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A STUDY OF FUZZY $\mathcal{P}$-ESSENTIAL SUBMODULES

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Abstract. In this paper, we introduce the concept of fuzzy $\mathcal{P}$-essential submodule of an R-module M. This concept is a generalization of the concept of fuzzy essential submodule. Also, we investigate various properties of $\mathcal{P}$-essential submodules concerning fuzzy multiplication modules over a commutative ring.

Keywords: Fuzzy essential submodules, fuzzy $\mathcal{P}$-essential submodule, fuzzy fully $\mathcal{P}$-essential submodule, fuzzy $\mathcal{P}$-uniform module.

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1. Introduction

In 1965, Zadeh [15] introduced the concept of a fuzzy subset as a generalization of the characteristic function in classical set theory. Rosenfled [12] in 1971 applied this concept to the theory of groupoids and group. Negoita and Ralescu [7] were the first ones to introduce a fuzzy submodule. Kalita [4] defined a fuzzy essential submodule and proved some characteristics of such submodules. Nimbhorkar and khubchandani [8] applied this concept of essential submodules to fuzzy essential submodules with respect to arbitrary submodules. Also, Nimbhorkar and khubchandani [9] studied fuzzy semi-essential submodules and fuzzy small-essential submodules. Finally, Nimbhorkar and khubchandani in [10] and [11] studied fuzzy semi-essential submodules, fuzzy closed submodules and L-fuzzy hollow modules, L- fuzzy multiplication modules respectively.

The purpose of this paper is to define fuzzy $\mathcal{P}$-essential submodule and study some of its properties.

2. Preliminaries

Throughout in this paper R denotes a commutative ring with identity, M a unitary R-module with zero element θ. We use the notations "$\subseteq$ "and " $\leq$ " to denote inclusion and submodule respectively. We recall some definitions from Moderson and Malik [6].

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Definition 2.1. [6] *A fuzzy subset of an R-module M, is a mapping* $\mathcal{U} : M \to [0,1]$.

The support of a fuzzy set $\mathcal{U}$, denoted by $\mathcal{U}^*$, is the set $\mathcal{U}^* = \{x \in M \mid \mu(x) > 0\}$. We denote by $\mathcal{U}_*$ the set $\mathcal{U}_* = \{x \in M \mid \mathcal{U}(x) = 1\}$.

Definition 2.2. [6] *If* $N \subseteq M$ *and* $\alpha \in [0,1]$, *then* α_N *is defined as,*

$$\alpha_N(x) = \begin{cases} \alpha, & if\ x \in N, \\ 0, & otherwise. \end{cases}$$

If $N = \{x\}$, then α_x is often called a fuzzy point and is denoted by χ_α.
If $\alpha = 1$, then 1_N is known as the characteristic function of N and is denoted by χ_N.

Definition 2.3. [6] *Let X and Y be two nonempty sets and* $f : X \to Y$ *be a mapping. Let* $\mathcal{U} \in [0,1]^X$ *and* $\mathcal{V} \in [0,1]^Y$. *Then the image* $f(\mathcal{U}) \in [0,1]^Y$ *and the inverse image* $f^{-1}(\mathcal{V}) \in [0,1]^X$ *are defined as follows: for all* $y \in Y$,

$$f(\mathcal{U})(y) = \begin{cases} \vee\{\mathcal{U}(x) \mid x \in X, f(x) = y\}, & if\ f^{-1}(y) \neq \phi, \\ 0, & otherwise. \end{cases}$$

and $f^{-1}(\mathcal{V})(x) = \mathcal{V}(f(x))$ *for all* $x \in X$.

Definition 2.4. [6] *Let M be an R-module. An fuzzy subset* μ *of an R-module M is said to be a fuzzy submodule, if for every* $x, y \in M$ *and* $r \in R$ *the following conditions are satisfied:*

(1) $\mathcal{U}(\theta) = 1$,
(2) $\mathcal{U}(x - y) \geq \mathcal{U}(x) \wedge \mathcal{U}(y)$,
(3) $\mathcal{U}(rx) \geq \mathcal{U}(x)$.

The set of all fuzzy submodules of M is denoted by $F(M)$.

Lemma 2.1. [6] $\mathcal{U}_*$ *is a submodule of an R-module M if and only if* $\mathcal{U}$ *is a fuzzy submodule of M.*

Theorem 2.1. [4] *A submodule A of an R-module M is essential in M if and only if* χ_A *is an essential fuzzy submodule of M.*

Theorem 2.2. [4] *Let* $\mathcal{U}$ *be a non-zero fuzzy submodule of an R-module M. Then* $\mathcal{U} \trianglelefteq M$ *if and only if* $\mathcal{U}^* \trianglelefteq M$.

Corollary 2.1. [2] *Let* $\mathcal{V}$ *be an L-fuzzy prime submodule of M. Then* $\mathcal{V}_* = \{x \in M \mid \mathcal{V}(x) = \mathcal{V}(0_M)\}$ *is a prime submodule of M.*

Theorem 2.3. [2] *a). Let N be a prime submodule of M and* α *a prime element in L. If* $\mathcal{U}$ *is the fuzzy subset of M defined by*

$$\mathcal{U}(x) = \begin{cases} 1, & if\ x \in N, \\ \alpha, & if\ otherwise \end{cases}$$

for all $x \in M$, *then* $\mathcal{U}$ *is an L-fuzzy prime submodules of M.*
b). Conversely, any L-fuzzy prime submodule can be obtained as in (a).

