frac{min\{|\vartheta(\mathfrak{s})-\varpi(\mathfrak{s})|,|\vartheta(\mathfrak{s})-\tilde{t}(\mathfrak{s})|,|\varpi(\mathfrak{s})-\tilde{t}(\mathfrak{s})|\}}{\mathfrak{K}(\mathfrak{t})(\tau\mathcal{M}_b(\vartheta,\varpi,\tilde{t})+1)^{\frac{1}{3}}}.sup_{\mathfrak{t}\in[0,1]} \int_0^1 \mathfrak{S}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}$
which implies

$$\eta(\mathcal{H}\vartheta,\mathcal{H}\varpi,\tilde{t}) \leq max_{\mathfrak{t}\in[0,1]}|\mathcal{H}\vartheta(\mathfrak{t})-\mathcal{H}\varpi(\mathfrak{t})| \leq \frac{\eta(\vartheta,\varpi,\tilde{t})}{(\tau\mathcal{M}_b(\vartheta,\varpi,\tilde{t})+1)^{\frac{1}{3}}}.$$

From here we have,

$$\delta(\mathcal{H}\vartheta,\mathcal{H}\varpi,\tilde{t}) \leq \frac{\delta(\vartheta,\varpi,\tilde{t})}{\tau\mathcal{M}_b(\vartheta,\varpi,\tilde{t})+1} \leq \frac{\mathcal{M}_b(\vartheta,\varpi,\tilde{t})}{\tau\mathcal{M}_b(\vartheta,\varpi,\tilde{t})+1}.$$

which yields

$$\tau - \frac{1}{\delta(\mathcal{H}\vartheta,\mathcal{H}\varpi,\tilde{t})} \leq -\frac{1}{\mathcal{M}_b(\mathcal{H}\vartheta,\mathcal{H}\varpi,\tilde{t})} \leq -\frac{1}{\mathcal{M}_b(\mathcal{H}\vartheta,\mathcal{H}\varpi,\tilde{t})} + L\mathcal{N}_b(\Upsilon,\mho,\tilde{t})$$

On choosing $\mathcal{F} = \frac{-1}{\mu}$ in the above, we get

$$\tau + \mathcal{F}(\delta(\mathcal{H}\vartheta,\mathcal{H}\varpi,\tilde{t})) \leq \mathcal{F}(\mathcal{M}_b(\Upsilon,\mho,\tilde{t})) + L\mathcal{N}_b(\Upsilon,\mho,\tilde{t}).$$

This shows that $\mathcal{H}$ satisfies condition (27) of Corollary 4.1. Consequently, all the hypotheses of Corollary 4.1 are verified and we conclude that $\mathcal{H}$ has a unique fixed point, which is a solution of periodic differential equation (28). □

## 6. Conclusions

In this paper, we introduce the notion of almost $\mathcal{F}\tilde{\mathcal{R}}\Im$-contraction type mappings and establish results concerning the existence of common coincident and fixed points for such mappings within the structure of a $\mathbb{B}2$-metric space equipped with a binary relation. We provide examples to illustrate our findings and discuss potential applications in solving nonlinear fractional differential equations.

Our methodology is distinguished by its reliance on more flexible conditions. Specifically, we assume $(\Im,\tilde{\mathcal{R}})$-continuity of $\mathcal{H}$ rather than standard continuity, and we impose $\tilde{\mathcal{R}}$-completeness only on specific subspaces instead of requiring completeness across the entire metric space. This approach allows the contraction condition to be applied selectively to related elements rather than universally. Moreover, in cases where the binary relation is universal, these contraction conditions reduce to classical forms.

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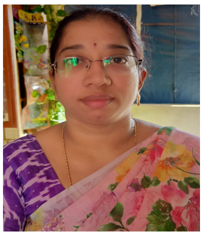

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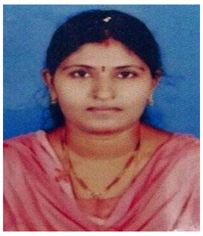

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