Definition 2.5. [9] *Let M be an R-module and let* $\mathcal{U} \in L(M)$. *Then* $\mathcal{U}$ *is said to be an essential L-submodule of M, if for any* $\mathcal{V} \in L(M)$ *satisfying* $\mathcal{U} \cap \mathcal{V} = \chi_\theta$ *implies* $\mathcal{V} = \chi_\theta$. *If* $L = [0,1]$, *then* $\mathcal{U}$ *is called a fuzzy essential submodule of M and is denoted by* $\mathcal{U} \trianglelefteq_f M$.

Definition 2.6. [1] *A fuzzy submodule $\mathcal{U}$ of an R-module M is called a fuzzy semi-essential submodule of M if for any nonzero fuzzy prime submodule $\mathcal{V}$ of M,*
$\mathcal{U} \cap \mathcal{V} \neq \chi_\theta$ and then we write $\mathcal{U} \trianglelefteq_{semi} M$.

3. Fuzzy $\mathcal{P}$-essential submodules

In this section, we define fuzzy $\mathcal{P}$-essential submodule, study some of its properties and examples. Now onwards all the fuzzy sets involved in this paper have finite images.

Definition 3.1. *Let M be an R-module and $\mathcal{P}$ be a non-constant fuzzy prime submodule of M. A fuzzy submodule $\mathcal{U}$ of M is called $\mathcal{P}$-essential if for any non-constant fuzzy submodule $\mathcal{V}$ of $\mathcal{P}$ satisfying $\mathcal{U} \cap \mathcal{V} = \chi_\theta$ implies $\mathcal{V} = \chi_\theta$, and is denoted by $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$.*

The above definition can also be stated as:

Definition 3.2. *A non-constant fuzzy submodule $\mathcal{U}$ of M is called $\mathcal{P}$-essential if $\mathcal{U} \cap \mathcal{V} \neq \chi_\theta$ for any non-constant fuzzy submodule $\mathcal{V}$ subset of $\mathcal{P}$ ($\chi_\theta \neq \mathcal{V} \subseteq \mathcal{P}$), where $\mathcal{P}$ be a non-constant fuzzy prime submodule of M and is denoted by $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$.*

Remark 3.1. *Every fuzzy essential submodule is $\mathcal{P}$-essential.*

Proof. Let $\mathcal{U}$ be a fuzzy submodule of an R-module M and $\mathcal{V}$ be an non-constant fuzzy submodule of non-constant fuzzy prime submodule $\mathcal{P}$, then $\mathcal{U} \cap \mathcal{V} \neq \chi_\theta$ as $\mathcal{U}$ is essential submodule of M. Hence, $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$. □

Theorem 3.1. *(Kalita* [4]*, Theorem 3.2.7, p.71) If A is a submodule of a non-zero prime submodule P if and only if χ_A is fuzzy submodule of non-constant fuzzy prime submodule χ_P.*

Theorem 3.2. *Let $\mathcal{U}$ be a non-constant fuzzy submodule of M. Then $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$ if and only if $\mathcal{U}^* \trianglelefteq_{\mathcal{P}_*} M$.*

Proof. Let $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$ and A be submodule of non-zero prime submodule $\mathcal{P}_*$. Then by Theorem 3.1, χ_A is fuzzy submodule of non-constant fuzzy prime submodule $\chi_{\mathcal{P}_*}$.
Suppose that $\mathcal{U}^* \cap A = \theta$. Then $(\mathcal{U} \cap \chi_A)^* = \theta$. Hence, $\mathcal{U} \cap \chi_A = \chi_\theta$. But $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$ implies that $\chi_A = \chi_\theta$. Hence, $A = \theta$. Thus, $\mathcal{U}^* \trianglelefteq_{\mathcal{P}_*} M$.
Conversely, assume that $\mathcal{U}^* \trianglelefteq_{\mathcal{P}_*} M$.
let χ_A be a fuzzy submodule of a non-constant fuzzy prime submodule $\chi_{\mathcal{P}_*}$, then by Theorem 3.1 A is a submodule of non-zero prime submodule $\mathcal{P}_*$. Suppose, $\mathcal{U} \cap \chi_A = \chi_\theta$ implies $(\mathcal{U} \cap \chi_A)^* = \theta$ gives $\mathcal{U}^* \cap A = \theta$. But $\mathcal{U}^* \trianglelefteq_{\mathcal{P}_*} M$, we get $A = \theta$ implies $\chi_A = \chi_\theta$.
Thus, $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$. □

The converse of Remark 3.1 may not be true.

Example 3.1. *Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{24}$.*
Define fuzzy submodule $\mathcal{U} : M \to [0,1]$ as follows:

$$\mathcal{U}(x) = \begin{cases} 1, & if\ x = 0, \\ 0.7, & if\ x \in \{6, 12, 18\}, \\ 0, & if\ x \notin \{6, 12, 18\}. \end{cases}$$

Then $\mathcal{U}^* = \{0, 6, 12, 18\}$.
Also, we define fuzzy submodule $\mathcal{P} : M \to [0,1]$ as follows:

$$\mathcal{P}(x) = \begin{cases} 1, & if\ x \in <\bar{3}>, \\ 0.5, & otherwise. \end{cases}$$

Then by Theorem 2.3, $\mathcal{P}$ is fuzzy prime submodule of $\mathbb{Z}_{24}$ and by Corollary 2.1, $\mathcal{P}_* =< \bar{3} >$ is prime submodule of M.
Then $\mathcal{U}^*$ is $< \bar{3} >$-essential submodule of M, that is $\mathcal{U}^* \trianglelefteq_{<\bar{3}>} M$, as for proper submodules $< \bar{6} >, < \bar{12} >$ of $\mathbb{Z}_{24}$ are submodules of $< \bar{3} >$ and intersection of these with $\mathcal{U}^*$ is non-zero. Then by Theorem 3.2, $\mathcal{U} \trianglelefteq_{\mathcal{P}} \mathbb{Z}_{24}$. But $\mathcal{U}^* \ntrianglelefteq \mathbb{Z}_{24}$ as $\mathcal{U}^* \cap < 8 >= 0$.
Thus by Theorem 2.2, $\mathcal{U} \ntrianglelefteq \mathbb{Z}_{24}$.

Remark 3.2. *A fuzzy semi-essential submodule may not be $\mathcal{P}$-essential submodule of M.*

Example 3.2. *Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{30}$.*
Define fuzzy submodule $\mathcal{V} : M \to [0, 1]$ as follows:

$$\mathcal{V}(x) = \begin{cases} 1, & if\ x \in < \bar{3} >, \\ 0, otherwise. \end{cases}$$

By example 3.1 of [10], $\mathcal{V}$ is semi-essential submodule of M.
Again we define fuzzy submodule $\mathcal{P} : M \to [0, 1]$ as follows:

$$\mathcal{P}(x) = \begin{cases} 1, & if\ x\ \in < \bar{5} >, \\ 0.8, & otherwise. \end{cases}$$

Then by Theorem 2.3, $\mathcal{P}$ is fuzzy prime submodule of $\mathbb{Z}_{30}$ and by Corollary 2.1, $\mathcal{P}_* =< \bar{5} >$ is prime submodule of M. Now, $\mathcal{V}^* =< \bar{3} >$ is not $\mathcal{P}_*$-essential submodule of M as $\{0, 10, 20\}$ is the only proper submodule of $< \bar{5} >$ but $\mathcal{V}^* \cap \{0, 10, 20\} = 0$. Implies $\mathcal{V}^* \ntrianglelefteq_{<\bar{5}>} M$. Hence by Theorem 3.2, $\mathcal{V} \ntrianglelefteq_{\mathcal{P}} M$.

Proposition 3.1. *Let M be an R-module, $\mathcal{P}$ be a non-constant fuzzy prime submodule and $\mathcal{U}$ be any fuzzy submodule of M. If $\mathcal{P} \trianglelefteq M$, then $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$ if and only if $\mathcal{U} \trianglelefteq M$.*

Proof. Suppose that $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$.
Let $\mathcal{P}$ be a non-constant fuzzy prime submodule of M and $\mathcal{V} \leq \mathcal{P}$ such that $\mathcal{U} \cap \mathcal{V} = \chi_\theta$ implies $\mathcal{U} \cap (\mathcal{P} \cap \mathcal{V}) = \chi_\theta$. As $\mathcal{P} \cap \mathcal{V} \leq \mathcal{P}$ and $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$, then $\mathcal{P} \cap \mathcal{V} = \chi_\theta$. By hypothesis, $\mathcal{P} \trianglelefteq M$, thus $\mathcal{V} = \chi_\theta$ implies $\mathcal{U} \trianglelefteq M$.
The converse is obvious. □

Proposition 3.2. *Let M be an R-module and $\mathcal{U}_1, \mathcal{U}_2 \in F(M)$ such that $\mathcal{U}_1 \leq \mathcal{U}_2$. If $\mathcal{U}_1 \trianglelefteq_{\mathcal{P}} M$, then $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$.*

Proof. let $\mathcal{V}$ be fuzzy submodule of a fuzzy prime submodule $\mathcal{P}$ of M such that $\mathcal{U}_2 \cap \mathcal{V} = \chi_\theta$. As $\mathcal{U}_1 \leq \mathcal{U}_2$ implies $\mathcal{U}_1 \cap \mathcal{V} \leq \mathcal{U}_2 \cap \mathcal{V} = \chi_\theta$. Implies $\mathcal{U}_1 \cap \mathcal{V} = \chi_\theta$. But $\mathcal{U}_1 \trianglelefteq_{\mathcal{P}} M$, so $\mathcal{V} = \chi_\theta$. Thus, $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$. □

The following example shows that the converse of Proposition 3.2 need not be true.

Example 3.3. *Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{24}$.*
Define fuzzy submodules $\mathcal{U}, \mathcal{V} : M \to [0, 1]$ as follows:

$$\mathcal{U}(x) = \begin{cases} 1, & if\ x = 0, \\ 0.9, if\ x \in \{4, 8, 12, 16, 20\}, \\ 0, if\ x \notin \{4, 8, 12, 16, 20\}. \end{cases}$$

$$\mathcal{V}(x) = \begin{cases} 1, & if\ x = 0, \\ 0.7, if\ x \in \{8, 16\}, \\ 0, if\ x \notin \{8, 16\}. \end{cases}$$

Here we observe that $\mathcal{V} \subseteq \mathcal{U}$.
Again we define $\mathcal{P} : M \to [0, 1]$ as,

$$\mathcal{P}(x) = \begin{cases} 1, & if\ x \in <\bar{2}>, \\ 0.3, & otherwise. \end{cases}$$

Then by Theorem 2.3, $\mathcal{P}$ is fuzzy prime submodule of M.
Here $\mathcal{P}_* =<\bar{2}>$ is prime submodule of M by Corollary 2.1 .
Also, $\mathcal{U}^* = \{0, 4, 8, 12, 16, 20\}$ then $\mathcal{U}^* \trianglelefteq_{\mathcal{P}_*} M$, then by Theorem 3.2 $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$.
Thus $\mathcal{V} \trianglelefteq_{\mathcal{P}} M$, as for $\zeta : M \to [0, 1]$

$$\zeta(x) = \begin{cases} 1, & if\ x = 0, \\ 0.5 & if\ x \in \{6, 12, 18\}, \\ 0, & otherwise. \end{cases}$$

We observe that $\mathcal{V} \cap \zeta = \chi_\theta$ but $\zeta \neq \chi_\theta$.

Corollary 3.1. *Let M be an R-module and $\mathcal{U}_1, \mathcal{U}_2 \in F(M)$. If $\mathcal{U}_1 \cap \mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$, then $\mathcal{U}_1 \trianglelefteq_{\mathcal{P}} M$ and $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$.*

Proof. We know that $\mathcal{U}_1 \cap \mathcal{U}_2 \leq \mathcal{U}_1$ and $\mathcal{U}_1 \cap \mathcal{U}_2 \leq \mathcal{U}_2$ and given that $\mathcal{U}_1 \cap \mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$, then by Proposition 3.2, $\mathcal{U}_1 \trianglelefteq_{\mathcal{P}} M$ and $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$. □

The following example shows that the converse of Cor 3.1 need not be true.

Example 3.4. *Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{24}$.*
Define fuzzy submodules $\mathcal{U}_1, \mathcal{U}_2 : M \to [0, 1]$ as follows:

$$\mathcal{U}_1(x) = \begin{cases} 1, & if\ x = 0, \\ 0.9, & if\ x \in \{4, 8, 12, 16\}, \\ 0, & if\ x \notin \{4, 8, 12, 16\}. \end{cases}$$

$$\mathcal{U}_2(x) = \begin{cases} 1, & if\ x = 0, \\ 0.6, & if\ x \in \{6, 12, 18\}, \\ 0, & if\ x \notin \{6, 12, 18\}. \end{cases}$$

Then ${\mathcal{U}_1}^* = \{0, 4, 8, 12, 16\}$ and ${\mathcal{U}_2}^* = \{0, 6, 12, 18\}$
Also, we define fuzzy submodule $\mathcal{P} : M \to [0, 1]$ as follows:

$$\mathcal{P}(x) = \begin{cases} 1, & if\ x \in <\bar{2}>, \\ 0.5, & otherwise. \end{cases}$$

Then $\mathcal{P}_* =<\bar{2}>$ is prime submodule by Corollary 2.1. Also, $\mathcal{P}$ is fuzzy prime submodule by Theorem 2.3. Then ${\mathcal{U}_1}^* \trianglelefteq_{\mathcal{P}_*} M$ and ${\mathcal{U}_2}^* \trianglelefteq_{\mathcal{P}_*} M$. But ${\mathcal{U}_1}^* \cap {\mathcal{U}_2}^* \ntrianglelefteq_{\mathcal{P}_*} M$ and here we observe that ${\mathcal{U}_1}^* \cap {\mathcal{U}_2}^* = (\mathcal{U}_1 \cap \mathcal{U}_2)^*$. Thus, $(\mathcal{U}_1 \cap \mathcal{U}_2)^* \ntrianglelefteq_{\mathcal{P}_*} M$. Then by Theorem 3.2, $\mathcal{U}_1 \trianglelefteq_{\mathcal{P}} M$, $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$ and $\mathcal{U}_1 \cap \mathcal{U}_2 \ntrianglelefteq_{\mathcal{P}} M$.

Proposition 3.3. *Let M be an R-module and $\mathcal{U}_1, \mathcal{U}_2 \in F(M)$. If $\mathcal{U}_1 \trianglelefteq M$ and $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$, then $\mathcal{U}_1 \cap \mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$.*

Proof. Let $\mathcal{P}$ be a fuzzy prime submodule of M and $\chi_\theta \neq \mathcal{V} \subseteq \mathcal{P}$.
Since, $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$, then $\mathcal{U}_2 \cap \mathcal{V} \neq \chi_\theta$.
Again as $\mathcal{U}_1 \trianglelefteq M$, then $\mathcal{U}_1 \cap (\mathcal{U}_2 \cap \mathcal{V}) \neq \chi_\theta$, so $(\mathcal{U}_1 \cap \mathcal{U}_2) \cap \mathcal{V} \neq \chi_\theta$.
This implies $\mathcal{U}_1 \cap \mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$. □

Proposition 3.4. *Let f be an R-module epimorphism from M to M_1. If $\mathcal{U} \trianglelefteq_{\mathcal{P}} M_1$, then $f^{-1}(\mathcal{U}) \trianglelefteq_{\mathcal{P}} M$.*

Proof. From Theorem 3.6 of [2], if $\mathcal{U}$ is an fuzzy prime submodule of M_1, then $f^{-1}(\mathcal{U})$ is an fuzzy prime submodule of M. let $\chi_\theta \neq \mathcal{V} \subseteq \mathcal{P}$ and $f^{-1}(\mathcal{U}) \cap \mathcal{V} = \chi_\theta$.

To prove, $\mathcal{V} = \chi_\theta$.
As $\mathcal{U} \cap f(\mathcal{V}) = \chi_\theta$ and as $\mathcal{U} \trianglelefteq_{\mathcal{P}} M_1$ and $f(\mathcal{V}) \subseteq \mathcal{P}$, then $f(\mathcal{V}) = \chi_\theta$ implies $\mathcal{V} \subseteq f^{-1}(\chi_\theta) = kerf \leq f^{-1}(\mathcal{U})$. But, $f^{-1}(\mathcal{U}) \cap \mathcal{V} = \chi_\theta$, gives $\mathcal{V} = \chi_\theta$. Thus, $f^{-1}(\mathcal{U}) \trianglelefteq_{\mathcal{P}} M$. □

Proposition 3.5. *Let M be an R-module and $\mathcal{U}, \mathcal{V}, \mathcal{W} \in F(M)$ such that $\mathcal{U} \leq \mathcal{V} \leq \mathcal{W}$. If $\mathcal{U} \trianglelefteq_{\mathcal{P}} \mathcal{V}$ and $\mathcal{V} \trianglelefteq_{\mathcal{P}} \mathcal{W}$, then $\mathcal{U} \trianglelefteq_{\mathcal{P}} \mathcal{W}$.*

Proof. Let $\mathcal{P}$ be a fuzzy prime submodule of $\mathcal{W}$ and $\mathcal{A}$ be a fuzzy submodule of $\mathcal{P}$ such that $\mathcal{U} \cap \mathcal{A} = \chi_\theta$.
Also we can write,

$$\begin{aligned} \chi_\theta &= \mathcal{U} \cap \mathcal{A} \\ &= (\mathcal{U} \cap \mathcal{A}) \cap \mathcal{V} \\ &= \mathcal{U} \cap (\mathcal{A} \cap \mathcal{V}). \end{aligned}$$

If $\mathcal{V} \subseteq \mathcal{A}$, then $\chi_\theta = \mathcal{U} \cap (\mathcal{A} \cap \mathcal{V}) = \mathcal{U} \cap \mathcal{V}$, hence $\mathcal{U} \cap \mathcal{V} = \chi_\theta$. But $\mathcal{U} \leq \mathcal{V}$, so $\mathcal{U} \cap \mathcal{V} = \mathcal{U}$ implies that $\mathcal{U} = \chi_\theta$, a contradiction. Thus, $\mathcal{V} \nleq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{V} = \mathcal{P}$. But $\mathcal{U} \trianglelefteq_{\mathcal{P}} \mathcal{V}$, therefore $\mathcal{A} \cap \mathcal{U} = \chi_\theta$ and since $\mathcal{V} \leq \mathcal{W}$, then $\mathcal{A} = \chi_\theta$, that is, $\mathcal{U} \trianglelefteq_{\mathcal{P}} \mathcal{W}$. □

The following example shows that the converse of Proposition 3.5 need not be true.

Example 3.5. *Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{24}$.*
Define fuzzy submodules $\mathcal{U}, \mathcal{V} : M \to [0, 1]$ as follows:

$$\mathcal{U}(x) = \begin{cases} 1, & if\ x = 0, \\ 0.5, & if\ x \in \{6, 12, 18\}, \\ 0, & if\ x \notin \{6, 12, 18\}. \end{cases}$$

$$\mathcal{V}(x) = \begin{cases} 1, & if\ x = 0, \\ 0.7, & if\ x \in \{2, 4, 6, ..., 22\}, \\ 0, & if\ x \notin \{2, 4, 6, ..., 22\}. \end{cases}$$

Here we observe that $\mathcal{U} \subseteq \mathcal{V}$.
Also, we define fuzzy submodule $\mathcal{P} : M \to [0, 1]$ as follows:

$$\mathcal{P}(x) = \begin{cases} 1, & if\ x \in <\bar{3}>, \\ 0.5, & otherwise. \end{cases}$$

Then $\mathcal{P}_* =<\bar{3}>$ is prime submodule by Corollary 2.1. Also, $\mathcal{P}$ is fuzzy prime submodule by Theorem 2.3. Here, $\mathcal{U}^* = \{0, 6, 12, 18\} =<\bar{6}>$, then $\mathcal{U}^* \trianglelefteq_{\mathcal{P}_*} M$. Then by Theorem 3.2, $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$. Again, $\mathcal{V}^* = \{0, 2, 4, 6, ..., 22\} =<\bar{2}>$. We observe that $\mathcal{U}^* \ntrianglelefteq_{\mathcal{P}_*} M$ and $\mathcal{V}^* \trianglelefteq_{\mathcal{P}_*} M$. Again by applying Theorem 3.2, $\mathcal{U} \ntrianglelefteq_{\mathcal{P}} M$ and $\mathcal{V} \trianglelefteq_{\mathcal{P}} M$.

Proposition 3.6. *Let M be an R-module and $\mathcal{U}_1, \mathcal{U}_2 \in F(M)$ such that $\mathcal{U}_1 \trianglelefteq_{\mathcal{P}} M$, $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$ and $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \chi_\theta$, then $\mathcal{U}_1 \cap \mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$.*

Proof. Let $\mathcal{P}$ be fuzzy prime submodule of M and $\mathcal{V} \in F(M)$ such that $\mathcal{V} \leq \mathcal{P}$ and $(\mathcal{U}_1 \cap \mathcal{U}_2) \cap \mathcal{V} = \chi_\theta$. This can be written as $\mathcal{U}_2 \cap (\mathcal{U}_1 \cap \mathcal{V}) = \chi_\theta$.
If $\mathcal{U}_1 \leq \mathcal{V}$, then we get a contradiction to the assumption, so $\mathcal{U}_1 \nleq \mathcal{V}$. This implies $\mathcal{U}_1 \cap \mathcal{V}$ is a submodule of M. As $\mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$ and $\mathcal{U}_1 \cap \mathcal{V}$ is a submodule of M, then $\mathcal{U}_1 \cap \mathcal{V} = \chi_\theta$. But $\mathcal{U}_1 \trianglelefteq_{\mathcal{P}} M$, therefore $\mathcal{V} = \chi_\theta$, hence $\mathcal{U}_1 \cap \mathcal{U}_2 \trianglelefteq_{\mathcal{P}} M$. □

4. $\mathcal{P}$-essential submodules in fuzzy multiplication modules

In this section we study some properties of $\mathcal{P}$-essential submodules concerning fuzzy multiplication modules over a ring.

Theorem 4.1. *Let M be a faithful fuzzy multiplication R-module, $\mathcal{I} \leq R$ and $\mathcal{U} \leq M$. Then $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$ if and only if $\mathcal{I} \trianglelefteq_{\mathcal{P}} R$.*

Proof. Assume $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$.
let $\mathcal{V}_1$ be a fuzzy ideal and $\mathcal{P}$ be a fuzzy prime ideal of R such that $\mathcal{V}_1 \leq \mathcal{P}$ and $\mathcal{I} \cap \mathcal{V}_1 = \chi_\theta$. Since, M is faithful fuzzy multiplication module, then $(\mathcal{I} \cap \mathcal{V}_1)\chi_M = \mathcal{I}\chi_M \cap \mathcal{V}_1\chi_M = \chi_\theta$. Now by Theorem 17 of [3], $\mathcal{V}_1\chi_M$ is fuzzy prime submodule of M.
Also, $\mathcal{V}_1\chi_M \subseteq \mathcal{P}\chi_M$ and $\mathcal{I}\chi_M = \mathcal{V}_1$ is $\mathcal{P}$-essential submodule of M, implies $\mathcal{V}_1\chi_M = \chi_\theta$. Since M is faithful fuzzy multiplication module, then $\mathcal{V}_1 = \chi_\theta$. Therefore, $\mathcal{I} \trianglelefteq_{\mathcal{P}} R$.

Conversely, assume $\mathcal{I} \trianglelefteq_{\mathcal{P}} R$.
let $\mathcal{P}$ be fuzzy prime submodule of M and $\mathcal{V}_2$ fuzzy submodule of $\mathcal{P}$ such that $\mathcal{U} \cap \mathcal{V}_2 = \chi_\theta$. Now by Proposition 5 of [3] there exists an fuzzy ideal ζ of R with $\zeta(0)_R = 1$ such that $\mathcal{V}_2 = \zeta\chi_M$. Hence, $\mathcal{U} \cap \mathcal{V}_2 = \mathcal{I}\chi_M \cap \zeta\chi_M = (\mathcal{I} \cap \zeta)\chi_M = \chi_\theta$, as M is faithful so $\mathcal{I} \cap \zeta = \chi_\theta$. But $\mathcal{I} \trianglelefteq_{\mathcal{P}} R$, then $\zeta = \chi_\theta$ therefore $\mathcal{V}_2 = \zeta\chi_M = \chi_\theta$. Thus, $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$. □

Proposition 4.1. *Assume M is faithful fuzzy multiplication R-module. If $\mathcal{C} \trianglelefteq_{\mathcal{P}} \mathcal{F}$, then $\mathcal{C}\chi_M \trianglelefteq_{\mathcal{P}} \mathcal{F}\chi_M$, for every fuzzy ideal $\mathcal{C}$ and $\mathcal{F}$ of R.*

Proof. Let $\mathcal{P}$ be a fuzzy prime submodule of $\mathcal{F}\chi_M$ such that $\mathcal{P} = \mathcal{D}\chi_M$ for some fuzzy prime ideal $\mathcal{D}$ of R and $\mathcal{D} \leq \mathcal{F}$. Let $\mathcal{U}$ be a fuzzy submodule of $\mathcal{P}$ such that

$$\mathcal{C}\chi_M \cap \mathcal{U} = \chi_\theta \tag{4.1}$$

Since, M is fuzzy multiplication R-module, then $\mathcal{C} = \mathcal{E}\chi_M$ for some fuzzy ideal $\mathcal{E}$ of R. So equation (4.1) becomes $\mathcal{C}\chi_M \cap \mathcal{E}\chi_M = \chi_\theta$ this can be written as $(\mathcal{C} \cap \mathcal{E})\chi_M = \chi_\theta$. Since, M is faithful R-module, then $\mathcal{C} \cap \mathcal{E} = \chi_\theta$. Since, $\mathcal{C}\chi_M \subseteq \mathcal{D}\chi_M$, $\mathcal{D}\chi_M \neq \chi_M$ and as M is faithful fuzzy multiplication R-module, then by Proposition 18 of [3], $\mathcal{E} \subseteq \mathcal{D}$. Since, $\mathcal{C} \trianglelefteq_{\mathcal{P}} \mathcal{F}$, then $\mathcal{E} = \chi_\theta$ and hence $\mathcal{U} = \chi_\theta$. Thus, $\mathcal{C}\chi_M \trianglelefteq_{\mathcal{P}} \mathcal{F}\chi_M$. □

Theorem 4.2. *let M be a faithful fuzzy multiplication R-module. If there exists an fuzzy essential ideal $\mathcal{U}$ of $F(R)$ such that $\mathcal{C} = \mathcal{U}\chi_M$, where $\mathcal{C}$ is fuzzy submodule of M, then $\mathcal{C}$ is essential.*

Proof. Let $\mathcal{V} \in F(M)$ such that $(\mathcal{U}\chi_M) \cap \mathcal{V} = \chi_\theta$. There exists an fuzzy ideal $\mathcal{F}$ of $F(R)$ with $\mathcal{V} = \mathcal{F}\chi_M$ and hence, $(\mathcal{U} \cap \mathcal{F})\chi_M \subseteq (\mathcal{U}\chi_M) \cap \mathcal{V} = \chi_\theta$. Since M is faithful, it follows that $\mathcal{U} \cap \mathcal{F} = \chi_\theta$ and hence, $\mathcal{F} = \chi_\theta$. Thus, $\mathcal{U}\chi_M$ is fuzzy essential submodule of M. □

Definition 4.1. *A non-zero ring R is called fuzzy fully $\mathcal{P}$-essential if every non-constant fuzzy $\mathcal{P}$-essential ideal of R is essential ideal of R.*

Definition 4.2. *let M be an non-zero module over a commutative ring R. M is called fuzzy fully $\mathcal{P}$-essential if every non-constant fuzzy $\mathcal{P}$-essential submodule of M is essential submodule of M.*

Remark 4.1. *Every fuzzy fully essential submodule is fully $\mathcal{P}$-essential but converse may not be true.*

Example 4.1. *Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{12}$.*
Define fuzzy submodules $\mathcal{U}, \mathcal{P} : M \to [0,1]$ as follows:

$$\mathcal{U}(x) = 1, \quad for\ all\ x \in \mathbb{Z}_{12}.$$

$$\mathcal{P}(x) = \begin{cases} 1, & if\ x \in <\bar{3}>, \\ 0, & if\ x \notin <\bar{3}>. \end{cases}$$

Then $\mathcal{P}_* = <\bar{3}>$ is prime submodule by Corollary 2.1. Then by Theorem 2.3, $\mathcal{P}$ is fuzzy prime submodule of M.

Here we observe that $\mathcal{U}$ is not fuzzy fully $\mathcal{P}$-essential because if we define fuzzy submodules $\mathcal{V} : \mathbb{Z}_{12} \to [0,1]$ by:

$$\mathcal{V}(x) = \begin{cases} 1, & if\ x = 0, \\ 0.7, & if\ x \in <\bar{6}>, \\ 0, & if\ x \notin <\bar{6}>. \end{cases}$$

Then $\mathcal{V}^* = \{0, 6\}$ and $\mathcal{V}^* \trianglelefteq_{\mathcal{P}_*} \mathbb{Z}_{12}$. Then by Theorem 3.2, $\mathcal{V} \trianglelefteq_{\mathcal{P}} \mathbb{Z}_{12}$.
But if we define a fuzzy submodule $\mathcal{W} : \mathbb{Z}_{12} \to [0,1]$ by:

$$\mathcal{W}(x) = \begin{cases} 1, & if\ x = 0, \\ 0.3, & if\ x \in \{4, 8\}, \\ 0, & if\ x \notin \{4, 8\}. \end{cases}$$

Here we observe that $\mathcal{V} \cap \mathcal{W} = \chi_\theta$ but $\mathcal{V} \neq \chi_\theta$.

Theorem 4.3. *let M be a faithful fuzzy multiplication R-module where R is fuzzy fully $\mathcal{P}$-essential ring. Then M is fully $\mathcal{P}$-essential.*

Proof. Let $\mathcal{U}$ be a non-constant fuzzy submodule of M such that $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$.
Since M is faithful fuzzy multiplication R-module then $\mathcal{U} = \mathcal{V}\chi_M$ for some fuzzy $\mathcal{P}$-essential ideal $\mathcal{V}$ of R. By hypothesis, $\mathcal{V}$ is fuzzy essential ideal of R. But M is faithful fuzzy multiplication module, then $\mathcal{U}$ is essential submodule of M, by Theorem 4.2.
Thus, M is fully $\mathcal{P}$-essential module. □

Definition 4.3. *A fuzzy module M is called $\mathcal{P}$-uniform if every non-constant submodule of M is $\mathcal{P}$-essential.*

Remark 4.2. *Each fuzzy uniform R-module is $\mathcal{P}$-uniform but converse is not true.*

Example 4.2. *Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{15}$.*
Define fuzzy submodules $\mathcal{U}, \mathcal{P} : M \to [0,1]$ as follows:

$$\mathcal{U}(x) = 1, \quad for\ all\ x \in \mathbb{Z}_{15}.$$

$$\mathcal{P}(x) = \begin{cases} 1, & if\ x \in <\bar{3}>, \\ 0, & if\ x \notin <\bar{3}>. \end{cases}$$

Then $\mathcal{P}_* =< 3 >$ is prime submodule of M by Corollary 2.1.
Also by Theorem 2.3 $\mathcal{P}$ is fuzzy prime submodule of M and $\mathcal{U}^* = \mathbb{Z}_{15}$. Then $\mathcal{U}^* \trianglelefteq_{\mathcal{P}_*} M$.
Hence by Theorem 3.2, $\mathcal{U} \trianglelefteq_{\mathcal{P}} M$ implies $\mathcal{U}$ is $\mathcal{P}$-uniform.
Now define fuzzy submodule $\mathcal{V} : M \to [0,1]$ such that $\mathcal{V} \subseteq \mathcal{U}$ as follows:

$$\mathcal{V}(x) = \begin{cases} 1, & if\ x = 0, \\ 0.7, & if\ x \in \{5, 10\}, \\ 0, & if\ x \notin \{5, 10\}. \end{cases}$$

we observe that $\mathcal{P} \cap \mathcal{V} = \chi_\theta$ but $\mathcal{V} \neq \chi_\theta$. Thus, $\mathcal{V} \ntrianglelefteq M$. Hence, $\mathcal{U}$ is not uniform.

Proposition 4.2. *Let M be an R-module. Then M is uniform if and only if M is $\mathcal{P}$-uniform and fully $\mathcal{P}$-essential.*

Theorem 4.4. *let M be a faithful fuzzy multiplication R-module. Then M is $\mathcal{P}$-uniform R-module if and only if R is a $\mathcal{P}$-uniform ring.*

Proof. Assume M is $\mathcal{P}$-uniform and let $\mathcal{U}$ be a non-constant fuzzy ideal of R.
Then $\mathcal{U}\chi_M$ is $\mathcal{P}$-essential submodule of M. By Theorem 4.1, $\mathcal{U}$ is $\mathcal{P}$-essential ideal of R.

Conversely, assume that R is a $\mathcal{P}$-uniform ring and $\mathcal{V} \in F(M)$. Since, M is fuzzy multiplication R-module, then there exists an fuzzy ideal $\mathcal{I}$ of R such that $\mathcal{V} = \mathcal{I}\chi_M$.
But R is a $\mathcal{P}$-uniform, so $\mathcal{I}$ is $\mathcal{P}$-essential. Thus, $\mathcal{V}$ is $\mathcal{P}$-essential by Theorem 4.1. □

5. Conclusion

In this article, we have defined $\mathcal{P}$-essential submodules and some of its properties are investigated. Also, some properties of $\mathcal{P}$-essential submodules concerning fuzzy multiplication modules over a ring are studied.

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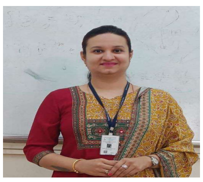
References

[1] Abbas, H. H. and Al-Aeashi, S. N., (2012), A fuzzy semi-essential submodules of a fuzzy module, J. of Kufa for Math. and Computer, 1(5), pp. 31-37.
[2] Acar, U., (2005), On L-fuzzy prime submodules, Hacettepe Journal of Mathematics and Statistics, 34, pp. 17-25.
[3] Atani, S.E. and Sarai, F. E. K., (2017), On L-fuzzy multiplication modules, Discuss. Math. Gen. Algebra Appl., 37, pp. 209-221.
[4] Kalita, M. C., (2007), A study of fuzzy algebraic structures: Some Special Types, Ph.D Thesis, Gauhati University, Gauhati, India.
[5] Khubchandani P. A. and Khubchandani J. A., (2024), Fuzzy biatomic pair in fuzzy lattices, TWMS J. App and Eng. Math., 14(4), pp. 1485-1494.
[6] Moderson, J. N. and Malik D. S., (1998), Fuzzy commutative algebra, World scientific, River Edge, NJ, USA, 1998.
[7] Nagoita, C. V. and Ralesca, D. A., (1975), Applications of fuzzy sets in system analysis, Birkhauser, Basel, Switzerland.
[8] Nimbhorkar, S. K. and Khubchandani, J. A., (2022), Fuzzy essential submodules with respect to an arbitrary fuzzy submodule, TWMS J. App. and Eng. Math., 12(2), pp. 435-444.
[9] Nimbhorkar S. K. and Khubchandani J. A., (2020), Fuzzy semi-essential submodules and fuzzy small-essential submodules, Journal of hyperstructures, 9(2), pp. 52-67.
[10] Nimbhorkar S. K. and Khubchandani J. A., (2023), Fuzzy semi-essential submodules and fuzzy semi-closed submodules, TWMS J. App and Eng. Math., 13(2), pp. 568-575.
[11] Nimbhorkar S. K. and Khubchandani J. A., (2024), L-fuzzy hollow modules and L- fuzzy multiplication modules, Kragujevac Journal of Mathematics, 48(3), pp. 423-432.
[12] Rosenfeld, A., (1971), Fuzzy groups, J. Math. Anal. Appl. 35, pp. 512-517.
[13] Wasadikar M. P. and Khubchandani P. A., (2022), Fuzzy distributive pairs in fuzzy Lattices, Discuss. Math. Gen. Algebra Appl., 42(1), pp. 179-199.
[14] Wasadikar M. P. and Khubchandani P. A., (2022), 'Del' relation and parallelism in fuzzy lattices, TWMS J. App and Eng. Math., 12(3), pp. 864-875.
[15] Zadeh, L. A., (1965), Fuzzy sets, Inform. and Control, 8, pp. 338-353.



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